Canad. Math. Bull. Vol. 66 (2), 2023, pp. 544–552
http://dx.doi.org/10.4153/S0008439522000510

A note on the relative growth of products of multiple partial quotients in the plane

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Abstract. Let $r = [a_1(r), a_2(r), \ldots]$ be the continued fraction expansion of a real number $r \in \mathbb{R}$. The growth properties of the products of consecutive partial quotients are tied up with the set admitting improvements to Dirichlet's theorem. Let $(t_1, \ldots, t_m) \in \mathbb{R}_+^m$, and let $\Psi : \mathbb{N} \to (1, \infty)$ be a function such that $\Psi(n) \to \infty$ as $n \to \infty$. We calculate the Hausdorff dimension of the set of all $(x, y) \in [0,1)^2$ such that

$$\max\left\{\prod_{i=1}^{m} a_{n+i}^{t_i}(x), \prod_{i=1}^{m} a_{n+i}^{t_i}(y)\right\} \geq \Psi(n)$$

is satisfied for all $n \geq 1$.

1 Introduction

The theory of continued fractions is simple yet extremely useful in characterizing irrational numbers. It is well known that every irrational $x \in [0,1)$ can be uniquely expressed as a simple infinite continued fraction expansion of the form

$$x = [a_1(x), a_2(x), \ldots,],$$

where $a_n(x) \in \mathbb{N}, n \geq 1$, are known as the partial quotients of x. The theory of continued fractions plays a pivotal role in giving quantitative information on how well a real number can be approximated by rationals. The main connection is that the nth convergents of a real number x are good rational approximates for x, summarized as

$$\frac{1}{(2 + a_{n+1}(x))q_n^2(x)} \leq \left|x - \frac{p_n(x)}{q_n(x)}\right| \leq \frac{1}{a_{n+1}(x)q_n^2(x)}.$$

The nth convergent of x, $\frac{p_n(x)}{q_n(x)}$, is a rational number obtained by truncating the continued fraction expansion of x at the nth term, that is, $\frac{p_n(x)}{q_n(x)} := [a_1(x), a_2(x), \ldots, a_n(x)]$. This, in turn, gives an alternative form of the famous Jarník–Besicovitch set; for any $\tau > 0$,

$$\{x \in [0,1) : a_n(x) \geq q_n^\tau(x) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Received by the editors June 27, 2022; revised August 8, 2022; accepted August 9, 2022.
Published online on Cambridge Core August 19, 2022.
This research is supported by the Australian Research Council Discovery Project (ARC DP200100994).
AMS subject classification: **11K50**, 11K55.
Keywords: Metric continued fractions, Hausdorff dimension, uniform Diophantine approximation.

The Hausdorff dimension, denoted by $\dim_{\mathcal{H}}$, of this set is $\frac{2}{2+\tau}$ (see [16] for more details). There have been plenty of work regarding the metrical properties of the growth of partial quotients, for instance, the classical Borel–Bernstein theorem states that the Lebesgue measure of the set

$$\{x \in [0,1) : a_n(x) \geq \Psi(n) \text{ for infinitely many } n \in \mathbb{N}\},$$

is either zero or full according to the convergence or divergence of the series $\sum_n 1/\Psi(n)$. For rapidly increasing functions, Ψ gives no information regarding the size of this set other than that its Lebesgue measure zero. One of the most appropriate tools to distinguish between Lebesgue zero sets is the notion of Hausdorff dimension. Good [6] proved the Hausdorff dimension for the set $\{x \in [0,1) : a_n(x) \to \infty$ as $n \to \infty\}$ to be $1/2$. Later, Łuczak [14] extended Good's result to the functions of the type $\Psi(n) = c^{b^n}$, where $b, c > 1$. The Hausdorff dimension for an arbitrary function Ψ was comprehensively established by Wang and Wu in [16].

Since the nth convergents are the best approximations, the continued fractions approach has proved to be extremely useful in analyzing the approximation properties of real numbers by rational numbers. However, this "standard approach" is not applicable in higher dimensions. There are various alternative tools proposed to replace the continued fraction approach to tackle the higher-dimensional approximation properties of real points by rational points. The dynamics on the space of lattices, for example, has proved to be useful, but the efficacy of continued fractions is yet to be matched. There have been many attempts to construct higher-dimensional analogue of the Gauss map, so that it captures all the features of simultaneous approximation. The theory is not well developed yet.

Lü and Zhang [13] used the Continued Fraction algorithm to compute the Hausdorff dimension of a set of points in the plane with certain growth conditions on the partial quotients in their continued fraction expansion. To be precise, they considered the following set:

$$E = \{(x,y) \in [0,1)^2 : \max\{a_n(x), a_n(y)\} \to \infty \text{ as } n \to \infty\},$$

and calculated its Hausdorff dimension to be $\frac{3}{2}$. As stated above, this set is a generalization to the plane of a classical result of Good [6].

Recently, it has been shown that the products of the consecutive partial quotients are associated with the improvements to Dirichlet's theorem (*uniform approximation*). To be precise, building on a work of Davenport and Schmidt [4], Kleinbock and Wadleigh [12] considered the set

$$D(\psi) := \left\{ x \in \mathbb{R} : \begin{array}{l} \exists\, N \text{ such that the system } |qx - p| < \psi(t),\ |q| < t \\ \text{has a nontrivial integer solution for all } t > N \end{array} \right\},$$

calling it a set of ψ-Dirichlet improvable numbers, where ψ is a nonincreasing function. A simple calculation shows the following simple yet extremely important criterion:

$$x \in D(\psi) \iff |q_{n-1}x - p_{n-1}| < \psi(q_n) \quad \text{for all } n \gg 1.$$

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One of the consequences of this observation is the following inclusion of sets:

$$G(\Phi) \subset D^c(\psi) \subset G(\Phi/4),$$

where

$$G(\Phi) := \Big\{ x \in [0,1) : a_n(x)a_{n+1}(x) > \Phi\big(q_n(x)\big) \text{ for infinitely many } n \in \mathbb{N} \Big\},$$

and $\Phi(t) := \frac{t\psi(t)}{1 - t\psi(t)}$. We refer the reader to [1–3, 5, 7–9, 17] for the background and the metrical results related to $G(\Phi)$.

In this article, we expand on the work of Lü and Zhang [13] by considering the relative growth properties of the products of consecutive partial quotients whose exponents are not necessarily units. Let $\Psi : \mathbb{N} \to (1, \infty)$ be a function such that $\Psi(n) \to \infty$ as $n \to \infty$. Let $(t_1, \dots, t_m) \in \mathbb{R}_+^m$. Define the set

$$\Lambda(\Psi) := \left\{ (x, y) \in [0,1]^2 : \max \left\{ \prod_{i=1}^{m} a_{n+i}^{t_i}(x), \prod_{i=1}^{m} a_{n+i}^{t_i}(y) \right\} \geq \Psi(n) \text{ for all } n \geq 1 \right\}.$$

We prove the following result.

Theorem 1.1 *Let Ψ be a positive function. Then,*

$$\dim_{\mathcal{H}}(\Lambda(\Psi)) = \frac{2 + \tau}{1 + \tau} \quad \textit{where} \quad \log \tau = \limsup_{n \to \infty} \frac{\log \log \Psi(n)}{n}.$$

2 Preliminaries

2.1 Hausdorff measure and dimension

Let $s > 0$, and let $E \subset \mathbb{R}^n$. Then, for any $\rho > 0$, a countable collection $\{B_i\}$ of balls in $\mathbb{R}^n$ with diameters $\operatorname{diam}(B_i) \leq \rho$ such that $E \subset \bigcup_i B_i$ is called a ρ-cover of E. Let

$$\mathcal{H}^s_\rho(E) = \inf \sum_i \operatorname{diam}(B_i)^s,$$

where the infimum is taken over all possible ρ-covers $\{B_i\}$ of E. Note that $\mathcal{H}^s_\rho(E)$ increases as ρ decreases and so approaches a limit as $\rho \to 0$. This limit could be zero or infinity, or take a finite positive value. Accordingly, the *s-Hausdorff measure $\mathcal{H}^s$* of E is defined to be

$$\mathcal{H}^s(E) = \lim_{\rho \to 0} \mathcal{H}^s_\rho(E).$$

It is easily verified that the Hausdorff measure is monotonic, countably subadditive, and $\mathcal{H}^s(\varnothing) = 0$. Thus, it is an outer measure on $\mathbb{R}^n$.

For any subset E, one can verify that there exists a unique critical value of s at which $\mathcal{H}^s(E)$ “jumps” from infinity to zero. The value taken by s at this discontinuity is referred to as the *Hausdorff dimension of E* and is denoted by $\dim_{\mathcal{H}} E$;

$$\dim_{\mathcal{H}} E := \inf\{ s \in \mathbb{R}_+ \ : \ \mathcal{H}^s(E) = 0 \}.$$

When $s = n$, $\mathcal{H}^n$ coincides with standard Lebesgue measure on $\mathbb{R}^n$.

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2.2 Continued fractions and Diophantine approximation

The Gauss transformation $T : [0,1) \to [0,1)$ is defined as

$$T(0) := 0, \quad T(x) := \frac{1}{x} \pmod 1, \quad \text{for } x \in (0,1).$$

For $x \in [0,1) \setminus \mathbb{Q}$ with continued fraction expansion $x = [a_1, a_2, \dots]$, as in Section, we have $a_n(x) = [1/T^{n-1}(x)]$ for each $n \geq 1$. Recall the sequences $p_n := p_n(x)$, $q_n := q_n(x)$ has the recursive relation

$$p_{n+1} = a_{n+1}p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad n \geq 0. \tag{2.1}$$

Thus, p_n, q_n are determined by the partial quotients $a_1, \dots, a_n$ which we may write $p_n = p_n(a_1, \dots, a_n)$, $q_n = q_n(a_1, \dots, a_n)$. When it is clear which partial quotients are involved, we denote them by p_n, q_n for simplicity.

For any integer vector $(a_1, \dots, a_n) \in \mathbb{N}^n$ with $n \geq 1$, write

$$I_n := I_n(a_1, \dots, a_n) := \{x \in [0,1) : a_1(x) = a_1, \dots, a_n(x) = a_n\}$$

for the corresponding "cylinder of order n," i.e., the set of all real numbers in $[0,1)$ whose continued fraction expansions begin with $(a_1, \dots, a_n)$.

We will frequently use the following well-known properties of continued fraction expansions. They are explained in the standard texts [10, 11].

Proposition 2.1 *For any positive integers $a_1, \dots, a_n$, let $p_n = p_n(a_1, \dots, a_n)$ and $q_n = q_n(a_1, \dots, a_n)$ be defined recursively by (2.1). Then,*

$$I_n = \begin{cases} \left[\frac{p_n}{q_n}, \frac{p_n+p_{n-1}}{q_n+q_{n-1}}\right) & \text{if } n \text{ is even,} \\ \left(\frac{p_n+p_{n-1}}{q_n+q_{n-1}}, \frac{p_n}{q_n}\right] & \text{if } n \text{ is odd.} \end{cases}$$

Thus, its length is given by

$$\frac{1}{2q_n^2} \leq |I_n| = \frac{1}{q_n(q_n + q_{n-1})} \leq \frac{1}{q_n^2} \leq \left(\prod_{i=1}^{n} a_i\right)^{-2},$$

since

$$p_{n-1}q_n - p_nq_{n-1} = (-1)^n, \quad \text{for all } n \geq 1.$$

The following result is due to Łuczak [14].

Lemma 2.2 (Łuczak [14]) *For any $b, c > 1$, the sets*

$$\left\{x \in [0,1) : a_n(x) \geq c^{b^n} \textit{ for infinitely many } n \in \mathbb{N}\right\},$$
$$\left\{x \in [0,1) : a_n(x) \geq c^{b^n} \textit{ for all } n \geq 1\right\}$$

have the same Hausdorff dimension $\frac{1}{b+1}$.

Lemma 2.3 (Good [6])

$$\dim_{\mathcal{H}} \{x \in [0,1] : a_n(x) \to \infty \ as \ n \to \infty\} = 1/2.$$

The following lemma proved by Marstrand [15] is well known.

Lemma 2.4 *For any measurable sets A and B,*

$$\dim_{\mathcal{H}}(A \times B) \geq \dim_{\mathcal{H}} A + \dim_{\mathcal{H}} B.$$

3 Proof of Theorem 1.1

To prove Theorem 1.1, we first prove the following proposition for the set

$$\Lambda := \left\{(x,y) \in [0,1]^2 : \max\left\{\prod_{i=1}^{m} a_{n+i}^{t_i}(x), \prod_{i=1}^{m} a_{n+i}^{t_i}(y)\right\} \to \infty \text{ as } n \to \infty\right\}.$$

Proposition 3.1

$$\dim_{\mathcal{H}}(\Lambda) = 3/2.$$

Proof. It is trivial that

$$\begin{aligned}\Lambda &\supset \left\{x \in [0,1] : \prod_{i=1}^{m} a_{n+i}^{t_i}(x) \to \infty \text{ as } n \to \infty\right\} \times [0,1) \\ &\supset \left\{x \in [0,1] : a_{n+1}^{t_1}(x) \to \infty \text{ as } n \to \infty\right\} \times [0,1).\end{aligned}$$

From Lemmas 2.3 and 2.4, it follows that $\dim_{\mathcal{H}} \Lambda \geq 3/2$. For the upper bound, we proceed as follows. Let $M \in \mathbb{R}$ be fixed. It is clear that

$$\Lambda \subseteq \bigcup_{N=1}^{\infty} \left\{(x,y) \in [0,1]^2 : \max\left\{\prod_{i=1}^{m} a_{n+i}^{t_i}(x), \prod_{i=1}^{m} a_{n+i}^{t_i}(y)\right\} \geq M \text{ for all } n \geq N\right\}$$

and the Hausdorff dimension of every set on the right-hand side is the same. Therefore, we only need to consider one of the sets,

$$\Lambda(M) := \left\{(x,y) \in [0,1]^2 : \max\left\{\prod_{i=1}^{m} a_{n+i}^{t_i}(x), \prod_{i=1}^{m} a_{n+i}^{t_i}(y)\right\} \geq M \text{ for all } n \geq 1\right\}.$$

Let (x,y) be an element in $\Lambda(M)$. Then, for any integer N and any $1 \leq n \leq N$, either $\prod_{i=1}^{m} a_{n+i}^{t_i}(x) \geq M$ or $\prod_{i=1}^{m} a_{n+i}^{t_i}(y) \geq M$. For this reason, either

$$\#\left\{1 \leq n \leq N : \prod_{i=1}^{m} a_{n+i}^{t_i}(x) \geq M\right\} \geq \left[\frac{N}{2}\right]$$

or

$$\#\left\{1 \leq n \leq N : \prod_{i=1}^{m} a_{n+i}^{t_i}(y) \geq M\right\} \geq \left[\frac{N}{2}\right],$$

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where the notation $[x]$ denotes the largest integer not greater than x. Thus, if we write

$$\Lambda_N(M) := \left\{x \in [0,1] : \#\left\{1 \le n \le N : \prod_{i=1}^{m} a_{n+i}^{t_i}(x) \ge M\right\} \ge \left[\frac{N}{2}\right]\right\},$$

then it is straightforward to see that

$$\Lambda_M \subseteq \bigcap_{N=1}^{\infty} (\Lambda_N(M) \times [0,1)) \bigcup ([0,1) \times \Lambda_N(M)).$$

Next, we find a cover for $\Lambda_N(M) \times [0,1)$. The cover for $[0,1) \times \Lambda_N(M))$ can be estimated similarly.

Let $l = \left[\frac{N}{2}\right]$ and $\mathcal{A}$ be all the possible choices $\omega = \{n_1 < n_2 < \cdots < n_l\}$, then the cardinality of $\mathcal{A}$ is bounded from above by 2^N. Denote the integers in $[2, N] \setminus \omega$ by ω^c. For any $n \ge 1$, set

$$D_n(\omega) := \left\{(a_1, \dots, a_n, \dots a_{n+m}) \in \mathbb{N}^{n+m} : \begin{array}{l} \prod_{i=1}^{m} a_{r+i}^{t_i}(x) \ge M \text{ for } r \in \omega \\ \prod_{i=1}^{m} a_{r+i}^{t_i}(y) \ge M \text{ for } r \in \omega^c \end{array}\right\}.$$

By the definition of $\Lambda_N(M)$, we see that

$$\begin{aligned}\Lambda_N(M) \times [0,1) &\subseteq \bigcup_{\omega \in \mathcal{A}} \left\{x \in [0,1) : \prod_{i=1}^{m} a_{r+i}^{t_i}(x) \ge M \text{ for } r \in \omega\right\} \times [0,1) \\ &= \bigcup_{\omega \in \mathcal{A}} \bigcup_{(a_1,\dots,a_{N+m}) \in D_N(\omega)} I_{N+m}(a_1, \dots, a_{N+m}) \times [0,1).\end{aligned}$$

For each $(a_1, \dots, a_{N+m}) \in D_N(\omega)$, the set $I_{N+m}(a_1, \dots, a_{N+m}) \times [0,1)$ can be covered by

$$I_{N+m}(a_1, \dots, a_{N+m})^{-1}$$

many cubes of side length $|I_{N+m}(a_1, \dots, a_{N+m})|$. Furthermore, for each $(a_1, \dots, a_{N+m}) \in D_N(\omega)$, $\prod_{i=1}^{m} a_{r+i}^{t_i}(x) \ge M$ for each $r \in \{n_1, \dots, n_l\}$. Then we have

$$mt_{\max} \sum_{k=1}^{N+m} \log a_k \ge \left[\frac{N}{2}\right] \log M. \tag{3.1}$$

Define a family of probability measures $\{\mu_h\}_{h>1}$ on the unit interval $[0,1]$. For each $h > 1$ and any $(a_1, \dots, a_{N+m}) \in \mathbb{N}^{n+m}$, define

$$\mu_l(I_{n+m}) = e^{-(N+m)P(h) - h \sum_{k=1}^{N+m} \log a_k},$$

where $P(h) = \log \sum_{j=1}^{\infty} \frac{1}{j^h} < \infty$.

Fix $s > \frac{3}{2}$ and let $h = s - \frac{1}{2} > 1$. Choose M sufficiently large such that

$$\left(s - \frac{3}{2}\right) \frac{\log M}{2mt_{\max}} \ge 2p(h) + \log 2. \tag{3.2}$$

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Then the s-dimensional Hausdorff measure of this cover can be estimated as

$$\begin{aligned}
\mathcal{H}^s(\Lambda(M)) &\ll \liminf_{n\to\infty}\sum_{\omega\in\mathcal{A}}\sum_{(a_1,\ldots,a_{N+m})\in D_N(\omega)} |I_{N+m}(a_1,\ldots,a_{N+m})|^{-1}|I_{N+m}(a_1,\ldots,a_{N+m})|^s\\
&\le \liminf_{n\to\infty}\sum_{\omega\in\mathcal{A}}\sum_{(a_1,\ldots,a_{N+m})\in D_N(\omega)}\prod_{k=1}^{N+m}(a_k)^{-2(s-1)}\\
&\le \liminf_{n\to\infty}\sum_{\omega\in\mathcal{A}}\sum_{(a_1,\ldots,a_{N+m})\in D_N(\omega)} e^{-2(s-1)\sum_{k=1}^{N+m}\log a_k}\\
&\le \liminf_{n\to\infty}2^{-N}\sum_{\omega\in\mathcal{A}}\sum_{(a_1,\ldots,a_{N+m})\in D_N(\omega)} e^{-(s-\frac{3}{2})\sum_{k=1}^{N+m}\log a_k-(s-\frac{1}{2})\sum_{k=1}^{N+m}\log a_k+N\log 2}\\
&\overset{(3.1),(3.2)}{\le} \liminf_{n\to\infty}2^{-N}\sum_{\omega\in\mathcal{A}}\sum_{(a_1,\ldots,a_{N+m})\in D_N(\omega)} e^{-(N+m)p(h)-h\sum_{k=1}^{N+m}\log a_k}\\
&\ll \liminf_{n\to\infty}\sum_{(a_1,\ldots,a_{N+m})\in\mathbb{N}^{N+m}} e^{-(N+m)p(h)-h\sum_{k=1}^{N+m}\log a_k}=1.
\end{aligned}$$

In some of the estimates, every term has been evaluated over $\prod_{i=1}^m a_{r+i}^{t_i}(x)\ge M$, for $r\in\omega$, and $t_{\max}=\max\{t_1,t_2,\ldots,t_m\}$.

Thus, by the definition of Hausdorff measure, $\dim_{\mathcal{H}}\Lambda(M)\le 3/2$ and consequently $\dim_{\mathcal{H}}\Lambda\le 3/2$. ■

To complete the proof of Theorem 1.1, we consider three cases for τ.

3.1 $1<\tau<\infty$.

Let $1<c<\tau$. By the definition of τ, there exist infinitely many n in an infinite subset $\Omega\subset\mathbb{N}$ such that

$$\frac{\log\log\Psi(n)}{n}\ge\log c,\quad\text{i.e.,}\quad \Psi(n)\ge e^{c^n}\qquad\forall n\in\Omega.$$

Thus, for every $n\in\Omega$, either $\prod_{i=1}^m a_{n+i}^{t_i}(x)\ge e^{c^n}$ or $\prod_{i=1}^m a_{n+i}^{t_i}(y)\ge e^{c^n}$. Then, for at least one index $1\le i\le m$, we have either $a_{n+i}^{t_i}(x)\ge e^{\frac{c^n}{m}}$ or $a_{n+i}^{t_i}(x)\ge e^{\frac{c^n}{m}}$. Hence,

$$\Lambda(\Psi)\subseteq\bigcup_{i=1}^m(\Lambda_1\times[0,1])\bigcup\bigcup_{i=1}^m([0,1]\times\Lambda_2),$$

where

$$\Lambda_1:=\left\{x\in[0,1]: a_{n+i}^{t_i}(x)\ge e^{\frac{c^n}{m}}\text{ for i.m. }n\in\mathbb{N}\right\},$$

and

$$\Lambda_2:=\left\{y\in[0,1]: a_{n+i}^{t_i}(y)\ge e^{\frac{c^n}{m}}\text{ for i.m. }n\in\mathbb{N}\right\}.$$

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Therefore, by Lemma 2.2, we have

$$\dim_{\mathcal{H}}(\Lambda(\Psi)) \le 1 + \lim_{c\to\tau} \frac{1}{1+c} = 1 + \frac{1}{1+\tau}.$$

The lower bound is given by a similar argument as above. For this, fix $c > \tau$, then $\Psi(n) \le e^{c^n}$ holds for all $n \ge n_0$. Therefore, we have the inclusion

$$\begin{aligned}\Lambda(\Psi) &\supseteq \left\{x \in [0,1] : \prod_{i=1}^{m} a_{n+i}^{t_i}(x) \ge e^{c^n} \text{ for all } n \ge n_0\right\} \times [0,1]\\ &\supseteq \left\{x \in [0,1] : a_{n+1}^{t_1}(x) \ge e^{c^n} \text{ for all } n \ge n_0\right\} \times [0,1].\end{aligned}$$

Hence, by Lemmas 2.2 and 2.4, we get that

$$\dim_{\mathcal{H}}(\Lambda(\Psi)) \ge 1 + \lim_{c\to\tau} \frac{1}{1+c} = 1 + \frac{1}{1+\tau}.$$

3.1.1 $\tau = \infty$.

This case readily follows from the upper bound argument above, that is,

$$\Lambda(\Psi) \le 1 + \lim_{\tau\to\infty} \frac{1}{\tau+1} = 1.$$

3.1.2 $\tau = 1$.

In this case, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, we have $\Psi(n) \le e^{(1+\varepsilon)^n}$. Then,

$$\begin{aligned}\Lambda(\Psi) &\supset \left\{x \in [0,1) : \prod_{i=1}^{m} a_{n+i}^{t_i} \ge \Psi(n) \text{ for all } n \ge n_0\right\} \times [0,1]\\ &\supset \left\{x \in [0,1) : a_{n+1}^{t_1}(x) \ge e^{(1+\varepsilon)^n} \text{ for all } \ n \ge n_0\right\} \times [0,1).\end{aligned}$$

Hence, by using Lemmas 2.2 and 2.4, we have

$$\dim_{\mathcal{H}} \Lambda(\Psi) \ge \lim_{\varepsilon\to 0} \frac{1}{1+1+\varepsilon} + 1 = \frac{3}{2}.$$

The upper bound follows from Proposition 3.1 as $\Lambda(\Psi) \subseteq \Lambda$. ■

Acknowledgment We thank the anonymous referee for the careful reading of the paper and helpful suggestions for improvement.

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