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DISTANCES BETWEEN CONVEX SUBSETS OF STATE SPACES

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Let L be a closed linear space of continuous real-valued functions, containing constants, on a compact Hausdorff space Ω. This paper gives some new criteria for a closed subset E of Ω to be an L-interpolation set, or more generally for $L|E$ to be uniformly closed or simplicial, in terms of distances between certain compact convex subsets of the state space of L. These criteria involve the facial structure of the state space and hence are of a geometric nature. The results sharpen some standard results of Glicksberg.

1. Introduction

The object of study will be a uniformly closed linear subspace L of continuous real-valued functions on a compact Hausdorff space Ω, such that L contains the constant functions and separates the points of Ω. We will denote by K the *state space* of L, so that

$$K = \{\varphi \in L^* : \|\varphi\| = 1 = \varphi(1)\}$$

endowed with the w^*-topology. There is a natural isometric isomorphism between L and $A(K)$, the Banach space of all continuous real-valued affine functions on K with the supremum norm, and a natural homeomorphic embedding of Ω into K (see Alfsen [1, II.2]).

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We shall see that if E is a closed subset of Ω then the uniform-closedness of the restriction space $L|E$ is equivalent to properties involving distances between certain convex subsets of K. Similarly, the simplicial nature of $L|E$ may also be interpreted by distance properties.

2. Closed restrictions

If A and B are non-empty subsets of L^* we write

$$d_L(A, B) = \inf\{\|x-y\| : x \in A, y \in B\} .$$

If there exists some f in L with $f \geq 1$ on A, $f \leq -1$ on B then it is clear that $d_L(A, B) \geq 2/\|f\|$, and in fact that $d_L(\overline{\text{co}}\, A, \overline{\text{co}}\, B) \geq 2/\|f\|$. Here, for example, $\overline{\text{co}}\, A$ denotes the w^*-closed convex hull of A in K. The following lemma gives a useful converse result.

LEMMA. *Let A and B be non-empty closed convex subsets of K such that $d_L(A, B) = d > 0$. Then there exists a function f in L such that $f \geq 1$ on A, $f \leq -1$ on B and $\|f\| < 6/d$.*

Proof. We write

$$A_1 = \{x \in L^* : d_L(x, A) \leq d/3\} , \quad B_1 = \{y \in L^* : d_L(y, B) \leq d/3\} .$$

Then A_1 and B_1 are disjoint w^*-closed convex sets, and so there exist some $g \in L$ and constants α, β with $\alpha > \beta$ such that $g \geq \alpha$ on A_1, $g \leq \beta$ on B_1. It follows that $g \geq \alpha + (d/3)\|g\|$ on A, $g \leq \beta - (d/3)\|g\|$ on B. We write

$$A_2 = \{x \in K : g(x) \geq \alpha+(d/3)\|g\|\} , \quad B_2 = \{y \in K : g(y) \leq \beta-(d/3)\|g\|\} ,$$

so that A_2 and B_2 are disjoint w^*-closed convex sets. In fact we have

$$d_L(A_2, B_2) \geq \|g\|^{-1}\{\alpha+(d/3)\|g\|-(\beta-(d/3)\|g\|)\} > 2d/3 .$$

Since L contains the constant functions we can find a function f in L with

$$A_2 = \{x \in K : f(x) \geq 1\} , \quad B_2 = \{y \in K : f(y) \leq -1\} .$$

In order to estimate $\|f\|$ we take $x_1 \in A_2$, $y_1 \in B_2$ such that $f(x_1) - f(y_1) > \|f\|$ and choose $\lambda, \lambda' \in [0, 1]$ so that $f(x_2) = 1$, $f(y_2) = -1$, where

$$x_2 = \lambda x_1 + (1-\lambda)y_1 \ , \quad y_2 = \lambda' x_1 + (1-\lambda')y_1 \ .$$

Since $\lambda > \lambda'$ we therefore have

$$(\lambda-\lambda')\|f\| < f\big((\lambda-\lambda')(x_1-y_1)\big) = f(x_2-y_2) = 2 \ ,$$

while

$$2(\lambda-\lambda') \geq \|(\lambda-\lambda')(x_1-y_1)\| = \|x_2-y_2\| \geq 2d/3 \ .$$

Consequently it follows that $\|f\| < 6/d$ as required.

If E is a closed subset of Ω (or of K) we will denote by $M(E)$ the family of all Radon measures on E, and by $M_1^+(E)$ the subset of all probability measures on E. Any measure $\mu \in M(E)$ may be considered also as a member of $M(\Omega)$ (or of $M(K)$). If $\mu \in M(\Omega)$ then we write

$$\begin{aligned}\|\mu\|_L &= \sup\{|\mu(f)| : f \in L, \|f\| \leq 1\} \\ &= \inf\{\|\mu+\nu\| : \nu \in L^\perp\}\end{aligned}$$

where $L^\perp$ denotes the family of measures in $M(\Omega)$ which annihilate L. Similarly, for $\mu \in M(E)$, we may define $\|\mu\|_{L|E}$.

If A, B are closed subsets of Ω then, since $x \in \overline{\text{co}}\, A$ if and only if there exists some $\mu \in M_1^+(A)$ with resultant x, we see that

$$d_L(\overline{\text{co}}\, A, \overline{\text{co}}\, B) = \inf\{\|\mu-\nu\|_L : \mu \in M_1^+(A), \nu \in M_1^+(B)\} \ .$$

The closed set E will be called an *L-interpolation set* if $L|E = C_{\mathbb{R}}(E)$. Glicksberg [8] gave necessary and sufficient conditions for E to be an L-interpolation set or for $L|E$ just to be uniformly closed; namely

(i) E is an L-interpolation set if and only if there exists a constant $C \geq 1$ such that $\|\mu|E\| \leq C\|\mu|(\Omega\backslash E)\|$ whenever $\mu \in L^\perp$,

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(ii) $L|E$ is uniformly closed if and only if there exists a constant $C \geq 1$ such that $\|\mu\|_{L|E} \leq C\|\mu\|_L$ for all $\mu \in M(E)$.

Using the above lemma we may re-formulate condition (i) as follows.

PROPOSITION. *A closed set E is an L-interpolation set if and only if there exists an $\varepsilon > 0$ such that $d_L(\overline{\text{co}}\, A, \overline{\text{co}}\, B) \geq \varepsilon$ whenever A and B are disjoint closed subsets of E .*

Proof. If E is an L-interpolation set then there exists an extension constant $C \geq 1$ such that each $f \in C_{\mathbb{R}}(E)$ has an extension $g \in L$ with $\|g\| \leq C\|f\|$. By Urysohn's Lemma we may take $\varepsilon = 2/C$.

Conversely, take $\mu \in L^{\perp}$ and, given $\delta > 0$, choose disjoint closed subsets A, B of E such that $\mu^+(E\backslash A) < \delta$, $\mu^-(E\backslash B) < \delta$ (where $\mu = \mu^+ - \mu^-$ is the Jordan decomposition of μ). Using the hypothesis and the lemma we may choose an $f \in L$ with $f \geq 1$ on A , $f \leq -1$ on B and $\|f\| < 6/\varepsilon$. Therefore we have

$$\begin{aligned} \|\mu|E\| &\leq \|\mu^+|A\| + \|\mu^-|B\| + 2\delta \\ &\leq \int_{A\cup B} f d\mu + 2\delta \leq \int_E f d\mu + 2\delta(1+\|f\|) . \end{aligned}$$

Using the facts that $\int_\Omega f d\mu = 0$ and that $\delta > 0$ is arbitrary we obtain $\|\mu|E\| \leq (6/\varepsilon)\|\mu|(\Omega\backslash E)\|$ so that E is an L-interpolation set by condition (i).

In the above proposition it is not sufficient to know only that $d_L(\overline{\text{co}}\, A, \overline{\text{co}}\, B) > 0$ (see McDonald [*10*, p. 432]).

We now consider analogous results for the situation when $L|E$ is uniformly closed, and for these we take a more geometric approach. We replace L by $A(K)$ and $L|E$ by $A(K)|F$, where $F = \overline{\text{co}}\, E$, and note that $L|E$ is uniformly closed if and only if $A(K)|F = A(F)$.

THEOREM 1. *Let F be a closed convex subset of K . Then $A(K)|F = A(F)$ if and only if there exists an $\varepsilon > 0$ such that*

$$d_{A(K)}(G, H) \geq \varepsilon d_{A(F)}(G, H)$$

whenever G *and* H *are disjoint peak faces of* F .

Proof. If $A(K)|F = A(F)$ then there exists an extension constant $C \geq 1$ such that each $f \in A(F)$ has an extension $g \in A(K)$ with $\|g\| \leq C\|f\|$. Consequently we may take $\varepsilon = 1/C$.

Conversely, we will show that the hypothesis involving ε implies condition (ii) of Glicksberg. Let $\mu \in M(F)$ such that $\|\mu\|_{A(F)} \neq 0$, and write $\psi \in A(F)^*$ such that $\psi(u) = \int u d\mu$ for $u \in A(F)$. If we take $\delta = (\varepsilon/8)\|\mu\|_{A(F)}$ then, by the Bishop-Phelps theorem [4], we may find some $\varphi \in A(F)^*$ and $f \in A(F)$ such that $\|f\| = 1$, $\|\varphi\| = \varphi(f)$ and $\|\varphi-\psi\| < \delta$. Let $\nu \in M(F)$ represent φ and satisfy $\|\nu\| = \|\varphi\|$. If we write $G = f^{-1}(1)$, $H = f^{-1}(-1)$ in F then G and H are (possibly empty) peak faces of F which respectively support ν^+ and ν^- .

If either G or H is empty then we have

$$\|\nu\| = |\nu(1)| = \|\nu\|_{A(F)} = \|\nu\|_{A(K)} ,$$

so that, since $\varepsilon \leq 1$,

$$\begin{aligned}\|\mu\|_{A(K)} &\geq |\mu(1)| = |\psi(1)| > |\varphi(1)| - \delta = \|\varphi\| - \delta \\ &> \|\psi\| - 2\delta = \|\mu\|_{A(F)} - 2\delta \geq (3/4)\|\mu\|_{A(F)} .\end{aligned}$$

Otherwise, since $d_{A(F)}(G, H) = 2$, the hypothesis and the lemma enable us to find $w \in A(K)$ such that $\|w\| = 1$, $w \geq \varepsilon/3$ on G and $w \leq -\varepsilon/3$ on H . Consequently we have

$$\begin{aligned}\|\mu\|_{A(K)} &\geq \int w d\mu = \int w d\nu + (\psi-\varphi)(w|F) \\ &\geq (\varepsilon/3)(\|\nu^+\|+\|\nu^-\|) - \delta = (\varepsilon/3)\|\varphi\| - \delta \\ &\geq (\varepsilon/3)\|\psi\| - \delta\big(1+(\varepsilon/3)\big) = (\varepsilon/3)\|\mu\|_{A(F)} - \delta\big(1+(\varepsilon/3)\big) \\ &\geq (\varepsilon/6)\|\mu\|_{A(F)} .\end{aligned}$$

In either case Glicksberg's condition (ii) clearly holds.

The direct analogue of the result of the proposition may now be deduced.

COROLLARY 1. *Let* E *be a closed subset of* Ω . *Then* $L|E$ *is uniformly closed if and only if there exists an* $\varepsilon > 0$ *such that*

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$d_L(\overline{\text{co}}\, A, \overline{\text{co}}\, B) \geq \varepsilon d_{L|E}(\overline{\text{co}}\, A, \overline{\text{co}}\, B)$ *whenever* A *and* B *are disjoint closed subsets of* E .

Proof. The necessity of the condition follows easily by again using the existence of an extension constant.

To prove the sufficiency we take $F = \overline{\text{co}}\, E$ and consider disjoint closed faces G and H of F. If we write $A = G \cap \Omega$, $B = H \cap \Omega$ then, since Ω contains the set of extreme points of K, we must have $G = \overline{\text{co}}\, A$, $H = \overline{\text{co}}\, B$. The result now follows immediately from Theorem 1.

Now let A denote a uniformly closed linear subspace of $C_{\mathbb{C}}(\Omega)$ containing constants and separating points of Ω. Write S for the state-space of A endowed with the w^*-topology, so that

$$S = \{\varphi \in \mathsf{A}^* : \|\varphi\| = 1 = \varphi(1)\} .$$

Then, if $K = \text{co}(S \cup -iS)$, the map $\Theta : \mathsf{A} \to A(K)$ defined by $(\Theta f)(k) = \text{re}\, k(f)$, $f \in \mathsf{A}$, $k \in K$, gives a real-linear homeomorphism of A onto $A(K)$ such that $\|\Theta f\| \leq \|f\| \leq \sqrt{2}\|\Theta f\|$ (see Asimow and Ellis [2, 4.0]). Moreover $\text{re}\, \mathsf{A}$ is naturally isometrically isomorphic to $A(K)|S$. If A and B are subsets of Ω we may write $d_{\text{re}\mathsf{A}}(\overline{\text{co}}\, A, \overline{\text{co}}\, B)$ for $d_{A(K)|S}(\overline{\text{co}}\, A, \overline{\text{co}}\, B)$ and if we define

$$d_{\mathsf{A}}(\overline{\text{co}}\, A, \overline{\text{co}}\, B) = \inf\{\|x-y\| : x \in \overline{\text{co}}\, A,\ y \in \overline{\text{co}}\, B\}$$

then we have

$$d_{\mathsf{A}}(\overline{\text{co}}\, A, \overline{\text{co}}\, B) \leq d_{A(K)}(\overline{\text{co}}\, A, \overline{\text{co}}\, B) \leq \sqrt{2} d_{\mathsf{A}}(\overline{\text{co}}\, A, \overline{\text{co}}\, B) .$$

Following this discussion it is now easy to derive from Corollary 1 a condition equivalent to $\text{re}\, \mathsf{A}$ being uniformly closed. In particular in the case when A is a uniform algebra on Ω this condition, together with the Hoffman-Wermer theorem [9], gives the following result.

COROLLARY 2. *Let* A *be a uniform algebra on* Ω . *Then* $\mathsf{A} = C_{\mathbb{C}}(\Omega)$ *if and only if there exists* $\varepsilon > 0$ *such that* $d_{\mathsf{A}}(\overline{\text{co}}\, A, \overline{\text{co}}\, B) \geq \varepsilon d_{\text{re}\mathsf{A}}(\overline{\text{co}}\, A, \overline{\text{co}}\, B)$ *whenever* A *and* B *are disjoint subsets of* Ω *which are peak sets for* $\overline{\text{re}\, \mathsf{A}}$.

It would be of interest to know whether in Corollary 2 peak sets for

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$\overline{\text{re}}\ A$ may be replaced by peak sets for $\text{re}\ A$. In this context the result of Briem [5] is relevant. However the closed convex hulls are essential in Corollary 2 since, by using the exponential function on A, it is easy to show the existence of an $\varepsilon > 0$ such that $d_A(A, B) \geq \varepsilon d_{\text{re}A}(A, B)$ whenever A and B are disjoint closed subsets of Ω and A is any uniform algebra on Ω.

We note that if we strengthen the hypothesis of Theorem 1 by assuming the existence of an $\varepsilon > 0$ such that $d_{A(K)}(G, H) \geq \varepsilon$ whenever G and H are disjoint peak faces of F then we can not conclude that F is a Bauer simplex (that is, that $A(K)|(F \cap \Omega) = C_{\mathbb{R}}(F \cap \Omega)$ where Ω denotes in this case the closure of the set of extreme points of K). Indeed if F is any finite-dimensional polytope then the strengthened hypothesis clearly holds. We now consider a further strengthening of the hypothesis which will imply that F is a simplex.

THEOREM 2. *Let F be a closed convex subset of K. Then F is a simplex such that $A(K)|F = A(F)$ if and only if there exists an $\varepsilon > 0$ such that $d_{A(K)}(\overline{\text{co}}\ A, \overline{\text{co}}\ B) \geq \varepsilon$ whenever A and B are disjoint closed extremal subsets of F.*

Proof. If F is a simplex and if A, B are disjoint closed extremal subsets of F then a result of Effros [6, Theorem 3.3] shows that $\overline{\text{co}}\ A$ and $\overline{\text{co}}\ B$ are disjoint closed faces of F. Therefore we have $d_{A(F)}(\overline{\text{co}}\ A, \overline{\text{co}}\ B) = 2$. If we assume further that $A(K)|F = A(F)$ then the existence of an extension constant C leads to the required $\varepsilon = 2/C$.

Conversely, suppose that $\varepsilon > 0$ exists such that $d_{A(K)}(G, H) \geq \varepsilon$ whenever G and H are disjoint closed extremal subsets of F. Applying Theorem 1 in the special case where G and H are closed faces of F gives $A(K)|F = A(F)$. Suppose that there exists a non-zero boundary measure μ on F annihilating $A(F)$. If $\mu = \mu^+ - \mu^-$ denotes the Jordan decomposition of (the Baire-restriction of) μ then we can find disjoint Baire sets D and E which respectively support μ^+ and μ^-. Given $\delta > 0$ the result of Teleman [11, corollary to Theorem 1] (see also Batty [3, Proposition 5]) shows that there exist disjoint closed extremal sets $A \subseteq D$ and $B \subseteq E$ such that $\mu^+(D\backslash A) < \delta$ and $\mu^-(E\backslash B) < \delta$.

Applying the hypothesis to A and B, the lemma gives $f \in A(K)$ such that $\|f\| \le 1$, $f \ge \varepsilon/6$ on A, $f \le -\varepsilon/6$ on B. Consequently we obtain

$$\begin{aligned} 0 = \int_F f d\mu &= \int_A f d\mu^+ - \int_B f d\mu^- + \int_{F\setminus(A\cup B)} f d\mu \\ &\ge (\varepsilon/6)(\|\mu^+\|+\|\mu^-\|-2\delta) - 2\delta > 0 , \end{aligned}$$

for $\delta > 0$ sufficiently small. This contradiction proves that F is a simplex.

In the case where F is metrisable the result of Theorem 2 may be simplified by using compact subsets of extreme points of F in place of compact extremal subsets of F. In the non-metrisable case the example [7, Theorem 1] shows that this simplified hypothesis fails to imply that F is a simplex. The result of Teleman on which Theorem 2 depends uses his theory of metrisable reductions for compact convex sets.

We note that in Theorem 2 the result is unchanged if we replace ε by 2. In order to obtain an interpolation result we must strengthen the hypothesis to include sets other than extremal subsets. For this purpose we revert to our original space L with state space K.

COROLLARY 3. *A closed set* E *is an* L*-interpolation set if and only if there exists an* $\varepsilon > 0$ *such that* $d_L(\overline{\text{co}}\, A, \overline{\text{co}}\, B) \ge \varepsilon$ *whenever the disjoint sets closed sets* A *and* B *are either singletons in* E *or closed extremal subsets of* $\overline{\text{co}}\, E$.

Proof. It follows directly from Theorem 2 that $L|E$ is a simplex space. To complete the proof we need to show that each $x \in E$ is an extreme point of $\overline{\text{co}}\, E$ so that $\overline{\text{co}}\, E$ is a Bauer simplex.

If $x \in E$ is not extreme then the maximal measure μ representing x has zero mass at x, and hence, given $\delta > 0$, we may find a closed extremal subset B of $\overline{\text{co}}\, E$ such that $\mu(B) > 1 - \delta$. By the lemma there exists some $f \in L$ such that $f(x) \ge 1$, $f(y) \le -1$ for $y \in B$ and $\|f\| < 6/\varepsilon$. But then we have

$$\begin{aligned} 1 \le f(x) = \int f d\mu &= \int_B f d\mu + \int_{E\setminus B} f d\mu \\ &\le -(1-\delta) + 6\delta/\varepsilon . \end{aligned}$$

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Since $\delta > 0$ was arbitrary we obtain the desired contradiction.

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