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Quantum Limits of Eisenstein Series and Scattering States

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Abstract. We identify the quantum limits of scattering states for the modular surface. This is obtained through the study of quantum measures of non-holomorphic Eisenstein series away from the critical line. We provide a range of stability for the quantum unique ergodicity theorem of Luo and Sarnak.

1 Introduction

An important problem of quantum chaos is to describe the limiting behavior of eigenfunctions. On a compact negatively curved Riemannian manifold X, Shnirelman [17], Colin de Verdière [2], and Zelditch [21] have proved that for a 'generic' family of eigenfunctions $\{\phi_j\}$ of the Laplacian the associated measures $d\mu_j(z) = \left|\phi_j(z)\right|^2 d\mu(z)$ converge weakly to the standard volume element $d\mu(z)$ of X, which we write as

$$d\mu_j(z) \to d\mu(z) \text{ as } j \to \infty. \tag{1.1}$$

Zelditch [22] extended the result to finite volume hyperbolic surfaces. Lindenstrauss and Soundararajan [11, 18] have proved that for $X = \Gamma\backslash\mathbb{H}^2$ where $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$ is of a certain *arithmetic* type, (1.1) holds if ϕ_j runs through the set of Hecke–Maaß cusp forms. Earlier Luo and Sarnak [12] investigated the question of quantum chaos for Eisenstein series $E(z, 1/2 + it)$, *i.e., generalized* eigenfunctions on $X = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2$. Since this series is not square integrable, a certain normalization is needed. The actual statement in [12] is the following: Let A and B be compact Jordan measurable subsets of X. Then

$$\lim_{t\to\infty} \frac{\int_A |E(z, 1/2 + it)|^2 \, d\mu(z)}{\int_B |E(z, 1/2 + it)|^2 \, d\mu(z)} = \frac{\mu(A)}{\mu(B)}. \tag{1.2}$$

In fact, see [12], this follows from the result

$$\int_A |E(z, 1/2 + it)|^2 \, d\mu(z) \sim \frac{6}{\pi} \cdot \mu(A) \log t, \quad t \to \infty. \tag{1.3}$$

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A general cofinite subgroup likely has few embedded eigenvalues, possibly finite, so (1.1) may be irrelevant. So far the quantum unique ergodicity of Eisenstein series is unproven for a general cofinite subgroup. A good substitute for the embedded eigenvalues are the scattering poles (resonances). A natural question is to study the quantum limits of these scattering states. We address this question for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. As these states are not in $L^2(\Gamma\backslash\mathbb{H}^2)$, some normalization is also needed. Consider a simple pole ρ of the scattering matrix. By the explicit calculation of the scattering matrix, see (2.2), such a pole is equal to half a zero of the Riemann zeta function. The Eisenstein series also has a pole at this point, and the residue has Fourier expansion

$$\operatorname*{res}_{s=\rho} E(z,s) = \left(\operatorname*{res}_{s=\rho} \phi(s)\right) y^{1-\rho} + \sum_{m\neq 0} c_m \sqrt{y} K_{\rho-1/2}(2\pi m y) e^{2\pi i m x}.$$

These scattering states are formal eigenfunctions of the Laplace operator. We choose to normalize them as follows: Set

$$u_\rho(z) = (\operatorname*{res}_{s=\rho} \phi(s))^{-1} \operatorname*{res}_{s=\rho} E(z,s),$$

so that the scattering functions have the simplest possible growth behavior at infinity, namely $y^{1-\rho}$.

We let $\{\gamma_n\}$ be a sequence of zeroes of the Riemann zeta function with $1/2 \leq \Re(\gamma_n)$, which satisfies $\lim_n \Re(\gamma_n) = \gamma_\infty < 1$. Automatically $\Re(\gamma_n) < 1$, and the Riemann hypothesis is equivalent to $\Re(\gamma_n) = \gamma_\infty = 1/2$, but we shall not assume it. The points $\rho_n = \gamma_n/2$ are poles of the scattering matrix $\phi(s)$.

Theorem 1.1 *Let A be a compact Jordan measurable subset of X. Then*

$$\int_A |u_{\rho_n}(z)|^2 \, d\mu(z) \to \int_A E(z, 2-\gamma_\infty) d\mu(z)$$

as $n \to \infty$. *This means that the quantum limit of the measures* $|u_{\rho_n}(z)|^2 \, d\mu(z)$ *is the invariant, absolutely continuous measure* $E(z, 2-\gamma_\infty) d\mu(z)$.

Remark 1.2 We know that a positive proportion of the zeros ρ of $\zeta(s)$ lie on the critical line and are simple. In fact, this proportion is at least 40.58%; see [1]. Under the Riemann hypothesis and the conjectured simplicity of the Riemann zeros, there is only one quantum limit, the one described in the theorem above: $E(z, 3/2) d\mu(z)$.

Theorem 1.1 follows rather easily by studying the quantum limits of Eisenstein series off the critical line. We present two such theorems. The first addresses the stability of (1.2) if, instead of real spectral value $1/4 + t^2$, we move in the complex plane. To be precise let

$$d\mu_{s(t)}(z) = \left| E\big(z, s(t)\big) \right|^2 d\mu(z),$$

where $s(t) = \sigma_t + it$, $\sigma_t > 1/2$. We investigate what happens in the limit as $t \to \infty$, assuming that $\sigma_t \to \sigma_\infty \geq 1/2$. We find qualitative differences depending on whether or not $\sigma_\infty = 1/2$. If $\sigma_\infty = 1/2$ the situation is very similar to that of [12].

Theorem 1.3 *Assume that $\sigma_\infty = 1/2$. Let A, B be compact Jordan measurable subsets of X. Then*

$$\frac{\mu_{s(t)}(A)}{\mu_{s(t)}(B)} \to \frac{\mu(A)}{\mu(B)}$$

as $t \to \infty$. In fact we have

$$\mu_{s(t)}(A) \sim \mu(A)\frac{3}{\pi(2\sigma_t - 1)}.$$

We see that the rate of increase depends on the rate at which σ_t tends to $\sigma_\infty = 1/2$. Moreover, Theorem 1.3 implies that the quantum unique ergodicity of the Eisenstein series holds in quite a big region in the complex plane (physical plane). For spectral value λ tending to infinity, the results holds as long as $\Im(\lambda) = o(\sqrt{\Re(\lambda)})$ in the region $\Re(\lambda) \geq 0$. To see this we write $\lambda = s(1-s)$ with $s = \sigma + it$ and $\sigma \geq 1/2$. Then $\Im(\lambda) = o(\sqrt{\Re(\lambda)})$ implies $\Re(\lambda) = \sigma(1-\sigma) + t^2 \to \infty$. We easily deduce that σ is bounded. Then $(1-2\sigma)t = o(\sqrt{\sigma(1-\sigma)+t^2})$ gives $\sigma \to 1/2$, so we can apply Theorem 1.3.

Surprisingly the situation is qualitatively different when $\sigma_\infty > 1/2$. We prove the following theorem.

Theorem 1.4 *Assume that $\sigma_\infty > 1/2$. Let A be a compact Jordan measurable subset of X. Then*

$$\mu_{s(t)}(A) \to \int_A E(z, 2\sigma_\infty)d\mu(z)$$

as $t \to \infty$.

This proves that, when $\sigma_\infty > 1/2$, the measures $d\mu_{s(t)}$ do not become equidistributed. In fact it suggests in this case to consider different measures

$$d\nu_{s(t)}(z) = \left|\frac{E(z, s(t))}{\sqrt{E(z, 2\sigma_\infty)}}\right|^2 d\mu(z).$$

We note that, since $2\sigma_\infty > 1$, we have $E(z, 2\sigma_\infty) > 0$. The downside of this definition is that the function $E(z, s(t))/\sqrt{E(z, 2\sigma_\infty)}$ is not an eigenfunction of the Laplacian in contrast to $E(z, s(t))$. The upside is that the corresponding measures become equidistributed.

Corollary 1.5 *Assume that $\sigma_\infty > 1/2$. Let A be a compact Jordan measurable subset of X. Then*

$$\nu_{s(t)}(A) \to \mu(A), \quad t \to \infty.$$

The result of Theorem 1.4 looks similar to [5, Theorem 1], where the authors consider the equidistribution of Eisenstein series for convex co-compact subgroups Γ of Iso($\mathbb{H}^{n+1}$) with Hausdorff dimension of the limit set δ_Γ satisfying $\delta_\Gamma < n/2$. In both theorems the Eisenstein series $E(z, 2\sigma_\infty)$ is well defined and $E(z, 2\sigma_\infty)d\mu(z)$ is the quantum limit.

Similar results to Theorem 1.1 and Theorem 1.4 for more general surfaces with cuspidal ends have recently been announced by Dyatlov [3].

Remark 1.6 The crucial ingredients in [12] are

(i) a subconvex estimate for the L-series of a Maaß cusp form on its critical line, *e.g.,* $L(\phi_j, 1/2 + it) \ll (1 + |t|)^{1/3+\epsilon}$ (see [14]);
(ii) a subconvex estimate for the Riemann zeta function on its critical line, *e.g.,* $\zeta(1/2 + it) \ll (1 + |t|)^{1/6+\epsilon}$;
(iii) estimates for $\zeta(1 + it)$ and $(\zeta'/\zeta)(1 + it)$.

For Theorem 1.3 we use subconvex bounds on L-functions and $\zeta(s)$. When $\sigma_\infty > 1/2$, *i.e.,* in Theorem 1.4, only *convexity* bounds are used. While we use estimates on $\zeta(1+it)$ and $1/\zeta(1+it)$ in both cases, the estimate for $(\zeta'/\zeta)(1+it)$ is required only for the theorem of Luo and Sarnak. Our results clarify the mechanism for quantum unique ergodicity of Eisenstein series.

Remark 1.7 Equation (1.2) was extended by Jakobson [9] to the unit tangent bundle of X. Koyama [10] extended the result to Eisenstein series for $\mathrm{PSL}_2(\mathbb{Z}[i])$, and Truelsen [20] to Eisenstein series for $\mathrm{PSL}_2(\mathcal{O}_K)$, with $\mathcal{O}_K$ the integers of a totally real field K of finite degree over $\mathbb{Q}$ with narrow class number one.

In both cases bounds of the type (i), (ii), and (iii) are used. In the case of $K = \mathbb{Q}(i)$ the subconvex estimate analogous to (i) was established by Petridis and Sarnak [15], and the general case was established by Michel and Venkatesh [13]. As a substitute for (ii) and (iii), one uses estimates for the Dedekind zeta function ζ_K.

The analogous question for holomorphic Hecke cusp form of weight k has recently been resolved by Holowinsky and Soundararajan [6]. Let f_k be a sequence of L^2-normalized holomorphic Hecke cusp forms for the group $\mathrm{SL}_2(\mathbb{Z})$ of weight k and let $\phi_k(z) := y^{k/2} f_k(z)$. Then the measures $|\phi_k(z)|^2 d\mu(z)$ converge weakly to $d\mu(z)$, as previously conjectured by Rudnick and Sarnak. We note that in this case ϕ_k is an eigenfunction of the weight k Laplacian with eigenvalue $k/2(1 - k/2)$.

2 Proofs

The non-holomorphic Eisenstein series $E(z, s)$, $(z, s) \in \mathbb{H}^2 \times \mathbb{C}$ is defined for $\Re(s) > 1$ by

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s.$$

Here $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, and Γ_∞ is the cyclic subgroup generated by $z \mapsto z+1$. The Eisenstein series $E(z, s)$ admits a Fourier expansion of the cusp $i\infty$ (see *e.g.,* [7, (3.25)]),

$$(2.1)\quad E(z, s) = \sum_{n \in \mathbb{Z}} a_n(y, s) e^{2\pi i n x} = y^s + \phi(s) y^{1-s} + \frac{2y^{1/2}}{\xi(2s)} \sum_{n \neq 0} |n|^{s-1/2} \sigma_{1-2s}(|n|) K_{s-1/2}(2\pi |n| y) e^{2\pi i n x}.$$

Here $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function satisfying the functional equation $\xi(s) = \xi(1-s)$; $\sigma_c(n)$ is the sum of the c-th powers of the divisors of n, and $K_s(y)$ is the K-Bessel function. The scattering matrix is

$$\phi(s) = \frac{\xi(2-2s)}{\xi(2s)}. \tag{2.2}$$

We notice that the corresponding expression in [12] is missing a factor of 2 in the non-zero terms. This is irrelevant for their purpose but becomes crucial for ours.

The spectral decomposition of $L^2(\Gamma\backslash\mathbb{H}^2)$ allows us to consider Maaß cusp forms and incomplete Eisenstein series separately.

2.1 Maaß Cusp Forms

Since there is a basis of the cuspidal eigenspaces consisting of Hecke–Maaß cusp forms, we restrict our attention to those.

Lemma 2.1 *Let ϕ_j be a Hecke–Maaß cusp form. Then*

$$\int_{\Gamma\backslash\mathbb{H}^2} \phi_j \left|E(z, \sigma_t + it)\right|^2 d\mu(z) \to 0,$$

as $t \to \infty$.

Proof The Maaß cusp form ϕ_j has a Fourier expansion

$$\phi_j(z) = y^{1/2} \sum_{n\neq 0} \lambda(n) K_{it_j}(2\pi n y) e(nx),$$

with $\lambda(1) = 1$. We assume that it is even, since, if it were odd, $\langle \phi_j, \mu_{s(t)} \rangle = 0$. Being a Hecke eigenform, ϕ_j has an L-series with Euler product

$$L(\phi_j, s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \left(1 - \lambda(p)p^{-s} + p^{-2s}\right)^{-1}.$$

We want to understand the behavior as $t \to \infty$ of

$$J_j(t) = \int_{\Gamma\backslash\mathbb{H}^2} \phi_j(z) \left|E\left(z, s(t)\right)\right|^2 d\mu(z).$$

We calculate

$$I_j(s) = \int_{\Gamma\backslash\mathbb{H}^2} \phi_j(z) E\left(z, s(t)\right) E(z, s) d\mu(z),$$

and set $s = \overline{s(t)}$ to recover $J_j(t)$. For fixed s, $I_j(s)$ is a holomorphic function of $w = s(t)$. In [12] this function is identified for $w = 1/2 + it$, so we use the principle of analytic continuation to deduce that

$$I_j(s) = \frac{R(s)}{\xi(2s(t))} \frac{\prod \Gamma\left(\frac{s \pm it_j \pm (s(t)-1/2)}{2}\right)}{2\pi^s \Gamma(s)},$$

with

$$R(s) = \frac{L(\phi_j, s - s(t) + 1/2)L(\phi_j, s + s(t) - 1/2)}{\zeta(2s)}.$$

We plug $s = \overline{s(t)}$ to get

$$J_j(t) = \quad 2^{-1}\pi^{s(t)-\overline{s(t)}} L(\phi_j, 1/2 - 2it)L(\phi_j, 2\sigma_t - 1/2) \frac{\prod \Gamma\left(\frac{\overline{s(t)} \pm it_j \pm (s(t)-1/2)}{2}\right)}{|\Gamma(s(t))\zeta(2s(t))|^2}.$$

We apply Stirling's formula [8, 5.112] in the form

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}\, |t|^{\sigma - 1/2}\, e^{-\frac{\pi}{2}|t|} \left(1 + O(|t|^{-1})\right) \tag{2.3}$$

uniformly for $|\sigma| \le M$. Using this we find that the quotient of Gamma factors is $\ll_j |t|^{1/2 - 2\sigma_t}$.

If σ_t is bounded away from 1/2, the function $|\zeta(2s(t))|^{-2}$ is bounded and the convexity estimate $L(\phi_j, 1/2 + it) \ll t^{1/2}$ suffices to guarantee that $\lim_t J_j(t) = 0$.

If σ_t is not bounded away from 1/2, then we need non-trivial estimates for $\zeta(2s(t))^{-1}$ and $L(\phi_j, 1/2 + it)$ to reach the same conclusion. Such estimates are certainly available: the estimate

$$\log^{-1} |t| \ll |\zeta(2s(t))| \ll \log |t| \tag{2.4}$$

is classical in the theory of the Riemann zeta function (see [19, 3.6.5 and 3.11.8]), and the subconvex estimate

$$L(\phi_j, 1/2 + it) = O_{j,\epsilon}\left(|t|^{1/3+\epsilon}\right)$$

was proved by Meurman [14]. We note that *any* subconvex estimate of the form $L(\phi_j, 1/2 + it) = O(|t|^{1/2-\epsilon})$ suffices to show that $\lim_t J_j(t) = 0$. ■

2.2 Incomplete Eisenstein Series

We now concentrate on the contribution of the incomplete Eisenstein series. Let $h(y) \in C^\infty(\mathbb{R}^+)$ be a function that decreases rapidly at 0 and ∞. This means that $h(y) = O_N(y^N)$ for $0 < y \le 1$ and $h(y) = O(y^{-N})$ for $y \gg 1$ for all $N \in \mathbb{N}$. Its Mellin transform is

$$H(s) = \int_0^\infty h(y) y^{-s} \frac{dy}{y},$$

and the Mellin inversion formula gives

$$h(y) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} H(s) y^s \, ds \tag{2.5}$$

for any $a \in \mathbb{R}$. The function $H(s)$ is entire and $H(\sigma + it)$ is in the Schwartz space in the t variable for any $\sigma \in \mathbb{R}$. We consider the incomplete Eisenstein series

$$F_h(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h(\Im(\gamma z)) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} H(s) E(z,s)\, ds.$$

Lemma 2.2 *Let F_h be an incomplete Eisenstein series as above. Then*

$$\int_{\Gamma\backslash\mathbb{H}^2} F_h(z) \left|E(z,\sigma_t + it)\right|^2 d\mu(z) \sim \begin{cases} \int_{\Gamma\backslash\mathbb{H}^2} F_h(z) E(z, 2\sigma_\infty) d\mu(z), & \text{if } \sigma_\infty \neq 1/2, \\ \int_{\Gamma\backslash\mathbb{H}^2} F_h(z) d\mu(z) (4\xi(2)(\sigma_t - 1/2))^{-1}, & \text{if } \sigma_\infty = 1/2, \end{cases}$$

as $t \to \infty$.

Proof We choose a such that $a > 2\sigma_t$ for all t. The function $F_h(z)$ is smooth and rapidly decreasing in the cusp. Then unfolding and using Parseval we get

$$\begin{aligned} \int_{\Gamma\backslash\mathbb{H}^2} F_h(z) d\mu_{s(t)}(z) &= \int_{\Gamma\backslash\mathbb{H}^2} F_h(z) \left| E\big(z, s(t)\big) \right|^2 d\mu(z) \\ &= \int_0^\infty \int_0^1 h(y) \left| E\big(z, s(t)\big) \right|^2 \frac{dx\, dy}{y^2} \\ &= \int_0^\infty h(y) \left(\sum_{n \in \mathbb{Z}} \left| a_n\big(y, s(t)\big) \right|^2 \right) \frac{dy}{y^2}. \end{aligned} \tag{2.6}$$

2.3 Contribution of the Constant Term

By (2.1) we have

$$\left| a_0\big(y, s(t)\big) \right|^2 = y^{2\sigma_t} + 2\Re\big(\phi\big(s(t)\big) y^{1-2it}\big) + \left|\phi\big(s(t)\big)\right|^2 y^{2-2\sigma_t}.$$

We analyze the three terms separately. The first term is

$$\int_0^\infty h(y) y^{2\sigma_t - 1} \frac{dy}{y} = H(1 - 2\sigma_t),$$

which converges to $H(1 - 2\sigma_\infty)$ when $t \to \infty$. Next

$$\phi\big(s(t)\big) \int_0^\infty h(y) y^{-2it} \frac{dy}{y} = \phi\big(s(t)\big) H(2it).$$

The function $H(2it)$ decays rapidly and $\phi(s(t))$ is bounded; see [16, (8.6)]. By analyzing the same expression with $\overline{\phi(s(t))} y^{1-2it}$ instead of $\phi(s(t)) y^{1+2it}$ we find that the term in (2.6) involving $\Re(\phi(s(t)) y^{1-2it})$ tends to zero.

The last expression coming from the constant term is

$$\left|\phi\big(s(t)\big)\right|^2 \int_0^\infty h(y) y^{1-2\sigma_t} \frac{dy}{y} = \left|\phi\big(s(t)\big)\right|^2 H(2\sigma_t - 1).$$

Certainly $H(2\sigma_t - 1) \to H(2\sigma_\infty - 1)$ as $t \to \infty$, and $|\phi(s(t))|$ is bounded.

Using the explicit expression for $\phi(s)$ in (2.2) we have better control of the behavior of $\phi(s(t))$ when $\sigma_\infty \neq 1/2$. We have

$$|\phi(s(t))| = \frac{|\xi(2-2s(t))|}{|\xi(2s(t))|} = \pi^{2\sigma_t - 1} \left|\frac{\zeta(2(1-\sigma_t) - 2it)}{\zeta(2\sigma_t + 2it)}\right| \left|\frac{\Gamma(1-\sigma_t - it)}{\Gamma(\sigma_t + it)}\right|.$$

Using the convexity bound $\zeta(\sigma + it) = O(|t|^{(1-\sigma)/2+\epsilon})$ we get

$$\zeta\big(2(1-\sigma_t) + it\big) = O\big(|t|^{\sigma_t - 1/2 + \epsilon}\big).$$

By (2.4),

$$\frac{1}{\zeta(2\sigma_t + 2it)} = O(\log|t|).$$

The quotient of Γ-factors is asymptotic to $|t|^{1-2\sigma_t}$ by (2.3). We therefore conclude that, when $\sigma_\infty \neq 1/2$, we have

$$|\phi(s(t))| \to 0 \tag{2.7}$$

as $t \to \infty$.

To summarize, we have proved that the contribution of the constant term in (2.6) converges to $H(1 - 2\sigma_\infty)$ if $\sigma_\infty \neq 1/2$ and is $O(1)$ if $\sigma_\infty = 1/2$.

2.4 Contribution of the Non-constant Terms

By (2.1) and (2.5) the contribution equals

$$A(t) = \int_0^\infty \frac{1}{2\pi i} \int_{\Re(s)=a} H(s) y^s ds \sum_{n=1}^\infty \frac{8y}{|\xi(2s(t))|^2} n^{2\sigma_t - 1}$$
$$\left|\sigma_{1-2s(t)}(n)\right|^2 \left|K_{s(t)-1/2}(2\pi n y)\right|^2 \frac{dy}{y^2}$$
$$= \int_0^\infty \frac{1}{2\pi i} \int_{\Re(s)=a} H(s) \sum_{n=1}^\infty \frac{y^s}{(2\pi n)^s} \frac{8}{|\xi(2s(t))|^2} n^{2\sigma_t - 1} \left|\sigma_{1-2s(t)}(n)\right|^2$$
$$\left|K_{s(t)-1/2}(y)\right|^2 ds \frac{dy}{y}$$
$$= \frac{1}{2\pi i} \int_{\Re(s)=a} H(s) \frac{1}{(2\pi)^s} \frac{8}{|\xi(2s(t))|^2} \sum_{n=1}^\infty \frac{\left|\sigma_{1-2s(t)}(n)\right|^2}{n^{s-(2\sigma_t - 1)}} \int_0^\infty y^s$$
$$\left|K_{s(t)-1/2}(y)\right|^2 \frac{dy}{y} ds.$$

We now use [4, 6.576 (4)] to calculate the integral involving the K-Bessel functions, and the Ramanujan identity

$$\sum_{n=1}^{\infty}\frac{\sigma_a(n)\sigma_b(n)}{n^s}=\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}$$

to see that

$$\begin{aligned}A(t)&=\frac{1}{2\pi i}\int_{\Re(s)=a}H(s)\frac{1}{|\xi(2s(t))|^2}\frac{\xi(s-2\sigma_t+1)\xi(s-2it)\xi(s+2it)\xi(s+2\sigma_t-1)}{\xi(2s)}ds\\&=\frac{1}{|\xi(2s(t))|^2\,2\pi i}\int_{\Re(s)=a}B(s)ds,\end{aligned}$$

where $B(s)$ equals $H(s)$ times the ξ-factors. Since $\xi(s)$ has poles at $s=0,1$, the poles of $B(s)$ in the region $\Re(s)\geq 1/2$ are at $1\pm 2it$, $2\sigma_t$, $2-2\sigma_t$, $\pm(2\sigma_t-1)$, and $\pm 2it$. We now move the line of integration to $\Re(s)=1/2$. By considering the Stirling asymptotics for the Γ-factors, convexity bounds for the zeta functions, equation (2.4), and the rapid decay of $H(s)$ we see that $B(s)$ decays rapidly in vertical strips, and this allows us to move the line of integration. We find that

$$\begin{aligned}A(t)=\frac{1}{|\xi(2s(t))|^2}\Big(&\operatorname*{res}_{s=1\pm 2it}B(s)+\operatorname*{res}_{s=2\sigma_t}B(s)\\&+\delta_t\cdot\operatorname*{res}_{s=2-2\sigma_t}B(s)+(1-\delta_t)\cdot\operatorname*{res}_{s=2\sigma_t-1}B(s)+\frac{1}{2\pi i}\int_{\Re(s)=1/2}B(s)ds\Big),\end{aligned}$$

with $\delta_t=1$ if $\sigma_t<3/4$ and 0 otherwise. We analyze these five terms.

(i) Using Stirling, convexity bounds on the zeta functions, (2.4), and the rapid decay of $H(1\pm 2it)$, the term

$$\frac{1}{|\xi(2s(t))|^2}\operatorname*{res}_{s=1\pm 2it}B(s)=\\ H(1\pm 2it)\frac{\xi(1\pm 4it)\xi(1\pm 2it-2\sigma_t+1)\xi(1\pm 2it+2\sigma_t-1)}{|\xi(2s(t))|^2\,\xi(2\pm 4it)}$$

tends to zero as $t\to\infty$.

(ii) We now consider the second term:

$$\frac{1}{|\xi(2s(t))|^2}\operatorname*{res}_{s=2\sigma_t}B(s)=H(2\sigma_t)\frac{\xi(4\sigma_t-1)}{\xi(4\sigma_t)},$$

which converges to $H(2\sigma_\infty)\frac{\xi(4\sigma_\infty-1)}{\xi(4\sigma_\infty)}$ when $t\to\infty$ and $\sigma_\infty\neq 1/2$. When $\sigma_t\to 1/2$ it behaves asymptotically like

$$H(1)\frac{1}{4\xi(2)(\sigma_t-1/2)}.$$

(iii) We then move on to the third term:

$$\frac{1}{|\xi(2s(t))|^2}\underset{s=2-2\sigma_t}{\mathrm{res}} B(s) = H(2-2\sigma_t)\frac{\xi(1-4\sigma_t)\xi(2-2\sigma_t-2it)\xi(2-2\sigma_t+2it)}{|\xi(2s(t))|^2\,\xi(4-4\sigma_t)}$$
$$= H(2-2\sigma_t)\,|\phi(s(t))|^2\,\frac{\xi(1-4\sigma_t)}{\xi(4-4\sigma_t)}.$$

If $\sigma_\infty \neq 1/2$, we can use (2.7) to conclude that this tends to zero, and if $\sigma_\infty = 1/2$, it is bounded, since $|\phi(s(t))|^2$ is bounded.

(iv) The fourth term is

$$\frac{1}{|\xi(2s(t))|^2}\underset{s=2\sigma_t-1}{\mathrm{res}} B(s) = H(2\sigma_t-1)\,|\phi(s(t))|^2\,.$$

When $\sigma_\infty \neq 1/2$ this tends to zero and when $\sigma_\infty = 1/2$ it is bounded, by the same arguments as for the third term.

(v) We now deal with the last term, *i.e.,*

$$\frac{1}{|\xi(2s(t))|^2\,2\pi i}\int_{\Re(s)=1/2} B(s)ds = \frac{1}{2\pi\,|\xi(2\sigma_t+2it)|^2}$$
$$\times\int_{-\infty}^{\infty} H(1/2+i\tau)\frac{|\xi(1/2+2\sigma_t-1+i\tau)|^2\,\xi(1/2+i(\tau-2t))\xi(1/2+i(\tau+2t))}{\xi(1+2i\tau)}d\tau.$$

We note that $H(1/2+i\tau)$ is of rapid decay. We study first the exponential behavior of the integral as a function of t. Stirling asymptotics for the integrand give

$$(e^{-\pi|\tau|/4})^2 e^{-\pi|\tau/2-t|/2} e^{-\pi|\tau/2+t|/2}(e^{\pi|\tau|/2}) \leq e^{-\pi t},$$

which cancels with the exponential growth of $1/\,|\xi(2s(t))|^2$. Using (2.4), the rapid decay of $H(1/2+i\tau)$, and any polynomial bound in τ of $\zeta(2\sigma_t-1/2+i\tau)$, we are reduced to estimate in t the integral

$$\frac{\log|t|}{(t^{-1/2+\sigma_t})^2}\int_{-\infty}^{\infty}\tilde{H}(\tau)\big(1+|\tau+2t|\big)^{-1/4}\big(1+|\tau-2t|\big)^{-1/4}$$
$$\Big|\,\zeta\big(1/2+i(\tau-2t)\big)\,\zeta\big(1/2+i(\tau+2t)\big)\,\Big|\,d\tau,$$

where $\tilde{H}$ is some function of rapid decay. We separate now the two cases $\sigma_\infty > 1/2$ and $\sigma_\infty = 1/2$. In the first case we use the convexity bound on the ζ function to estimate the expression as

$$\frac{\log|t|}{(t^{-1/2+\sigma_t})^2}\int_{-\infty}^{\infty}\tilde{H}(\tau)\big(1+|\tau+2t|\big)^{\epsilon}\big(1+|\tau-2t|\big)^{\epsilon}\,d\tau = o(1),$$

as $\sigma_\infty > 1/2$. For the second case we can use any subconvex bound $\zeta(1/2+it) = O(|t|^{1/4-\delta})$, for instance Weyl's bound [19, Theorem 5.5]

$$\zeta(1/2+it) \ll |t|^{1/6+\epsilon}.$$

We are reduced to estimate in t the integral

$$\frac{\log|t|}{(t^{-1/2+\sigma_t})^2}\int_{-\infty}^{\infty}\tilde{H}(\tau)(1+|\tau+2t|)^{-1/4+1/4-\delta}(1+|\tau-2t|)^{-1/4+1/4-\delta}d\tau,$$
$$= t^{1-2\sigma_t+\epsilon}\int_{-\infty}^{\infty}\tilde{H}(\tau)(1+|\tau+2t|)^{-\delta}(1+|\tau-2t|)^{-\delta}\,d\tau = o(1).$$

This concludes the evaluation of the non-constant terms in (2.6).

To summarize we have proved that if $\sigma_\infty \neq 1/2$ the function $\int_{\Gamma\backslash\mathbb{H}^2} F_h(z)d\mu_{s(t)}(z)$ converges to

$$H(1-2\sigma_\infty) + H(2\sigma_\infty)\frac{\xi(4\sigma_\infty-1)}{\xi(4\sigma_\infty)}$$

as $t\to\infty$, and, if $\sigma_\infty = 1/2$,

$$\int_{\Gamma\backslash\mathbb{H}^2} F_h(z)d\mu_{s(t)}(z) \sim \frac{H(1)}{4\xi(2)((\sigma_t-1/2)},$$

as $t\to\infty$. This finishes the proof once we notice that

$$\begin{aligned} H(1-2\sigma_\infty)+&H(2\sigma_\infty)\frac{\xi(4\sigma_\infty-1)}{\xi(4\sigma_\infty)}\\ &= \int_0^\infty h(y)\big(y^{(2\sigma_\infty-1)+1}+\phi(2\sigma_\infty)y^{-2\sigma_\infty+1}\big)\frac{dy}{y^2}\\ &= \int_0^\infty h(y)\left(\int_0^1 E(z,2\sigma_\infty)dx\right)\frac{dy}{y^2}\\ &= \int_{\Gamma\backslash\mathbb{H}^2} F_h(z)E(z,2\sigma_\infty)d\mu(z) \end{aligned}$$

and

$$H(1) = \int_{\Gamma\backslash\mathbb{H}^2} F_h(z)d\mu(z). \qquad \blacksquare$$

It is straightforward to verify that Theorems 1.3 and 1.4 follow from Lemmas 2.1 and 2.2 using an approximation argument as in the proof of [12, Proposition 2.3].

Remark 2.3 In [12] the point $s=1$ was a double pole, coming from $2\sigma_t$ and $2-2\sigma_t$ with $\sigma_t = 1/2$, and this explains the logarithm in (1.3).

Proof of Theorem 1.1 We have

$$\begin{aligned}|u_{\rho_n}|^2\, d\mu(z) &= \left|\Big(\operatorname*{res}_{s=\rho_n}\phi(s)\Big)^{-1}\operatorname*{res}_{s=\rho_n}E(z,s)\right|^2 d\mu(z)\\ &= \left|\Big(\operatorname*{res}_{s=\rho_n}\phi(s)\Big)^{-1}\operatorname*{res}_{s=\rho_n}\phi(s)E(z,1-s)\right|^2 d\mu(z)\\ &= |E(z,1-\rho_n)|^2\, d\mu(z).\end{aligned}$$

The result now follows from Theorem 1.4 with $\sigma_\infty = 1-\gamma_\infty/2$. ■

Proof of Corollary 1.5 Let f be a test function for the convergence in Corollary 1.5. Then we use

$$\frac{f(z)}{E(z,2\sigma_\infty)}$$

as test function for Theorem 1.4 to deduce that, as $t\to\infty$,

$$\int_{\Gamma\backslash\mathbb{H}^2}\frac{f(z)}{E(z,2\sigma_\infty)}d\mu_{s(t)} \to \int_{\Gamma\backslash\mathbb{H}^2}\frac{f(z)}{E(z,2\sigma_\infty)}E(z,2\sigma_\infty)d\mu(z) = \int_{\Gamma\backslash\mathbb{H}^2} f(z)\, d\mu(z).$$

Finally one uses the approximation argument in [12, Proposition 2.3] to complete the proof. ■

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