1
CATEGORIES

Category theory provides the language for the discussions in this text and it is also an inescapable foundation for any treatment of K-theory. In this chapter, we set out the fundamental ideas of category theory, and we give a first application of the power of categorical methods. These ideas are introduced in three stages, each of which occupies one section. To start, we define categories themselves. Next, we consider functors, which are the tools for moving between different categories, and then natural transformations, which allow us to compare different functors. In the final, fourth, section of the chapter, we apply these basic notions to investigate universal constructions and universal objects. The idea of 'universality' reveals the common properties of apparently diverse objects, such as free modules and free groups, and it provides the framework for many of the definitions and constructions that we make later in this text.

1.1 FUNDAMENTAL PROPERTIES OF CATEGORIES

We commence our discussion of the theory of categories with the axiomatic definition of a category, together with a selection of techniques for the manufacture of new categories from old.

Because our main interest in this text is the application of category theory to module theory, many of our illustrations are obtained by considering various types of module. We shall also see some categories based on other algebraic entities such as sets and groups.

The principal innovation in our discussion stems from our need to use both right modules and left modules. Our view is that it is best to write operators opposite scalars as far as is practicable, so that a homomorphism between right modules is to be a left operator, while a homomorphism of left modules is to be a right operator. We also wish to compose homomorphisms in the

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natural way, by 'associativity' – more formal definitions are given below in (1.1.3) and (1.1.4). The consequence is that we obtain two versions of the axiom for the composition of morphisms in an abstract category. Thus we arrive at two kinds of abstract category, a 'right' category and a 'left' category, which are modeled on the corresponding categories of modules. We call this phenomenon *chirality.* It is worth remarking that the distinction between left and right modules was made at a very early stage in the modern development of module theory [Noether & Schmeidler 1920].

The distinction between right and left categories does not seem to have been made explicit before now, and it could be avoided by using a technique given in (1.1.5). However, it seems to us to be more natural to allow both notations for categories, rather than suppressing one notation artificially.

We also discuss some points from set theory which arise from future applications in K-theory, where one needs to be able to work with 'small' categories, that is, categories whose objects can all be taken to be members of some set, which may itself be very big. In general, the objects of a category need not be contained in a set. We are particularly indebted to Wilfrid Hodges for his helpful comments on these questions.

1.1.1 The definition

Informally, a category consists of a collection of mathematical entities, such as the right R-modules over a given ring R, which can be recognised as sharing a common structure, together with a collection of mappings between these entities that respect this structure; for R-modules, we would expect these to be the R-module homomorphisms.

The entities which share the common structure are known as the 'objects' of the category, while the structure-preserving maps are the 'morphisms' of the category. Thus, to define a *category* $\mathcal{C}$ in general, we must specify the following data.

Cat 1. A class $\mathrm{Ob}\,\mathcal{C}$. Members of $\mathrm{Ob}\,\mathcal{C}$ are called the *objects* of $\mathcal{C}$.

Cat 2. For each ordered pair C, D of objects of $\mathcal{C}$, there is a set $\mathrm{Mor}_{\mathcal{C}}(C, D)$; the elements of $\mathrm{Mor}_{\mathcal{C}}(C, D)$ are called the *morphisms* from C to D in $\mathcal{C}$.

It may happen that $\mathrm{Mor}_{\mathcal{C}}(C, D)$ is the empty set.

Given $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, the object C is called the *domain* of α and D the *codomain.* An arrow

$$\alpha : C \longrightarrow D$$

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is often used to indicate that α is a morphism from C to D.

These objects and morphisms obey some axioms, but before listing these we note two deeply entrenched conventions and give some examples.

The first convention is to write $C \in \mathcal{C}$ rather than $C \in \mathrm{Ob}\,\mathcal{C}$ when C is an object of $\mathcal{C}$. This reflects the fact that a category is usually considered to be defined in terms of the class $\mathrm{Ob}\,\mathcal{C}$. For this reason, a category is often named after its objects.

Second, the term 'morphism' is used mostly when we are considering categories in the abstract; in concrete situations, the morphisms are often the homomorphisms between some familiar objects, such as modules, groups or rings. In such a case, one writes $\mathrm{Hom}(C, D)$ rather than $\mathrm{Mor}_{\mathcal{C}}(C, D)$.

1.1.2 *Some examples*

$\mathcal{S}_{\mathcal{ET}}$ The class $\mathrm{Ob}\,\mathcal{S}_{\mathcal{ET}}$ of objects is the class of all sets and the set of morphisms from X to Y is the set of all mappings from X to Y. (Some authors denote this category $\mathcal{E}_{\mathcal{NS}}$).

We write $\mathrm{Map}(X, Y)$ rather than $\mathrm{Mor}(X, Y)$.

$\mathcal{G}_{\mathcal{P}}$ The objects are the groups, with group homomorphisms as morphisms.

$\mathcal{R}_{\mathcal{ING}}$ The objects are the rings (which we require to have an identity element), the morphisms from R to S being the ring homomorphisms, which, we insist, send the identity of R to the identity of S.

$\mathcal{R}_{\mathcal{NG}}$ The objects are now the nonunital rings, which are rings except that they need not possess an identity element. A morphism from R to S is a homomorphism of nonunital rings, which is the same as a ring homomorphism except that there can be no requirement that the identity of R is sent to the identity of S, even when R and S are actually rings.

$\mathcal{M}_{\mathcal{OD}R}$ Given a ring R, we form the category of all right R-modules. Thus $\mathrm{Ob}(\mathcal{M}_{\mathcal{OD}R})$ is the class of all right R-modules and the set of morphisms from M to N is the set of all R-module homomorphisms from M to N, which we write $\mathrm{Hom}(M, N)$, $\mathrm{Hom}_R(M, N)$ or $\mathrm{Hom}(M_R, N_R)$ according to context.

${}_R\mathcal{M}_{\mathcal{OD}}$ Similarly, we form the category of left R-modules and homomorphisms of left R-modules.

$\mathcal{A}_{\mathcal{B}}$ The category of abelian groups – the objects are abelian groups, the morphisms are group homomorphisms.

Since an abelian group has a uniquely defined structure as a right $\mathbb{Z}$-module ([BK: IRM] (1.2.2)), $\mathcal{A}_{\mathcal{B}}$ is $\mathcal{M}_{\mathcal{OD}\mathbb{Z}}$ with another name. Despite

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the fact that any right $\mathbb{Z}$-module can be regarded equally as a left $\mathbb{Z}$-module, with $am = ma$ for any integer a and element m of M, it is not quite true that $\mathcal{A}_{\mathcal{B}}$ is the same as the category ${}_{\mathbb{Z}}\mathcal{M}_{\mathcal{OD}}$ of left $\mathbb{Z}$-modules. The reason for the distinction will be discussed further in (1.1.5).

$\mathcal{T}_{\mathcal{OP}}$ This important non-algebraic example of a category has topological spaces as objects and continuous maps as morphisms.

Λ It is sometimes convenient to regard a partially ordered set as a category. A *partially ordered* set is a set Λ together with an order relation $\leq$ on Λ which satisfies the following requirements.

PO 1. **Reflexivity:**
$\lambda \leq \lambda$ for each $\lambda \in \Lambda$.

PO 2. **Transitivity:**
if $\lambda \leq \mu$ and $\mu \leq \nu$ for λ, μ and ν in Λ, then $\lambda \leq \nu$ also.

A partially ordered set is said to be *proper* if the following axiom also holds.

PO 3. If $\lambda \leq \mu$ and $\mu \leq \lambda$ for λ and μ in Λ, then $\lambda = \mu$.

Here are two examples that will be generalized in (5.1).

(a) Let Λ be the set of nonzero ideals of $\mathbb{Z}$, ordered by reverse inclusion:

$$I \leq J \iff J \subseteq I.$$

Then Λ is a proper partially ordered set.

(b) Take Σ to be the set of nonzero elements of $\mathbb{Z}$, ordered by division:

$$a \leq b \iff ax = b \text{ for some } x \in \mathbb{Z}.$$

Then Σ is partially ordered, but not proper.

Given a partially ordered set Λ, we can view Λ as a category whose objects are the elements $\lambda, \mu, \ldots$ of Λ. If $\lambda \leq \mu$, then $\mathrm{Mor}(\lambda, \mu)$ contains a single element $\iota^{\lambda\mu}$, and $\mathrm{Mor}(\lambda, \mu)$ is empty otherwise.

$\mathcal{O}_{\mathcal{RD}}$ The class of all ordered sets can be considered to be a category. We say that a set Λ is *totally ordered*, or simply *ordered*, if it is a proper partially ordered set in which any two members are comparable, that is, the following axiom holds.

TO. If λ and μ are in Λ, then either $\lambda \leq \mu$ or $\mu \leq \lambda$.

Then the objects of $\mathcal{O}_{\mathcal{RD}}$ are the ordered sets Λ and a morphism $\alpha : \Lambda \to \Sigma$ of ordered sets is a mapping that preserves the order: if $\lambda \leq \mu$ in Λ, then $\alpha\lambda \leq \alpha\mu$ in Σ.

1.1.3 The axioms

Now that we have some examples in mind, we list the axioms that the morphisms of an abstract category $\mathcal{C}$ are required to satisfy. These correspond to the properties of the homomorphisms in the category $\mathcal{M}od_R$, provided that we write homomorphisms on the left and that we use the consequent natural convention for composing homomorphisms. Thus, if α is a right R-module homomorphism from M_R to N_R and m is in M, then αm denotes the image of m in N, and if β is a homomorphism from N_R to P_R, then

$$(\beta\alpha)m = \beta(\alpha m).$$

Cat 3. For each object C of $\mathcal{C}$, there is a distinguished morphism

$$id_C \in \mathrm{Mor}_{\mathcal{C}}(C, C),$$

called the *identity* morphism of C.

Cat 4. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$, then there is a morphism

$$\beta\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, E),$$

called the *product* or *composite* of α and β; α and β are said to be *composable*.

Cat 5. For any morphism $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$,

$$(id_D)\alpha = \alpha = \alpha(id_C).$$

Cat 6. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$ and $\gamma \in \mathrm{Mor}_{\mathcal{C}}(E, F)$, then

$$\gamma(\beta\alpha) = (\gamma\beta)\alpha.$$

With regard to Cat 4, note that the product of two morphisms α, β is defined precisely when the codomain of α is the same as the domain of β. Observe also that the manner of writing the product (as $\beta\alpha$ rather than $\alpha\beta$) forms a part of this axiom.

Statements such as Cat 6 are sometimes formulated as 'the product is associative when defined'.

1.1.4 Chirality

It seems to us that a category satisfying the axioms above should, strictly speaking, be called a *right* category, since the form of the axioms mimics the natural form of composition of homomorphisms of *right* R-modules when these homomorphisms are written on the left, as is our convention.

When we consider left modules, we prefer to put homomorphisms on the right. Thus, given a homomorphism of left R-modules $\alpha : {}_RM \to {}_RN$ and an element m in M, α sends m to $m\alpha$ rather than αm. If $\beta : {}_RN \to {}_RP$ is also a homomorphism, then the product $\alpha\beta$ is given by the rule

$$m(\alpha\beta) = (m\alpha)\beta.$$

With this convention, we obtain a modified list of axioms that defines a *left* category, as follows.

Cat 4^ℓ. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$, then there is a morphism

$$\alpha\beta \in \mathrm{Mor}_{\mathcal{C}}(C, E).$$

Cat 5^ℓ. For any morphism $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$,

$$(id_C)\alpha = \alpha = \alpha(id_D).$$

Cat 6^ℓ. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$ and $\gamma \in \mathrm{Mor}_{\mathcal{C}}(E, F)$, then

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

The prime example of a left category is ${}_R\mathcal{M}od$, the category of all left R-modules. Also, most group theorists in effect treat $\mathcal{G}p$ as a left category.

As we noted in (1.1.2), a partially ordered set Λ can be viewed as a category. It is convenient to regard Λ as a left category, since the left-category composition reads

$$\iota^{\lambda\mu}\iota^{\mu\nu} = \iota^{\lambda\nu},$$

where $\iota^{\lambda\mu}$ is the unique morphism from λ to μ (when $\lambda \leq \mu$).

We introduce the term *chirality* to distinguish the two kinds of category, a left or right category having correspondingly left or right chirality.

1.1.5 The mirror

There is a purely formal method that allows us to switch between categories of opposite chiralities. Given a left category $\mathcal{C}$, we manufacture a right category $\mathcal{C}^{\odot}$, the *mirror* of $\mathcal{C}$, as follows. The objects of $\mathcal{C}^{\odot}$ are symbols $C^{\odot}$, corresponding bijectively to the objects C of $\mathcal{C}$, and the morphisms from $C^{\odot}$ to $D^{\odot}$ in $\mathcal{C}^{\odot}$ are symbols $\alpha^{\odot}$ in bijective correspondence with the morphisms α from C to D in $\mathcal{C}$. Thus for each morphism

$$\alpha : C \longrightarrow D \text{ in } \mathcal{C},$$

there is exactly one morphism

$$\alpha^{\odot} : C^{\odot} \longrightarrow D^{\odot} \text{ in } \mathcal{C}^{\odot}.$$

The product in $\mathcal{C}^{\odot}$ is defined by putting

$$\beta^{\odot}\alpha^{\odot} = (\alpha\beta)^{\odot},$$

where $\beta : D \to E$ is a morphism in $\mathcal{C}$ that is composable with α.

Observe that $\mathcal{C}$ and $\mathcal{C}^{\odot}$ are genuinely distinct categories, rather than the same category with two notational conventions, since the order of a pair of morphisms in their product is specified by the axioms.

It is clear that given a right category $\mathcal{D}$, we can construct a left category $\mathcal{D}^{\odot}$ by a similar method, and that there is a formal identification of $\mathcal{C}^{\odot\odot}$ with $\mathcal{C}$ for any category $\mathcal{C}$.

In [BK: IRM] (1.2.6), we introduced a method of turning a left R-module M into a right R°-module M°, where R° is the opposite ring to R. A typical element of R° has the form r° for r in R, and addition and multiplication in the opposite ring are given by

$$r^{\circ} + s^{\circ} = (r+s)^{\circ} \quad \text{and} \quad r^{\circ}s^{\circ} = (sr)^{\circ}.$$

The elements m° of M° are in bijective correspondence with those of M, and the addition and scalar multiplication for M° are defined by

$$m^{\circ} + n^{\circ} = (m+n)^{\circ} \quad \text{and} \quad m^{\circ}r^{\circ} = (rm)^{\circ}.$$

If $\alpha : M \to N$ is a homomorphism of left R-modules, we define a homomorphism α° of right R°-modules by

$$\alpha^{\circ}m^{\circ} = (m\alpha)^{\circ}.$$

Thus we have an identification

$$({}_R\mathcal{M}\textsc{od})^{\odot} = \mathcal{M}\textsc{od}_{(R^{\circ})},$$

which shows that the use of the mirror category is a generalization of the use of the opposite ring. This identity was used implicitly in Exercise 1.2.14 of [BK: IRM], where we explored the consequences of this technique for switching between left R-modules and right R°-modules. Some authors use this method to avoid dealing both with left and with right modules.

Suppose now that the ring R is commutative. Then $R = R^{\circ}$, so that $({}_R\mathcal{M}\textsc{od})^{\odot} = \mathcal{M}\textsc{od}_R$. Again, this identity is used implicitly to write scalars and operators on the same side in many elementary textbooks that are concerned only with commutative rings.

We can now elucidate the comment in (1.1.2) about the category $\mathcal{A}\textsc{b}$ of abelian groups. For, we regard $\mathcal{A}\textsc{b}$ as the right category $\mathcal{M}\textsc{od}_{\mathbb{Z}}$, which is the mirror of the category ${}_{\mathbb{Z}}\mathcal{M}\textsc{od}$ of left $\mathbb{Z}$-modules.

We remind the reader that if a category has no obvious chirality, for example, $\mathcal{G}_{\mathcal{P}}$ or $\mathcal{R}_{\mathcal{ING}}$, we always take it to be a right category. However, many group theorists prefer to view $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{B}}$ as left categories.

Remark. Some authorities in category theory do not regard a category and its mirror as distinct categories, but simply as alternative notations for composition of morphisms in the one category. From this point of view, a left R-module and the corresponding right R°-module are the same object, in differing notations. However, a ring and its opposite are definitely different rings, as can be seen from the use of opposites in Brauer theory ([Cohn 1979], §10.3). We therefore feel that the distinction between a ring and its opposite should be extended to that between a category and its mirror.

1.1.6 The opposite category

We now give the definition of the *opposite* category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$. This is distinct from the notion of the mirror category that we introduced above.

Given a right category $\mathcal{C}$, the objects C^{op} of the right category $\mathcal{C}^{\mathrm{op}}$ are in bijective correspondence with the objects C of $\mathcal{C}$, and for each pair of objects C^{op}, D^{op} of $\mathcal{C}^{\mathrm{op}}$, there is a bijection between the morphisms α in $\mathrm{Mor}_{\mathcal{C}}(C, D)$ and the morphisms α^{op} in $\mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(D^{\mathrm{op}}, C^{\mathrm{op}})$; thus $\alpha : C \to D$ corresponds to $\alpha^{\mathrm{op}} : D^{\mathrm{op}} \to C^{\mathrm{op}}$. If $\beta : D \to E$ is a morphism in $\mathcal{C}$, then composition in $\mathcal{C}^{\mathrm{op}}$ is given by the rule

$$(\alpha^{\mathrm{op}})(\beta^{\mathrm{op}}) = (\beta\alpha)^{\mathrm{op}}.$$

(The definition for left categories is left to the reader, who, we are confident, will get it right ...)

Note that $\mathcal{C}^{\mathrm{op}}$ has the same chirality as $\mathcal{C}$.

1.1.7 The principle of duality

If we can make a definition or state a result by using only the objects and morphisms of a category $\mathcal{C}$, then we obtain a *dual* definition or result in the opposite category $\mathcal{C}^{\mathrm{op}}$ in which objects $C, D, \ldots$ of $\mathcal{C}$ are replaced by the corresponding objects $C^{\mathrm{op}}, D^{\mathrm{op}}, \ldots$ of $\mathcal{C}^{\mathrm{op}}$ and similarly morphisms $\alpha, \beta, \ldots$ are replaced by $\alpha^{\mathrm{op}}, \beta^{\mathrm{op}}, \ldots$. An example is provided by the relationship between the definitions of projective and injective modules.

A projective module may be defined as a right R-module P for which the following holds.

Pro. Given any surjective R-module homomorphism $\pi : M \to P$, there is a splitting homomorphism $\sigma : P \to M$, that is, $\pi\sigma = id_P$.

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On the other hand, a right R-module I is injective if instead we have the following.

Inj. Given any injective R-module homomorphism $\mu : I \to M$, there is a splitting homomorphism $\rho : M \to I$, that is, $\rho\mu = id_I$.

(More details are given in §2.5 of [BK: IRM].)

It is reasonable to assume that if we can generalize surjective and injective homomorphisms of modules to abstract categories, then a surjective map in $\mathcal{C}$ will correspond to an injective map in the opposite category $\mathcal{C}^{\mathrm{op}}$, and so a projective object in $\mathcal{C}$ will correspond to an injective object in the opposite category. (This assumption is justified in the next chapter – see (2.2.2).) Thus, provided that we confine ourselves to statements that involve only objects and morphisms, we need only prove results about projective objects, since the corresponding results for injective objects are then true by duality.

This technique for pairing together definitions, arguments, etc. in a category and its opposite is the *principle of duality*, and the phrase *by duality* is used to indicate this method of argument.

1.1.8 Subcategories

A *subcategory* $\mathcal{D}$ of a (right) category $\mathcal{C}$ is defined by the following data.

Sub 1. A subclass $\mathrm{Ob}\,\mathcal{D}$ of $\mathrm{Ob}\,\mathcal{C}$, which specifies the objects that belong to $\mathcal{D}$.

Sub 2. For each pair of objects C, D of $\mathcal{D}$, a subset $\mathrm{Mor}_{\mathcal{D}}(C, D)$ of $\mathrm{Mor}_{\mathcal{C}}(C, D)$; these are the morphisms from C to D in $\mathcal{D}$.

Sub 3. If C is an object of $\mathcal{D}$, then the identity morphism id_C in $\mathrm{Mor}_{\mathcal{C}}(C, C)$ shall belong to $\mathrm{Mor}_{\mathcal{D}}(C, C)$ (where it is again the identity morphism).

Sub 4. If $\alpha \in \mathrm{Mor}_{\mathcal{D}}(C, D)$ and $\beta \in \mathrm{Mor}_{\mathcal{D}}(D, E)$, then

$$\beta\alpha \in \mathrm{Mor}_{\mathcal{D}}(C, E),$$

(where the product is the product in $\mathcal{C}$).

It is clear that a subcategory of a right category is itself a right category with identities and products inherited from $\mathcal{C}$.

The corresponding definitions for a left category are obvious.

Examples of subcategories are $\mathcal{A}\mathcal{B}$ in $\mathcal{G}\mathcal{P}$ and $\mathcal{R}\mathcal{I}\mathcal{N}\mathcal{G}$ in $\mathcal{R}\mathcal{N}\mathcal{G}$. Note that any group homomorphism from G to H is also a morphism in $\mathcal{A}\mathcal{B}$ if the groups G and H happen to be abelian. In contrast, a morphism in $\mathcal{R}\mathcal{N}\mathcal{G}$ from a ring R to a ring S need not be allowable as a morphism in $\mathcal{R}\mathcal{I}\mathcal{N}\mathcal{G}$. This can be seen

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by computing $\mathrm{Mor}(0, S)$ in the two cases: if S is a ring other than the zero ring, then $\mathrm{Mor}(0, S) = \{0\}$ in the first case but $\mathrm{Mor}(0, S) = \emptyset$ in the second.

1.1.9 Full subcategories

A *full* subcategory $\mathcal{D}$ of $\mathcal{C}$ is one in which

$$\mathrm{Mor}_{\mathcal{D}}(C, D) = \mathrm{Mor}_{\mathcal{C}}(C, D)$$

for any two objects of $\mathcal{D}$.

It is obvious that any subclass $\mathcal{D}$ of the class of objects of $\mathcal{C}$ gives rise to a unique full subcategory of $\mathcal{C}$.

This method of constructing subcategories of $\mathcal{M}\textsc{od}_R$, for various rings R, will be very useful to us in the sequel. For this reason, we define a *module category* to be any category which is a full subcategory of $\mathcal{M}\textsc{od}_R$ (or ${}_R\mathcal{M}\textsc{od}$) for some ring R.

Here are some important examples.

$\mathcal{F}\textsc{ree}_R$, the category of all free right R-modules.
$\mathcal{P}\textsc{roj}_R$, the category of all projective right R-modules.
$\mathcal{M}_R$, the category of finitely generated right R-modules.
$\mathcal{F}_R$, the category of free right modules R^k of finite rank.
$\mathcal{P}_R$, the category of finitely generated projective right R-modules.

The corresponding subcategories of ${}_R\mathcal{M}\textsc{od}$ are denoted ${}_R\mathcal{F}\textsc{ree}$, ${}_R\mathcal{P}\textsc{roj}$, ${}_R\mathcal{M}$, ${}_R\mathcal{F}$ and ${}_R\mathcal{P}$ respectively.

1.1.10 Some remarks on set theory and small categories

In our definition of a category, we use the expression *'class'*, rather than the word *'set'* which the reader might have expected. The reason for this is that we wish to make a naïve distinction between sets, on which all mathematical constructions are allowed, and 'non-sets', which are too large to permit some operations. The aim is to avoid Russell's Paradox and similar traps: if X is a set of sets, it is sensible to ask whether or not the set X is itself in X. The paradox arises by taking X to be the set of all sets that do not contain themselves. Our avoiding action is to declare that this particular choice of X is a class but not a set; essentially, X is too large to be considered to be a set. The kind of set theory that we have in mind is expanded in detail in [Herrlich & Strecker 1979], [van Dalen, Doets & de Swart 1979], and [Levy 1979].

If the class $\mathrm{Ob}\,\mathcal{C}$ of objects of the category $\mathcal{C}$ is in fact a set, then the category is said to be *small*.

In K-theory there are technical reasons for working with small categories, or at least, categories which are equivalent to small categories in a sense to be made precise in (1.3.15). The purpose of the following remarks is to show that the categories of greatest concern to us can indeed be taken to be small.

To illustrate this point, consider $\mathcal{M}_R$, the category of finitely generated R-modules. From one point of view, the objects in $\mathcal{M}_R$ must form a class, rather than a set, for the trivial reason that any set can be regarded artificially as the single element of a zero module. However, we can impose a very natural restriction on the objects which are permitted in $\mathcal{M}_R$ to ensure that it is small, as follows.

An object M in $\mathcal{M}_R$ is the homomorphic image of some free module R^n of finite rank ([BK: IRM] Lemma 2.5.7), and so $M \cong R^n/\operatorname{Ker}\alpha$ for a suitable surjective homomorphism α, by the Induced Mapping Theorem ([BK: IRM] (1.2.11)). But, by construction ([BK: IRM] (1.2.10)), the quotient module $R^n/\operatorname{Ker}\alpha$ is a set of subsets of R^n. We can therefore regard each object of $\mathcal{M}_R$ as being a set of subsets of the countably infinite cartesian product $\prod_\omega R$. Since a fundamental axiom of set theory assures us that the class of subsets of a set is again a set, we see that $\mathcal{M}_R$ is small.

An alternative way of interpreting this argument is that it shows that $\mathcal{M}_R$ has a set of isomorphism classes of objects, and therefore that $\mathcal{M}_R$ is equivalent to a small category. In applications, it makes no difference whether we take $\mathcal{M}_R$ in the 'large', or replace it by some convenient, equivalent, small category, which we give the same name.

The category $\mathcal{M}_R$ is typical of the type of category that we deal with. The objects (and morphisms) of these categories are constructed from a given set of objects by operations which formally involve taking sets of subsets of cartesian products of the form $\prod_{\lambda\in\Lambda} X_\lambda$, where each X_λ is a set and Λ is some index set. Such a method of construction must generate a set of objects.

More detailed accounts of the bearing of set theory on category theory can be found in [Cameron 1999], [Herrlich & Strecker 1979], [Mac Lane 1971], §1.6, and [Schubert 1972], §3.

We now show how to construct some new categories from given categories.

1.1.11 Product categories

Given a pair of (right) categories $\mathcal{C}$ and $\mathcal{D}$, we form the *product* category $\mathcal{C}\times\mathcal{D}$ as follows. The objects of $\mathcal{C}\times\mathcal{D}$ are pairs (C, D), where C is an object of $\mathcal{C}$ and D is an object of $\mathcal{D}$, and a morphism

$$(\gamma,\delta) : (C, D) \longrightarrow (C'', D'')$$

in $\mathcal{C} \times \mathcal{D}$ is a pair of morphisms

$$\gamma : C \longrightarrow C'' \text{ and } \delta : D \longrightarrow D'',$$

belonging to $\mathcal{C}$ and $\mathcal{D}$ respectively.

Composition is given by the rule

$$(\gamma, \delta)(\gamma', \delta') = (\gamma\gamma', \delta\delta'),$$

where $(\gamma', \delta') : (C', D') \rightarrow (C, D)$.

It is easily verified that $\mathcal{C} \times \mathcal{D}$ is a right category, the identity map on a pair (C, D) being the morphism (id_C, id_D). It is clear that the product category is small if both $\mathcal{C}$ and $\mathcal{D}$ are small.

If instead $\mathcal{C}$ and $\mathcal{D}$ are both left categories, we make the obvious change in the definition of composition so that $\mathcal{C} \times \mathcal{D}$ is also a left category. Should we need to consider products of categories of mixed chiralities, we fix the notation (by using mirrors) so that the product is always a right category. The reader is invited to write out the various forms for composition. An illustration is encountered in (1.3.6).

We next show that if $\mathcal{C}$ and $\mathcal{D}$ are both module categories, so is their product. Given rings R and S, their *direct product* is

$$R \times S = \{(r, s) \mid r \in R, s \in S\},$$

with addition and multiplication defined componentwise:

$$(r, s) + (r', s') = (r + r', s + s'), \quad (r, s) \cdot (r, s) = (rr', ss').$$

We state the result only for right categories; the reader may supply the variations needed for left categories and for mixed chiralities. (Hint: 'it's all done by mirrors'.)

1.1.12 Lemma

Let $\mathcal{C}$ be a subcategory of $\mathcal{M}\text{OD}_R$ and let $\mathcal{D}$ be a subcategory of $\mathcal{M}\text{OD}_S$. Then $\mathcal{C} \times \mathcal{D}$ is a subcategory of $\mathcal{M}\text{OD}_{R \times S}$.

Proof

An object (C, D) in the product category can be made into an $R \times S$-module by the rule $(c, d) \cdot (r, s) = (cr, ds)$, and a morphism in $\mathcal{C} \times \mathcal{D}$ will then be an $R \times S$-module homomorphism. □

Remark. It is not difficult to show that an $R \times S$-module M must be (isomorphic to) a direct sum $L \oplus N$, in which L is an R-module and N is an S-module. Furthermore, a homomorphism of $R \times S$-modules $L \oplus N \rightarrow L' \oplus N'$ must arise from a pair of homomorphisms $L \rightarrow L'$ and $N \rightarrow N'$. (Details are spelt out in [BK: IRM] (2.6.6)ff.) Thus the categories $\mathcal{M}od_{R\times S}$ and $\mathcal{M}od_R \times \mathcal{M}od_S$ are in effect the same, but we cannot substantiate this claim formally until we have the language of equivalences at our disposal – see (1.3.16) and (1.3.17).

1.1.13 Infinite products

The generalization of the definition of a direct product to an arbitrary set of categories $\{\mathcal{C}_\lambda \mid \lambda \in \Lambda\}$, where Λ is any ordered index set, is evident. An object of the product category $\prod_\Lambda \mathcal{C}_\lambda$ is a sequence (C_λ), where each C_λ is an object of the corresponding category $\mathcal{C}_\lambda$, and a morphism $(\gamma_\lambda) : (C_\lambda) \to (C''_\lambda)$ is given by a sequence of morphisms $\gamma_\lambda : C_\lambda \to C''_\lambda$, $\lambda \in \Lambda$.

If each $\mathcal{C}_\lambda$ is a right or left category, then the product of morphisms in $\prod_\Lambda \mathcal{C}_\lambda$ is defined so that $\prod_\Lambda \mathcal{C}_\lambda$ is also right or left accordingly; again, if the chiralities are mixed, we make the product a right category.

In the case that the index set is finite, we usually take it to be $\{1, \dots, n\}$ for some natural number n, and write the product as $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n$. When the component categories $\mathcal{C}_\lambda$ are all the same category, $\mathcal{C}$ say, we use the natural notations $\mathcal{C}^n$ and $\mathcal{C}^\Lambda$.

Products of more than two categories arise naturally when we consider direct products of rings. If $R = R_1 \times \cdots \times R_n$ is a finite direct product of rings and $\mathcal{C}_i$ is a category of R_i-modules for each i, an obvious generalization of (1.1.12) shows that $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ is a category of right R-modules.

1.1.14 Morphism categories

We can form new categories by regarding some of the morphisms of a given category as themselves objects of a category. Such categories are called *morphism categories.* We give the constructions only for right categories, the left-handed versions being obvious.

The first of this type is the category $\mathcal{M}or\,\mathcal{C}$, the *morphism category* of the category $\mathcal{C}$.

The objects of $\mathcal{M}or\,\mathcal{C}$ are triples (C, D, χ) where χ is a morphism from C to D in $\mathcal{C}$, and a morphism from (C, D, χ) to (C'', D'', χ'') is a morphism $(\gamma, \delta) : (C, D) \to (C'', D'')$ in the product category $\mathcal{C}\times\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\ \gamma\ } & C'' \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi''} \\ D & \xrightarrow{\ \delta\ } & D'' \end{array}$$

commutes; that is, $\chi''\gamma = \delta\chi$.

Clearly, $\mathcal{M}\textsc{or}\,\mathcal{C}$ is a right category, which can be regarded as a subcategory of $\mathcal{C} \times \mathcal{C}$ by ignoring the morphism χ. If $\mathcal{C}$ is a module category, say a subcategory of $\mathcal{M}\textsc{od}_R$, then $\mathcal{M}\textsc{or}\,\mathcal{C}$ is also a module category, being a subcategory of $\mathcal{M}\textsc{od}_{R\times R}$.

It is also clear that $\mathcal{M}\textsc{or}\,\mathcal{C}$ is small if $\mathcal{C}$ is small.

Further morphism categories arise as full subcategories of $\mathcal{M}\textsc{or}\,\mathcal{C}$ which are defined by placing restrictions on the morphism appearing in (C, D, χ). Here are a few examples.

$\mathcal{E}\textsc{nd}\,\mathcal{C}$, the *endomorphism* category. The objects are morphisms (C, C, χ), that is, $\chi : C \to C$ must be an *endomorphism* of C.

$\mathcal{I}\textsc{so}\,\mathcal{C}$, the *isomorphism* category, which has objects (C, D, χ) where χ is an *isomorphism* – in an arbitrary category, this means that there is a morphism $\chi^{-1} : D \to C$ with $\chi^{-1}\chi = id_C$ and $\chi\chi^{-1} = id_D$, or, in other words, χ is *invertible*. We may then write $C \cong D$ to indicate that C and D are isomorphic.

$\mathcal{A}\textsc{ut}\,\mathcal{C}$, the *automorphism* category. Here, objects have the form (C, C, χ) with χ invertible.

Exercises

1.1.1 Discrete categories

A category $\mathcal{C}$ is *discrete* if

$$\mathrm{Mor}(C, D) = \begin{cases} \{id_C\} & \text{if } C = D, \\ \emptyset & \text{if } C \neq D. \end{cases}$$

Show that

(a) any set X can be regarded as a discrete category,

(b) if Y is any subclass of the objects of a given category $\mathcal{D}$, then Y defines a discrete subcategory of $\mathcal{D}$.

1.1.2 Let λ and μ be elements of a partially ordered set Λ. Show that the following are equivalent.

(a) $\lambda \cong \mu$ when we regard Λ as a category.

(b) Both $\lambda \leq \mu$ and $\mu \leq \lambda$.

1.1.3 A group as a category

Given a group G, we can define two categories, called $\mathcal{B}G$ and $\mathcal{E}G$, as follows.

$\mathcal{B}G$: this has one object, G itself, and $\mathrm{Mor}(G, G) = \{g \mid g \in G\}$. Composition is multiplication.

$\mathcal{E}G$: here, $\mathrm{Ob}\,\mathcal{E}G = \{g \mid g \in G\}$ is the set of elements of G, and $\mathrm{Mor}(g, h) = \{hg^{-1}\}$ has one element.

What are the subcategories of $\mathcal{B}G$ and of $\mathcal{E}G$?

Investigate the various morphism categories associated with $\mathcal{B}G$ and $\mathcal{E}G$.

For example, observe that the class $\mathrm{Mor}\,\mathcal{C}$ of all morphisms of a category $\mathcal{C}$ forms a group precisely when $\mathcal{C}$ may be identified with the category $\mathcal{B}(\mathrm{Mor}\ \mathcal{C})$.

1.1.4 A monoid as a category

A *monoid* (or *semigroup with identity*) is a set M with an associative multiplication for which there is an identity element 1 (see [BK: IRM] (1.1.1)). Repeat, as far as possible, the previous exercise, substituting a monoid M in place of a group. Since an element of M need not be invertible, we must take the morphisms in $\mathcal{E}M$ to be

$$\mathrm{Mor}(m, n) = \{x \in M \mid xm = n\},$$

so that $id_m = 1$ always.

Show that $m \cong 1$ in $\mathcal{E}M$ if and only if m has a (twosided) inverse in M.

1.1.5 Opposites of groups

Groups G and H are said to be *anti-isomorphic* if there is a bijective map $\phi : G \to H$ with $\phi(gh) = \phi(h)\phi(g)$ for all g, h in G.

Show that $\iota : g \to g^{-1}$ is an anti-isomorphism of G, and that ι is an isomorphism of G if and only if G is abelian.

Define the *opposite* of a group to be a formal set

$$G^{\mathrm{op}} = \{g^{\mathrm{op}} \mid g \in G\}$$

in bijective correspondence with G, with multiplication given by

$$g^{\mathrm{op}}h^{\mathrm{op}} = (hg)^{\mathrm{op}}.$$

Verify that G^{op} is a group and that

$$\tau : G \longrightarrow G^{\mathrm{op}}, \quad g \mapsto g^{\mathrm{op}},$$

is an anti-isomorphism.

Show also that any anti-isomorphism $\phi : G \to H$ factors as a product $\phi = \overline{\phi}\tau$ where $\overline{\phi}$ is a group isomorphism.

Deduce that $\mathrm{Mor}((\mathcal{B}G)^{\odot})$ and $\mathrm{Mor}((\mathcal{B}G)^{\mathrm{op}})$ are both isomorphic to G^{op}.

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Remark. The existence of an anti-isomorphism between a group and its opposite is sometimes used implicitly to identify the two groups. An example is given by the symmetric group S_n, the group of all permutations of the set $\{1, \ldots, n\}$. If we consider the permutations to act as right operators, we obtain one version of the symmetric group, while if we view the permutations as left operators, we obtain the opposite group. Usually, a group theorist will take the right-handed version to be S_n, so that the left-handed version is then properly S_n^{op}.

A practical situation which requires both the left and the right handed versions of S_n arises in Chapter 12 of [Loday 1992].

1.1.6 **Morphisms on morphisms**

Describe the category $\mathcal{M}\text{OR}^2\mathcal{C} = \mathcal{M}\text{OR}\,(\mathcal{M}\text{OR}\,\mathcal{C})$. Compare the categories $\mathcal{A}\text{UT}\,(\mathcal{M}\text{OR}\,\mathcal{C})$ and $\mathcal{M}\text{OR}\,(\mathcal{A}\text{UT}\mathcal{C})$, etc.

1.1.7 Compare $(\mathcal{C}^{\mathrm{op}})^{\odot}$ and $(\mathcal{C}^{\odot})^{\mathrm{op}}$.

1.1.8 Show that if each category $\mathcal{C}_\lambda$ is a module category, then the product $\prod_\Lambda \mathcal{C}_\lambda$ is also a module category.

1.1.9 **Groupoids**

A *groupoid* is a small category G in which every morphism is invertible.

Let $\{v_\lambda \mid \lambda \in \Lambda\}$ be the set of objects of G, where Λ is some set. Show that $G_\lambda = \mathrm{End}(v_\lambda, v_\lambda)$ is a group for each λ; G_λ is called the *vertex group* at λ.

Show also that if $\mathrm{Mor}(v_\lambda, v_\mu) \neq \emptyset$, then the groups G_λ and G_μ are isomorphic.

(Any set $\{G_\lambda\}$ of groups can be viewed as defining a groupoid in which $\mathrm{Mor}(v_\lambda, v_\mu) = \emptyset$ if $\lambda \neq \mu$.)

1.2 FUNCTORS

A major impact of category theory on mathematics is its emphasis on describing the morphisms between mathematical objects, rather than considering the objects in isolation. This philosophy can be applied to the study of categories themselves, leading to the definition of a functor, that is, a morphism between categories.

The language of functors plays an indispensable role in the remainder of this text, and in K-theory in general.

Up to this point, our distinction between right and left categories has not had much influence on the text, since we have been able to develop the theory for right categories and dismiss the left-handed version as obvious. However, the fact that categories may have different chiralities does now have an effect on our notation. The reason is that we have to take into account the possibility that a functor may well relate categories of different chiralities. The outcome is that some entrenched conventions must be treated with caution, since a covariant functor may act in a way that is superficially similar to a contravariant functor.

Nevertheless, we continue to work with right categories as far as possible, leaving statements and proofs in the left-handed case to the reader, unless there are particular points which we need to emphasise.

1.2.1 Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* F from $\mathcal{C}$ to $\mathcal{D}$ associates with each object C of $\mathcal{C}$ an object FC in $\mathcal{D}$, and with each morphism

$$\alpha : C' \longrightarrow C \ \text{ in } \mathcal{C}$$

it associates a morphism

$$F\alpha : FC' \longrightarrow FC \ \text{ in } \mathcal{D}.$$

Thus F is a collection of mappings of two types (all of which are, by custom, denoted by the same letter F): there is one map of the first type

$$F : \mathrm{Ob}\,\mathcal{C} \longrightarrow \mathrm{Ob}\,\mathcal{D},$$

and for each pair C' and C of objects of $\mathcal{C}$ there is a map of the second type

$$F : \mathrm{Mor}_{\mathcal{C}}(C', C) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(FC', FC).$$

These mappings are subject to the axioms

Fun 1 $F(id_C) = id_{FC}$

and

Fun 2 if $\mathcal{C}$ and $\mathcal{D}$ are both right categories and

$$\alpha : C' \longrightarrow C \ \text{ and } \ \beta : C \longrightarrow C''$$

are morphisms in $\mathcal{C}$, then

$$F(\beta\alpha) = F(\beta)F(\alpha) : FC' \longrightarrow FC''.$$

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A *contravariant functor* reverses the direction of morphisms. Thus for each morphism

$$\alpha : C' \longrightarrow C \ \text{ in } \mathcal{C}$$

there is a morphism

$$F\alpha : FC \longrightarrow FC' \ \text{ in } \mathcal{D}.$$

The second axiom now reads

Fun 2^o If $\mathcal{C}$ and $\mathcal{D}$ are both right categories and

$$\alpha : C' \longrightarrow C \ \text{ and } \ \beta : C \longrightarrow C''$$

are morphisms in $\mathcal{C}$, then

$$F(\beta\alpha) = F(\alpha)F(\beta) : FC'' \longrightarrow FC'.$$

The unqualified term *functor* will mean a covariant functor, unless the context demands otherwise.

1.2.2 Some examples

Here are some basic examples.

(i) The *identity* functor $\mathrm{Id}_{\mathcal{C}}$ from a category $\mathcal{C}$ to itself; this 'does nothing' to any object or morphism.

(ii) If $\mathcal{C}'$ is a subcategory of $\mathcal{C}$, there is the *inclusion* functor

$$\mathrm{Inc} : \mathcal{C}' \longrightarrow \mathcal{C};$$

again this leaves objects and morphisms unchanged.

(iii) On any category, there is a standard contravariant functor, the *opposite functor*

$$\mathrm{Op} : \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}},$$

which has $\mathrm{Op}(C) = C^{\mathrm{op}}$ and $\mathrm{Op}(\alpha) = \alpha^{\mathrm{op}}$ – see (1.1.6).

(iv) The *forgetful* or *underlying* functor

$$\Upsilon : \mathcal{C} \longrightarrow \mathcal{D};$$

this arises when each object and morphism of the category $\mathcal{C}$ can be regarded as belonging to another category $\mathcal{D}$ by forgetting some of its structure. Thus there are forgetful functors from $\mathcal{M}\textsc{od}_R$ to $\mathcal{A}\textsc{b}$, from $\mathcal{A}\textsc{b}$ to $\mathcal{G}\textsc{p}$ and from $\mathcal{G}\textsc{p}$ to $\mathcal{S}\textsc{et}$.

1.2.3 Free modules

Another important functor is

$$\mathrm{Fr}_R : \mathcal{S}\mathit{ET} \longrightarrow \mathcal{M}\mathit{OD}_R,$$

which associates to a set X the free right R-module $\mathrm{Fr}_R(X)$ on X, where R is a fixed ring. Since this functor will be encountered frequently in the sequel, we give a sketch of its construction. A fuller account is given in [BK: IRM] (2.1.16).

An element m of $\mathrm{Fr}_R(X)$ is a formal sum

$$m = \sum_{x \in X} x r_x(m) \ \text{ with } r_x(m) \in R,$$

in which only a finite set of coefficients $r_x(m)$ may be nonzero. (In other words, m is a function from X to R whose value at x we prefer to write as $r_x(m)$ rather than $m(x)$, and which takes the value 0 except possibly at a finite number of members of X.) Two such sums m, n are the same precisely when $r_x(m) = r_x(n)$ for all x in X, and addition and scalar multiplication are defined by the rules

$$m + n = \sum_{x \in X} x(r_x(m) + r_x(n)) \quad \text{and} \quad mr = \sum_{x \in X} x(r_x(m)r).$$

It is easy to verify that $\mathrm{Fr}_R(X)$ is a right R-module. We view X as a subset of $\mathrm{Fr}_R(X)$ by identifying an element x of X with the module element that has $r_x = 1$ and $r_y = 0$ for $y \neq x$.

Now, given a mapping $\alpha : X \to Y$ of sets, there is one and only one way to extend α to a homomorphism

$$\mathrm{Fr}_R(\alpha) : \mathrm{Fr}_R(X) \longrightarrow \mathrm{Fr}_R(Y),$$

since we must have

$$\mathrm{Fr}_R(\alpha)\left(\sum_{x \in X} x r_x(m)\right) = \sum_{x \in X} \alpha(x) r_x(m).$$

It is straightforward to check that Fr_R is a functor as claimed.

1.2.4 The product

Suppose that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors. Then the *composite* or *product* functor

$$GF : \mathcal{C} \longrightarrow \mathcal{E}$$

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is given by

$$GF(C) = G(F(C))$$

and

$$GF(\alpha) = G(F(\alpha)).$$

The product GF is sometimes written $G \circ F$ if we wish to emphasise that we are performing a composition of functors.

It is easy to verify that GF will be covariant if F and G are either both covariant or both contravariant, and contravariant otherwise.

Notice that the opposite functor $\mathrm{Op} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ provides a method of interchanging contravariant and covariant functors, since a contravariant functor $G : \mathcal{D} \to \mathcal{C}$ defines a covariant functor $\mathrm{Op} \circ G : \mathcal{D} \to \mathcal{C}^{\mathrm{op}}$, and vice versa.

1.2.5 Restriction

Given a subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$, the *restriction* of F to $\mathcal{C}'$ is the composite

$$F \circ \mathrm{Inc} : \mathcal{C}' \longrightarrow \mathcal{D}.$$

If $\mathcal{D}'$ is a subcategory of $\mathcal{D}$ such that FC and $F\alpha$ are in $\mathcal{D}'$ for every object and morphism of $\mathcal{C}'$, we can also regard F as a functor from $\mathcal{C}'$ to $\mathcal{D}'$, which we again call the restriction of F.

In applications, $\mathcal{D}'$ will often be a full subcategory of $\mathcal{D}$; in this case, F restricts to a functor from $\mathcal{C}'$ to $\mathcal{D}'$ if and only if $F(\mathrm{Ob}\,\mathcal{C}')$ is contained in $\mathrm{Ob}\,\mathcal{D}'$.

To save notation, a restriction of a functor is usually denoted by the same symbol as the original functor.

1.2.6 Appearances and chirality

So far, we have discussed functors between right categories only. When we have to consider left categories as well, we find that the appearance of axiom Fun 2 depends on the chiralities of the categories involved, as well the variance of the functor itself.

To help the reader through this tangle, we list the various possibilities; here, F is a functor from $\mathcal{C}$ to $\mathcal{D}$, and, as hitherto, $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C', C)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(C, C'')$.

Chirality of $\mathcal{C}$	of $\mathcal{D}$	F covariant	F contravariant
left	left	$F(\alpha\beta) = F(\alpha)F(\beta)$	$F(\alpha\beta) = F(\beta)F(\alpha)$
left	right	$F(\alpha\beta) = F(\beta)F(\alpha)$	$F(\alpha\beta) = F(\alpha)F(\beta)$
right	left	$F(\beta\alpha) = F(\alpha)F(\beta)$	$F(\beta\alpha) = F(\beta)F(\alpha)$
right	right	$F(\beta\alpha) = F(\beta)F(\alpha)$	$F(\beta\alpha) = F(\alpha)F(\beta)$

We have listed these variations in detail since they are ignored in all texts on category theory that we have come across. Presumably, this is because it is the usual practice in category theory to deal only with what we have called here right categories.

When concentrating on right categories only, it is tempting to characterize covariant and contravariant in terms of the appearance of axiom Fun 2: *in this case*, covariant functors are those which 'preserve' products, while contravariant functors 'reverse' products. However, as we have seen, this characterization does not adequately describe the intrinsic distinction between covariant and contravariant functors. It is an artefact of the notation and so needs to be treated with caution. The essential difference is that the *direction* of a morphism is preserved in one case but reversed in the other.

In an ideal world, there would be a systematic notation devised to handle functors of differing variances between categories of differing chiralities. In reality we are obliged to use well-established notations for functors that take no account of chirality and variance.

The above table can be simplified by introducing the notion of the *chirality* of a functor. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *cochiral* if the categories $\mathcal{C}$ and $\mathcal{D}$ have the same chirality, and *contrachiral* otherwise. Then, for any composable morphisms α, β in $\mathcal{C}$, we have

$$F(\alpha\beta) = F(\alpha)F(\beta)$$

whenever F has the same variance as chirality, that is, both are co- or both are contra-, and

$$F(\alpha\beta) = F(\beta)F(\alpha)$$

whenever the variance of F differs from its chirality.

One example of a product reversing functor is the opposite functor Op (1.2.2)(iii) which is contravariant and cochiral.

Another such functor is the *mirror* functor. For any category $\mathcal{C}$, we define $\text{Mir} : \mathcal{C} \to \mathcal{C}^{\odot}$ from $\mathcal{C}$ to its mirror category $\mathcal{C}^{\odot}$ (1.1.5) by putting $\text{Mir}(C) = C^{\odot}$ and $\text{Mir}(\alpha) = \alpha^{\odot}$. This functor is evidently covariant and contrachiral.

Reinterpreting the definition of the product in $\mathcal{C}^{\odot}$, we find that

$$\mathrm{Mir}(\alpha\beta) = \mathrm{Mir}(\beta)\,\mathrm{Mir}(\alpha).$$

As we noted above, the opposite functor can be used to convert a contravariant functor into a covariant functor. Similarly, $\mathrm{Mir} : \mathcal{D} \to \mathcal{D}^{\odot}$ converts a contrachiral functor $F : \mathcal{C} \to \mathcal{D}$ to a cochiral functor $\mathrm{Mir} \circ F : \mathcal{C} \to \mathcal{D}^{\odot}$, and likewise a cochiral functor can be changed into a contrachiral functor.

Thus, a judicious use of the functors Op and Mir allows one to work only with covariant and cochiral functors.

1.2.7 The morphism functors

Let $\mathcal{C}$ be a (right) category. The morphism sets $\mathrm{Mor}(L, X) = \mathrm{Mor}_{\mathcal{C}}(L, X)$ can be viewed as defining two functors, $\mathrm{Mor}(L, -)$ and $\mathrm{Mor}(-, X)$, from $\mathcal{C}$ to the category $\mathcal{S}_{\mathcal{ET}}$ of sets, by holding one term of $\mathrm{Mor}(L, X)$ constant, and allowing the other to vary, as the notation is meant to suggest. The functors arising from Mor in one way or another are called *morphism functors.*

Choose an object L of $\mathcal{C}$. The functor

$$\mathrm{Mor}(L, -) : \mathcal{C} \longrightarrow \mathcal{S}_{\mathcal{ET}}$$

sends an object X to the set $\mathrm{Mor}(L, X)$, and for a morphism $\xi : X \to Y$ in $\mathcal{C}$, the map

$$\mathrm{Mor}(L, \xi) : \mathrm{Mor}(L, X) \longrightarrow \mathrm{Mor}(L, Y)$$

is given by

$$\mathrm{Mor}(L, \xi)\alpha = \xi\alpha : L \longrightarrow Y \ \text{ for } \alpha : L \longrightarrow X \text{ in } \mathrm{Mor}(L, X),$$

so that $\mathrm{Mor}(L, -)$ is a covariant functor from $\mathcal{C}$ to $\mathcal{S}_{\mathcal{ET}}$.

(We write $\mathrm{Mor}(L, X)$ and $\mathrm{Mor}(L, \xi)$ in preference to $\mathrm{Mor}(L, -)(X)$ and $\mathrm{Mor}(L, -)(\xi)$.)

If $\vartheta : Y \to Z$ is another morphism, then

$$\mathrm{Mor}(L, \vartheta\xi)\alpha = \vartheta\xi\alpha = \mathrm{Mor}(L, \vartheta)\,\mathrm{Mor}(L, \xi)\alpha.$$

On the other hand, suppose that we fix the object X. Then $\mathrm{Mor}(-, X)(L) = \mathrm{Mor}(L, X)$ for any object L of $\mathcal{C}$, and for a morphism $\lambda : L \to M$, we define

$$\mathrm{Mor}(\lambda, X) : \mathrm{Mor}(M, X) \longrightarrow \mathrm{Mor}(L, X)$$

by

$$\mathrm{Mor}(\lambda, X)\beta = \beta\lambda \ \text{ for } \beta \in \mathrm{Mor}(M, X),$$

which shows that $\mathrm{Mor}(-, X)$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{S}_{ET}$.

If $\mu : M \to N$ is another morphism, then, for any $\gamma : N \to X$,

$$\mathrm{Mor}(\mu\lambda, X)(\gamma) = \gamma\mu\lambda = \mathrm{Mor}(\lambda, X)\,\mathrm{Mor}(\mu, X)\gamma.$$

When the fixed object L or X need not be mentioned, it is usual to make the abbreviations $\mathrm{Mor}(L, \xi) = \xi_*$ and $\mathrm{Mor}(\lambda, X) = \lambda^*$. The formulas for products then become

$$(\xi\vartheta)_* = \xi_*\vartheta_* \;\text{ for }\; \xi : X \longrightarrow Y \;\text{ and }\; \vartheta : Y \longrightarrow Z$$

and

$$(\mu\lambda)^* = \lambda^*\mu^* \;\text{ for }\; \lambda : L \longrightarrow M \;\text{ and }\; \mu : M \longrightarrow N.$$

Note that these formulas do depend on the fact that both the categories $\mathcal{C}$ and $\mathcal{S}_{ET}$ are assumed to be right categories, that is, the functors $\mathrm{Mor}(L, -)$ and $\mathrm{Mor}(-, X)$ are cochiral. If instead we take $\mathcal{C}$ to be a left category, then the product formulas read

$$(\vartheta\xi)_* = \xi_*\vartheta_* \;\text{ for }\; \xi : X \longrightarrow Y \;\text{ and }\; \vartheta : Y \longrightarrow Z$$

and

$$(\lambda\mu)^* = \lambda^*\mu^* \;\text{ for }\; \lambda : L \longrightarrow M \;\text{ and }\; \mu : M \longrightarrow N,$$

despite the fact that the variances of the functors $\mathrm{Mor}(L, -)$ and $\mathrm{Mor}(-, X)$ are unchanged. This is an instance of the variability of notation that we discussed in the preceding section.

1.2.8 The category $\mathcal{C}_{AT}$

Given a class of small categories which contains all the small categories that we wish to discuss, we can form the category $\mathcal{C}_{AT}$ of all such categories. The objects of this category are themselves categories $\mathcal{C}, \mathcal{D}, \ldots$, and the set of morphisms from $\mathcal{C}$ to $\mathcal{D}$ is the set $[\mathcal{C}, \mathcal{D}]$ of all covariant functors from $\mathcal{C}$ to $\mathcal{D}$.

Note that an isomorphism in the category $\mathcal{C}_{AT}$ is an invertible functor. Accordingly, we call any invertible functor an *isomorphism.*

1.2.9 Fibre categories

For a functor $F : \mathcal{D} \to \mathcal{C}$, there is a counterpart of the fundamental concept of the kernel of a module homomorphism, namely, that of a right-fibre category. However, as a general category has no distinguished object corresponding to the zero element of a module, we must define a right-fibre category $C\backslash F$ (of F at C) for each object C in $\mathcal{C}$.

An object of the *right-fibre category* consists of a pair (D, α), where D is an object of $\mathcal{D}$ and $\alpha : C \to F(D)$ is a morphism of $\mathcal{C}$; such a pair is said to be *under* C *with respect to* F, or F-*under* C.

A morphism in $C\backslash F$ from (D, α) to (D', α') is given by a morphism $\beta : D \to D'$ in $\mathcal{D}$ such that $\alpha' = F(\beta) \circ \alpha$. Diagrammatically, this may be visualized as:

$$\begin{array}{ccccc} C & \xrightarrow{\alpha} & F(D) & \qquad & D \\ & \searrow^{\alpha'} & \vdots\, F(\beta) & & \vdots\, \beta \\ & & \downarrow & & \downarrow \\ & & F(D') & & D' \end{array}$$

The definition of the dual notion of the *left-fibre category* F/C of pairs (D, α) F-*over* C, should be apparent.

The justification for the name right-fibre category (replacing comma category, surely one of the least descriptive names ever in mathematics) comes from Theorem B of [Quillen 1973] in the homotopy theory of categories, which, however, lies beyond the scope of this book. We use right-fibre categories in section 1.4, where we consider universal constructions.

Here are some simple examples which shed light on the definition, as does Exercise 1.2.1.

1.2.10 Examples

1. Let $\mathcal{C}$ be the category having a unique object C and a unique morphism, id_C. Then, for the unique functor $F : \mathcal{D} \to \mathcal{C}$, the category $C\backslash F$ is isomorphic to the category $\mathcal{D}$ through the functors

$$(D, id_C) \leftrightarrow D, \quad g \leftrightarrow g.$$

2. For the identity functor $\mathrm{Id}_\mathcal{C}$ of an arbitrary category $\mathcal{C}$ with C an object of $\mathcal{C}$, the right-fibre category $C\backslash \mathrm{Id}_\mathcal{C}$ is just the category of objects of $\mathcal{C}$ *under* C, which is sometimes called the *slice* category $C\backslash\mathcal{C}$. Similarly, there is the category $\mathcal{C}/C = \mathrm{Id}_\mathcal{C}/C$ of objects *over* C. These categories are widely used by topologists.
3. Given a group G, form the category $\mathcal{B}G$ with one object G, the morphisms in $\mathcal{B}G$ being simply the elements of the group, as in Exercise 1.1.3. If H

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is also a group viewed as a category in this way, a functor F from $\mathcal{B}H$ to $\mathcal{B}G$ is the same as a group homomorphism γ from H to G.

The right-fibre category $G\backslash\gamma$ consists of pairs (H, g) where H is the single object of $\mathcal{B}H$ and g is an arbitrary element of G; we can therefore identify $G\backslash\gamma$ with the set $\{g\}$ of elements of G. A morphism from g to g' is an element h of H with $\gamma(h) \circ g = g'$.

Thus the set of morphisms in $G\backslash\gamma$ from id_G to itself is the kernel of γ, while the set of morphisms from id_G to g' is $\{h \mid \gamma(h) = g'\}$, the inverse image of g'.

In particular, $G\backslash \mathrm{Id}_{\mathrm{Cat}(G)}$ is the category $\mathcal{E}G$ of Exercise 1.1.3.

Exercises

1.2.1 Let X and Y be sets regarded as discrete categories, as in Exercise 1.1.1.

Show that a functor from X to Y is simply a map $f : X \to Y$.

Verify that for any $y \in Y$, the fibre $y\backslash f$ is $f^{-1}(y)$, the inverse image of y.

1.2.2 We may regard a group G as a category, $\mathcal{B}G$ or $\mathcal{E}G$, in two ways as in Exercise 1.1.3. In (1.2.10), we noted that a functor from $\mathrm{Cat}(H)$ to $\mathrm{Cat}(G)$ is the same thing as a group homomorphism from H to G.

Are there any functors from $\mathcal{B}G$ to $\mathcal{E}G$ or vice versa?

Describe the functors from $\mathcal{E}H$ to $\mathcal{E}G$. In particular, show that each element of the group G acts, by multiplication, as a functor on $\mathcal{E}G$ that is transitive on the objects.

Remark. So, if we identify the objects of $\mathcal{E}G$ with one another via this multiplication, we obtain $\mathcal{B}G$. Given a method of passing from groups to topological spaces, this procedure may be used to form the classifying space BG of the group G [Segal 1968].

1.2.3 Recall (or look up in a standard text on group theory) that the commutator subgroup $[G, G]$ of a group G is the normal subgroup of G generated by all commutators $ghg^{-1}h^{-1}$, and that the *commutator quotient group* $G_{ab} = G/[G, G]$ is an abelian group. Sometimes the group G_{ab} is said to be *G made abelian.*

Define $\mathrm{Ab} : \mathcal{G}_{\mathcal{P}} \to \mathcal{A}_{\mathcal{B}}$ by $\mathrm{Ab}(G) = G_{ab}$, where $\mathcal{G}_{\mathcal{P}}$ is the category of groups and $\mathcal{A}_{\mathcal{B}}$ is the category of abelian groups. Verify that, with the obvious action on homomorphisms, Ab is a functor, sometimes called *abelianization.*

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1.3 NATURAL TRANSFORMATIONS

Functors themselves can be related to one another by natural transformations. This ability to compare functors enables us to define the concepts of multifunctors and of adjoint functors, vital for our discussion of universal properties in the next section. It also leads to the notion of a functor category, consisting of all the functors from one category to another. When the domain category has an especially simple form, this gives the important special case of a diagram in a category.

We also consider equivalences between categories; these are analogous to isomorphisms between modules or groups in the sense that equivalent categories can be regarded as identical for most purposes. The definition of isomorphism of categories, given in (1.2.8) above, is usually too restrictive for applications.

1.3.1 Natural transformations

Natural transformations are the morphisms between functors themselves.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be covariant functors. A *natural transformation*

$$\eta : F \longrightarrow G$$

consists of a family of morphisms

$$\eta_C : FC \longrightarrow GC,$$

one for each object C of $\mathcal{C}$, such that for each morphism

$$\alpha : C' \longrightarrow C,$$

the diagram

$$\begin{array}{ccc} FC' & \xrightarrow{\eta_{C'}} & GC' \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FC & \xrightarrow{\eta_C} & GC \end{array}$$

commutes. Somewhat informally, a typical member of this family is sometimes said to be a natural transformation.

If, instead, F and G are both contravariant, we require the following dia-

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gram to commute:

$$\begin{array}{ccc} FC & \xrightarrow{\eta_C} & GC \\ {\scriptstyle F\alpha}\downarrow & & \downarrow{\scriptstyle G\alpha} \\ FC' & \xrightarrow{\eta_{C'}} & GC' \end{array}$$

(There is little to be gained by attempting to compare functors of different variances, as can be seen from the example of 'unnatural' behaviour that is given in Exercise 1.3.18.)

1.3.2 Examples

(i) Basic and important examples of natural transformations arise from the morphism functors (1.2.7) associated with a (right) category $\mathcal{C}$.

Let $\lambda : L \to M$ and $\xi : X \to Y$ be morphisms in $\mathcal{C}$ and recall that

$$\xi_* = \xi_*^L : \mathrm{Mor}(L, X) \longrightarrow \mathrm{Mor}(L, Y)$$

is given by

$$\xi_*(\alpha) = \xi\alpha : L \longrightarrow Y \ \text{ for } \ \alpha : L \longrightarrow X \ \text{ in } \mathrm{Mor}(L, X),$$

and

$$\lambda^* = \lambda_X^* : \mathrm{Mor}(M, X) \longrightarrow \mathrm{Mor}(L, X)$$

is given by

$$\lambda^*\beta = \beta\lambda \ \text{ for } \ \beta \ \text{ in } \mathrm{Mor}(M, X).$$

It is easy to verify that there is a commutative square

$$\begin{array}{ccc} \mathrm{Mor}(M, X) & \xrightarrow{\xi_*^M} & \mathrm{Mor}(M, Y) \\ \downarrow{\scriptstyle \lambda_X^*} & & \downarrow{\scriptstyle \lambda_Y^*} \\ \mathrm{Mor}(L, X) & \xrightarrow{\xi_*^L} & \mathrm{Mor}(L, Y) \end{array}$$

which can be interpreted in two ways, one as showing that ξ_* is a natural transformation from $\mathrm{Mor}(-, X)$ to $\mathrm{Mor}(-, Y)$ and the other, that λ^* is a natural transformation from $\mathrm{Mor}(M, -)$ to $\mathrm{Mor}(L, -)$.

(ii) Let R be a ring and let $M_n(R)$ be the ring of $n \times n$ matrices over R. The inclusion map

$$\iota_R : R \longrightarrow M_n(R)$$

sends an element r of R to the scalar matrix $\operatorname{diag}(r, \dots, r)$. (The structure of matrix rings is discussed at length in section 2.2 of [BK: IRM].)

If $f : R \to S$ is a ring homomorphism, there is an obvious commuting diagram

$$\begin{array}{ccc} R & \xrightarrow{\iota_R} & M_n(R) \\ f \downarrow & & \downarrow M_n(f) \\ S & \xrightarrow{\iota_S} & M_n(S) \end{array}$$

of ring homomorphisms. Thus ι is a natural transformation from the identity functor on the category $\mathcal{R}\text{ING}$ to the functor $M_n(-)$ on $\mathcal{R}\text{ING}$. Typically, one says just that the embedding ι is natural.

(iii) A ring R contains a smallest subring, namely its *prime ring*, which is the subring $\langle 1_R \rangle$ comprising the multiples $n.1_R$, $n \in \mathbb{Z}$, of the multiplicative identity 1_R of R. The assignment $P : R \mapsto \langle 1_R \rangle$ is clearly a functor from the category $\mathcal{R}\text{ING}$ of rings to itself. Then the embedding of $\langle 1_R \rangle$ in R is a natural transformation from P to the identity functor because a ring homomorphism $f : R \to S$ gives a commuting square

$$\begin{array}{ccc} \langle 1_R \rangle & \hookrightarrow & R \\ Pf \downarrow & & \downarrow f \\ \langle 1_S \rangle & \hookrightarrow & S \end{array}$$

(iv) Given a nonunital ring R, the *enveloping ring* or *unitalization* of R is

$$\overline{R} = \{(r, a) \mid r \in R, a \in \mathbb{Z}\},$$

which is a ring under the addition and multiplication given by

$$(r, a) + (s, b) = (r + s, a + b) \quad \text{and} \quad (r, a) \cdot (s, b) = (rs + br + as, ab).$$

The identity of $\overline{R}$ is $(0, 1)$. Then, in the category $\mathcal{R}\text{NG}$ of nonunital rings, the embedding of R in its enveloping ring is natural.

(v) In the category of commutative domains, the embedding of a commutative domain in its field of fractions is also natural – see [BK: IRM] (1.1.12).

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(vi) For another example, where neither functor is the identity functor, we use the operation of transposition on matrices. Given a matrix $A = (a_{ij})$ over a ring R, the *transpose* of A is defined to be the matrix $A^t = (a^\circ_{ji})$, which has entries in the opposite ring R°. Applying this definition to matrices over R° rather than R itself, we obtain a map

$$(-)^t : M_n(R^\circ) \longrightarrow (M_n(R))^\circ$$

which is readily seen to be a ring isomorphism. (Further details can be found in Exercises 1.2.13 and 1.2.14 of [BK: IRM].)

Given a ring homomorphism $f : R \to S$ there is a commutative square

$$\begin{array}{ccc} M_n(R^\circ) & \xrightarrow{(-)^t} & (M_n(R))^\circ \\ \downarrow & & \downarrow \\ M_n(S^\circ) & \xrightarrow{(-)^t} & (M_n(S))^\circ \end{array}$$

so that $(-)^t$ is a natural transformation between the functors $M_n((-)^\circ)$ and $(M_n(-))^\circ$.

(vii) For a ring R, the general linear group $\mathrm{GL}_n(R)$ is the group of invertible $n \times n$ matrices over R, that is, the group of units of the matrix ring $M_n(R)$.

Evidently, $\mathrm{GL}_n(R)$ can be embedded in $\mathrm{GL}_{n+1}(R)$ by the insertion of an extra (last) row and column with diagonal entry 1 and other entries 0 to each matrix A in $GL_n(R)$:

$$A \longmapsto \left(\begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right).$$

We abbreviate the right-hand matrix to $A \oplus 1$. Since

$$AB \oplus 1 = (A \oplus 1)(B \oplus 1),$$

we obtain a group homomorphism $- \oplus 1 : \mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$. Moreover, $- \oplus 1$ is natural because any ring homomorphism $f : R \to S$ gives

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rise to a commuting square

$$\begin{array}{ccc} \mathrm{GL}_n(R) & \xrightarrow{-\oplus 1} & \mathrm{GL}_{n+1}(R) \\ {\scriptstyle \mathrm{GL}_n(f)}\downarrow & & \downarrow{\scriptstyle \mathrm{GL}_{n+1}(f)} \\ \mathrm{GL}_n(S) & \xrightarrow{-\oplus 1} & \mathrm{GL}_{n+1}(S) \end{array}$$

(viii) We continue with the above notation, but now restrict our attention to the category of commutative rings. We can then define the determinant map $\det : M_n(R) \to R$ by the usual formula involving sums and differences of products of entries. (See [Cohn 1982], Chapter 7 for example.) Because the determinant preserves products, that is, $\det(AB) = \det(A)\det(B)$ for $n \times n$ matrices A and B, it restricts to a homomorphism of the groups of units $\det : \mathrm{GL}_n(R) \to U(R)$. Again, this homomorphism is natural, meaning that there is a commuting square

$$\begin{array}{ccc} \mathrm{GL}_n(R) & \xrightarrow{\det} & U(R) \\ {\scriptstyle \mathrm{GL}_n(f)}\downarrow & & \downarrow{\scriptstyle U(f)} \\ \mathrm{GL}_n(S) & \xrightarrow{\det} & U(S) \end{array}$$

whenever f is a ring homomorphism.

1.3.3 Natural isomorphisms

If for each object C of $\mathcal{C}$ the morphism η_C is an isomorphism in $\mathcal{D}$, we say that the functors F and G are *naturally isomorphic* through the *natural isomorphism* η. The notation is $F \simeq G$. The terms *naturally equivalent* and *natural equivalence* are also used, especially in older texts.

Among the above examples, that in (vi) is a natural isomorphism because the matrix transpose operation is evidently invertible. In (ii) and (viii), we clearly have natural isomorphisms when $n = 1$.

1.3.4 Matrices and bases of free modules: a summary

Our next example gives a categorical interpretation of some results on the matrix representation of homomorphism between free modules. To get started, we need some facts about free modules and their bases which are given in detail in [BK: IRM], particularly §2.2. For future ease of reference in this volume, we review these facts in greater generality than we need for our immediate purposes.

Let R be a ring and let M be a right R-module. A ***basis*** of M is an ordered subset $\{f_\lambda\}_{\lambda\in\Lambda}$ (where Λ is an ordered indexing set), which satisfies the following property.

Bas. Given any element m of M, there are *unique* elements $r_\lambda(m)$, $\lambda \in \Lambda$, of R such that

$$m = \sum_{\lambda\in\Lambda} f_\lambda r_\lambda(m).$$

The set Λ may be infinite, but at most a finite number of coefficients $r_\lambda(m)$ can be nonzero for any particular m.

An arbitrary module M need not have a basis. If M does have a basis, then M is a *free right* R-module.

Note that a basis must be ordered. An 'unordered basis' is called a *free generating set*. Thus, the free module $\mathrm{Fr}_R(X)$ on an unordered set X (1.2.3) has X as a free generating set. If we impose an ordering on X, then we obtain a basis of $\mathrm{Fr}_R(X)$.

When the index set Λ is finite, we usually take it to be $\{1,\ldots,m\}$ for some integer m and write the basis as

$$\{f_j\} = \{f_1,\ldots,f_m\}.$$

We then say that the free module has *finite rank*.

Now let R^n be the standard free right R-module of finite rank n, which we view as the 'space of column vectors' of length n, with entries in R. The evident choice of 'unit' vectors $\{e_1,\ldots,e_n\}$ gives the *standard basis* $\{e_j\}$ of R^n. Next, take any finite basis

$$\{f_j\} = \{f_1,\ldots,f_m\}$$

of R^n. If it must be the case that $m = n$, the ring R is said to have *invariant basis number*. However, we wish to work with an arbitrary ring, so we must allow the possibility that $m \neq n$. Some examples and results on invariant basis number and the lack of it are to be found in §2.3 of [BK: IRM].

An element x of R^n has a *coordinate vector* ${}_F(x)$ with respect to the basis $F = \{f_j\}$, which is given by the formula

$${}_F(x) = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \quad \text{where } x = f_1y_1 + \cdots + f_my_m, \;\; y_1,\ldots,y_m \in R.$$

If R^ℓ is another standard free right R-module with basis $\{g_j\} = \{g_1, \ldots, g_k\}$ and $\alpha : R^n \to R^\ell$ is a homomorphism of right R-modules, we associate to α a $k \times m$ matrix $A = (a_{ij})$, which is determined by the formula

$$\alpha(f_j) = \sum_{i=1}^{k} g_i a_{ij}, \quad j = 1, \ldots, m.$$

The coordinate vector ${}_G(\alpha x)$ of αx with respect to $G = \{g_j\}$ is then given by the relation

$${}_G(\alpha x) = A_F(x).$$

The association of a matrix to a homomorphism is easily seen to be multiplicative; that is, if $\beta : R^p \to R^n$ is an R-module homomorphism that is composable with α, and if B is the matrix of β with respect to a basis $\{h_j\}$ of R^p and the given basis $\{f_j\}$ of R^n, then the matrix of $\alpha\beta$ with repect to the pair $\{h_j\}$, $\{g_j\}$ is AB.

The association is also additive, for if α and α' are homomorphisms from R^n to R^ℓ with matrices A and A' respectively (relative to the pair of bases $\{f_j\}$, $\{g_j\}$), then the matrix of $\alpha + \alpha'$ will be $A + A'$ (relative to the same pair of bases).

In particular, fix an n-element basis of R^n and use it at 'both ends'. Then the matrix A of an endomorphism α will be $n \times n$, and the map $\alpha \mapsto A$ is a ring isomorphism from $\mathrm{End}(R^n)$ to the matrix ring $M_n(R)$.

1.3.5 Example: matrices and bases of free modules

Now we interpret the above discussion in terms of categories and functors. We define $\mathcal{B}\textsc{ases}_R$ to be the category whose objects are free right R-modules of the form R^n equipped with a preferred basis $\{f_j\}$, that is, a pair $(R^n, \{f_j\})$. The morphisms in $\mathcal{B}\textsc{ases}_R$ are just the R-module homomorphisms from R^n to R^ℓ (here the choice of basis is ignored). The full subcategory $\mathcal{S}\textsc{tan}\mathcal{B}\textsc{ases}_R$ of $\mathcal{B}\textsc{ases}_R$ has as objects the pairs $(R^n, \{e_j\})$ where $\{e_j\}$ is the standard basis of R^n. (We use the notation $\{e_j\}$ ambiguously for a standard basis of any size.)

The fact that the operation of taking a matrix of a homomorphism is multiplicative now reveals that we obtain a functor

$$S : \mathcal{B}\textsc{ases}_R \longrightarrow \mathcal{S}\textsc{tan}\mathcal{B}\textsc{ases}_R$$

by setting

$$S(R^n, \{f_j\}) = (R^m, \{e_j\}), \quad m \text{ being the number of members of } F = \{f_j\},$$

and

$$S(\alpha) = A, \quad \text{the matrix of } \alpha.$$

For each object $(R^n, F = \{f_j\})$ of $\mathcal{B}\textsc{ases}_R$, there is an R-module isomorphism η_F from R^n to R^m which associates to each member x of R^n its coordinate vector ${}_F(x)$: thus

$$\eta_F(x) = {}_F(x).$$

(Properly speaking, η should have as suffix the pair of objects between which we map, but the notation is then unwieldy.)

To interpret these isomorphisms as defining a natural isomorphism of functors, we let T be the composite functor $T = \mathrm{Inc} \circ S$, where Inc is the inclusion of $\mathcal{S}\textsc{tan}\mathcal{B}\textsc{ases}_R$ in $\mathcal{B}\textsc{ases}_R$. For any morphism $\alpha : (R^n, F = \{f_j\}) \to (R^l, G = \{g_j\})$ in $\mathcal{B}\textsc{ases}_R$, there is a commutative diagram:

$$\begin{array}{ccc}
(R^n, \{f_j\}) & \xrightarrow{\quad \eta_F \quad} & (R^m, \{e_j\}) \\
\alpha \Big\downarrow & & \Big\downarrow T(\alpha) \\
(R^\ell, \{g_j\}) & \xrightarrow{\quad \eta_G \quad} & (R^k, \{e_j\})
\end{array}$$

which shows $\eta : \mathrm{Id} \to T$ to be a natural transformation, as desired. Note that the homomorphism $T(\alpha)$ is simply left multiplication by the matrix A of α.

1.3.6 Multifunctors

It is often necessary to consider functors of several variables. The guiding example is $\mathrm{Mor}_{\mathcal{C}}(-,-)$ for a (right) category $\mathcal{C}$, which attaches a set $\mathrm{Mor}_{\mathcal{C}}(C, D)$ to each pair of objects C, D of $\mathcal{C}$. By the remarks in (1.2.7), $\mathrm{Mor}_{\mathcal{C}}(-, D)$ is a contravariant functor, while $\mathrm{Mor}_{\mathcal{C}}(C, -)$ is covariant. So we wish to recognize $\mathrm{Mor}_{\mathcal{C}}(-,-) : \mathcal{C} \times \mathcal{C} \to \mathcal{S}\textsc{et}$ as simultaneously functorial in both variables, even though it is not actually a functor on the product category $\mathcal{C} \times \mathcal{C}$. The point is that, because of its mixed variances, the would-be functor $\mathrm{Mor}(-,-)$ does not obey either of the composition laws Fun 2 or Fun 2^o. The same difficulty arises when we consider $\mathrm{Op} \times \mathrm{Id} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$.

A related problem arises when we wish to take the variables from categories of differing chiralities, since the law of composition in the product of categories of mixed chiralities can be rather convoluted. For instance, let $\mathcal{C}$ be a right category, so that its mirror $\mathcal{C}^{\odot}$ (1.1.5) is a left category. Our rule then is to view the product category $\mathcal{C} \times \mathcal{C}^{\odot}$ as a right category (1.1.11).

To see how this works, recall that a morphism $\beta^{\odot} : B^{\odot} \to D^{\odot}$ in $\mathcal{C}^{\odot}$ is a formal symbol corresponding to a morphism $\beta : B \to D$ in $\mathcal{C}$, and that the composition of a pair of composable morphisms $\beta^\odot$ and $\delta^\odot$ in $\mathcal{C}^\odot$ is $\beta^\odot\delta^\odot = (\delta\beta)^\odot$.

Thus, if we have a pair of composable morphisms $(\alpha, \beta^\odot)$, $(\gamma, \delta^\odot)$ in $\mathcal{C} \times \mathcal{C}^\odot$, their product is

$$(\gamma, \delta^\odot)(\alpha, \beta^\odot) = (\gamma\alpha, \beta^\odot\delta^\odot) = (\gamma\alpha, (\delta\beta)^\odot).$$

Methods of handling mixed chiralities become important later (3.1.9), when we consider tensor products. Thus we need a definition which accommodates both mixed variances and mixed chiralities.

Our uniform treatment of the various possible mixtures of variances and chiralities is based on two observations. Firstly, if F is a contravariant functor on $\mathcal{C}$, then $F \circ \mathrm{Op}$ is a covariant functor on $\mathcal{C}^{\mathrm{op}}$; and secondly, if $F : \mathcal{C} \to \mathcal{D}$ is a contrachiral functor, then $F \circ \mathrm{Mir} : \mathcal{C}^\odot \to \mathcal{D}$ is a cochiral functor. (Here we rely on the obvious identifications $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$ and $(\mathcal{C}^\odot)^\odot = \mathcal{C}$ for any category $\mathcal{C}$.)

Let $\mathcal{D}, \mathcal{C}_1, \dots, \mathcal{C}_k$ be a finite set of (right) categories. Suppose that

$$F : \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \longrightarrow \mathcal{D}$$

is given by a function

$$F : \mathrm{Ob}(\mathcal{C}_1) \times \cdots \times \mathrm{Ob}(\mathcal{C}_k) \longrightarrow \mathrm{Ob}(\mathcal{D})$$

and, for each pair C_i', C_i of objects of each $\mathcal{C}_i$, a function

$$F : \mathrm{Mor}_{\mathcal{C}_1} \times \cdots \times \mathrm{Mor}_{\mathcal{C}_k} \longrightarrow \mathrm{Mor}_{\mathcal{D}}(F(C_1', \dots, C_k'), F(C_1, \dots, C_k)).$$

Then we say that F is a *multifunctor* or a *functor in k variables* on $\mathcal{C}_1, \dots, \mathcal{C}_k$ provided that for some (necessarily unique) choice of functors δ_i, ϵ_i on $\mathcal{C}_i$, with each δ_i as either Id or Op and each ϵ_i as either Id or Mir,

$$F \circ (\delta_1\epsilon_1 \times \cdots \times \delta_k\epsilon_k) : \epsilon_1\delta_1(\mathcal{C}_1) \times \cdots \times \epsilon_k\delta_k(\mathcal{C}_k) \longrightarrow \mathcal{D}$$

is a covariant and cochiral functor.

If δ_i is Id, then F is said to be *covariant in the i th variable*, while if δ_i is Op, F is said to be *contravariant in the i th variable.*

Likewise, if ϵ_i is Id, then F is said to be *cochiral in the i th variable*, while if ϵ_i is Op, F is *contrachiral in the i th variable.*

The terms *bifunctor*, *trifunctor*,... are also used when $k = 2, 3, \dots$. Note that a multifunctor in one variable is simply a functor of the given variance and chirality.

Thus $\mathrm{Mor}_{\mathcal{C}}(-,-)$ is a bifunctor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{S}\mathit{ET}$ which is contravariant in

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the first variable and covariant in the second. It is cochiral in both variables. A version which exhibits mixed chiralities is given in Exercise 2.1.1.

Natural transformations between multifunctors, of the same variance and chirality, are defined in the obvious way.

It is also possible to describe a multifunctor 'term-by-term'. Let F be a multifunctor, and choose any index $h = 1, \ldots, k$ and any set of objects $\{C_i \in \mathcal{C}_i \mid i \neq h\}$. Then F restricts to a functor

$$F(C_1, \ldots, C_{h-1}, -, C_{h+1}, \ldots, C_k) : \mathcal{C}_h \longrightarrow \mathcal{D},$$

which is covariant or contravariant according as δ_h is Id or Op, and cochiral or contrachiral according as ϵ_h is Id or Mir.

Furthermore, any collection of morphisms

$$\{\beta_i : \delta_i(B_i) \longrightarrow \delta_i(C_i) \in \delta_i(\mathcal{C}_i) \mid i \neq h\}$$

gives rise to a natural transformation

$$F(B_1, \ldots, B_{h-1}, -, B_{h+1}, \ldots, B_k) \longrightarrow F(C_1, \ldots, C_{h-1}, -, C_{h+1}, \ldots, C_k).$$

Conversely, given a fixed index h, a multifunctor can be defined by a collection of functors of the form $F(C_1, \ldots, C_{h-1}, -, C_{h+1}, \ldots, C_k)$, one for each set of objects from the categories $\mathcal{C}_i$, $i \neq h$, provided that the appropriate natural transformations exist.

1.3.7 Adjoint functors

We now introduce a concept which is very important in both the theory and the applications of categories, particularly in connection with universal constructions, which we consider in detail in the next section.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be (covariant) functors. We can form two bifunctors

$$\mathrm{Mor}_{\mathcal{C}}(-, G(-)) : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{S}\varepsilon\tau$$

and

$$\mathrm{Mor}_{\mathcal{D}}(F(-), -) : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{S}\varepsilon\tau,$$

both of which are contravariant in the first variable and covariant in the second.

The functors F and G are said to form an *adjoint pair*, or to be *adjoint*, if there is a natural isomorphism

$$\eta : \mathrm{Mor}_{\mathcal{D}}(F(-), -) \xrightarrow{\cong} \mathrm{Mor}_{\mathcal{C}}(-, G(-)).$$

If this is the case, F is said to be *left adjoint* to G and G is said to be *right adjoint* to F. (The terms 'left' and 'right' here are convincing only for covariant functors.)

Thus, to show that F and G are adjoint, it is necessary and sufficient to give a family of bijections

$$\eta_{C,D} : \mathrm{Mor}_{\mathcal{D}}(F(C), D) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(D))$$

which are natural in C and D.

The following example typifies the set-up.

1.3.8 Example: free modules

Let X be a set and let $\mathrm{Fr}_R(X)$ be the free right R-module on X. Then $\mathrm{Fr}_R : \mathcal{S}\varepsilon\tau \to \mathcal{M}od_R$ is a covariant functor (1.2.3). In the other direction, let $\Upsilon : \mathcal{M}od_R \to \mathcal{S}\varepsilon\tau$ be the forgetful functor, which assigns to a module M its underlying set $\Upsilon(M)$ of elements, as in (1.2.2)(iv). It is easy to see that Fr_R is left adjoint to Υ.

1.3.9 Functor categories

Given a small category $\mathcal{C}$ and an arbitrary category $\mathcal{D}$, the covariant functors from $\mathcal{C}$ to $\mathcal{D}$ can themselves be regarded as the objects of a category $[\mathcal{C}, \mathcal{D}]$. Such categories $[\mathcal{C}, \mathcal{D}]$ and their subcategories are called *functor* categories.

The objects of $[\mathcal{C}, \mathcal{D}]$ are the covariant functors

$$F : \mathcal{C} \longrightarrow \mathcal{D},$$

and a morphism between functors F and G is a natural transformation

$$\eta : F \longrightarrow G.$$

We write $\mathrm{Nat}(F, G)$ for the set of all such natural transformations. This is indeed a set, since $\eta = \{\eta_C \mid C \in \mathcal{C}\}$ is specified by a collection of data which is indexed by the set $\mathrm{Ob}\,\mathcal{C}$, and each η_C must belong to the set $\mathrm{Mor}(FC, GC)$. (The target category $\mathcal{D}$ need not be small.)

To see that we actually have a (right) category, we note that each functor F has an *identity* natural transformation id_F given by

$$(id_F)_C = id_{FC} \text{ for each } C \in \mathcal{C},$$

and that for $\eta \in \mathrm{Nat}(F, G)$ and $\zeta \in \mathrm{Nat}(G, H)$, their product

$$\zeta\eta : F \longrightarrow H$$

is defined by

$$(\zeta\eta)_C = \zeta_C \eta_C \text{ for each } C \in \mathcal{C}$$

when $\mathcal{D}$ is a right category.

The category $[\mathcal{C}^{\mathrm{op}}, \mathcal{D}]$ is the category of all contravariant functors from $\mathcal{C}$ to $\mathcal{D}$.

1.3.10 Diagrams

A useful application of the preceding discussion occurs when functors from $\mathcal{C}$ to $\mathcal{D}$ are interpreted as defining *diagrams* in $\mathcal{D}$.

A diagram in a category $\mathcal{D}$ is the same thing as the *graph* of a functor from some category $\mathcal{C}$ to $\mathcal{D}$. That is, it consists of a collection of objects and morphisms of $\mathcal{D}$ which are the images of the objects and morphisms of $\mathcal{C}$ and which satisfy the appropriate commutativity conditions. Although in principle any category $\mathcal{C}$ defines diagrams, the notion is most often used when $\mathcal{C}$ has few objects an morphisms.

For example, let $[n]$ be the totally ordered set $\{0, \ldots, n\}$, which we view as a category with exactly one morphism from i to j whenever $i \le j$. We can display $[n]$ as a diagram

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \quad \cdots \quad \longrightarrow n-1 \longrightarrow n,$$

in which the 'generating' morphisms are shown; the remaining morphisms in $[n]$ are identities or products of those indicated. (Note that this category is labelled according to the number of generating morphisms, rather than the number of objects. This is due to its topological applications. For example, see Exercise 1.3.17 below.)

A functor from $[n]$ to $\mathcal{D}$, that is, an object F of $[[n], \mathcal{D}]$, is then a sequence of morphisms in $\mathcal{D}$

$$F : D_0 \xrightarrow{\alpha_1} D_1 \xrightarrow{\alpha_2} D_2 \quad \cdots \quad D_{n-1} \xrightarrow{\alpha_n} D_n$$

and a morphism η from F' (with the obvious labelling) to F is a set of morphisms in $\mathcal{D}$

$$\eta_i : D'_i \longrightarrow D_i, \quad i = 0, \ldots, n,$$

such that the diagram

$$\begin{array}{cccccccccc}
F' : D'_0 & \xrightarrow{\alpha'_1} & D'_1 & \xrightarrow{\alpha'_2} & D'_2 & \cdots & D'_{n-1} & \xrightarrow{\alpha'_n} & D'_n \\
\downarrow \eta_0 & & \downarrow \eta_1 & & \downarrow \eta_2 & & \downarrow \eta_{n-1} & & \downarrow \eta_n \\
F : D_0 & \xrightarrow{\alpha_1} & D_1 & \xrightarrow{\alpha_2} & D_2 & \cdots & D_{n-1} & \xrightarrow{\alpha_n} & D_n
\end{array}$$

commutes, that is, each of its squares commutes.

Note that the category $[[0], \mathcal{D}]$ is just $\mathcal{D}$, while $[[1], \mathcal{D}]$ is the morphism category $\mathcal{M}\text{OR}\,\mathcal{D}$ under another name. Notice also that an object in $[[2], \mathcal{D}]$ can be interpreted as a commutative triangle.

1.3.11 Equivalence of categories

We now consider the circumstances in which two categories can be regarded as the same.

The most obvious of such circumstances is that the categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*, that is, we have a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with $GF = \mathrm{Id}_{\mathcal{C}}$ and $FG = \mathrm{Id}_{\mathcal{D}}$. This means that F and G are mutually inverse isomorphisms (1.2.8). If we have such a pair of functors, they induce mutually inverse bijections between the classes $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Ob}(\mathcal{D})$. But this is a relatively uncommon event, since the definition of the 'value' $F(C)$ of a functor F on an object C very often requires the choice of one object from many isomorphic candidates.

More likely, and still acceptable as a means of comparing categories, is for functors F and G to induce bijections of the collections of isomorphism classes of objects of $\mathcal{C}$ and $\mathcal{D}$.

For example, for a field $\mathcal{K}$, each isomorphism class of objects of $\mathcal{S}\text{TAN}\mathcal{B}\text{ASES}_{\mathcal{K}}$ is a singleton, comprising only the standard basis of $\mathcal{K}^n$, whereas $\mathcal{B}\text{ASES}_{\mathcal{K}}$ has many objects isomorphic to the standard basis of $\mathcal{K}^n$. Yet because *every* object of $\mathcal{B}\text{ASES}_{\mathcal{K}}$ is isomorphic to the standard basis, the subcategory $\mathcal{S}\text{TAN}\mathcal{B}\text{ASES}_{\mathcal{K}}$ is a good working approximation to $\mathcal{B}\text{ASES}_{\mathcal{K}}$.

Therefore, the appropriate notion is *equivalence* of categories: two categories $\mathcal{C}$ and $\mathcal{D}$ are (naturally) equivalent if there are (covariant) functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ so that GF is naturally isomorphic to $\mathrm{Id}_{\mathcal{C}}$ and FG is naturally isomorphic to $\mathrm{Id}_{\mathcal{D}}$. The functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are then said to be mutually inverse *equivalences* (of $\mathcal{C}$ and $\mathcal{D}$); the notation is $\mathcal{C} \simeq \mathcal{D}$.

If F and G are contravariant, they are called *dualities*, and the categories are said to be *dual.*

1.3.12 *Example: standard bases*

In (1.3.5), we introduced the category $\mathcal{B}_{\mathcal{ASES}_R}$ whose objects are free R-modules R^n together with a chosen basis, and its full subcategory $\mathcal{S}_{\mathcal{TAN}}\mathcal{B}_{\mathcal{ASES}_R}$ of free modules with standard bases. These categories are equivalent since, in our previous notation, we have shown that $T = \mathrm{Inc} \circ S$ is isomorphic to the identity functor on $\mathcal{B}_{\mathcal{ASES}_R}$, while $F \circ \mathrm{Inc}$ is clearly the identity functor on $\mathcal{S}_{\mathcal{TAN}}\mathcal{B}_{\mathcal{ASES}_R}$.

1.3.13 *Faithful, full and dense*

In general, a category $\mathcal{C}$ is equivalent to a full subcategory $\mathcal{C}'$ precisely when there is a natural way of associating to each object of $\mathcal{C}$ a representative in $\mathcal{C}'$ of its $\mathcal{C}$-isomorphism class. This idea may be developed as follows.

A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* if the map

$$F : \mathrm{Mor}_{\mathcal{C}}(C', C) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(FC', FC)$$

is an injection for every pair of objects in $\mathcal{C}$; F is *full* if the above map is a surjection, and F is *dense* or *representative* if for each object D in $\mathcal{D}$, we have $D \cong FC$ for some C in $\mathcal{C}$.

With these definitions, the following result is straightforward. (More details can be found in [Herrlich & Strecker 1979], (14.11), and [Mitchell 1965], II, particularly Proposition 10.1.)

1.3.14 Proposition

The categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if there is a full, faithful and dense functor from $\mathcal{C}$ to $\mathcal{D}$. □

1.3.15 *Skeletons*

A category $\mathcal{S}$ is called *skeletal* if no two of its objects are isomorphic. Trivial examples are provided by the categories $[n]$, $n \geq 0$.

Given any category $\mathcal{C}$, we can form a full skeletal subcategory $\mathrm{Sk}(\mathcal{C})$ of $\mathcal{C}$ by choosing one object from each isomorphism class of the objects of $\mathcal{C}$; naturally, $\mathrm{Sk}(\mathcal{C})$ is called the *skeleton* of $\mathcal{C}$. The skeleton is not, in general, unique. However, the inclusion functor from $\mathrm{Sk}(\mathcal{C})$ to $\mathcal{C}$ is evidently full, faithful and dense, so $\mathrm{Sk}(\mathcal{C})$ is equivalent to $\mathcal{C}$.

Notice that $\mathrm{Sk}(\mathcal{C})$ is small precisely when the class of isomorphism classes of objects in $\mathcal{C}$ is in fact a set. (This condition suffices for applications in K-theory.) Thus the outcome of the discussion in (1.1.10) can be rephrased as saying that, for a ring R, the category $\mathcal{M}_R$ of finitely generated right R-modules has a small skeleton.

Next we present some equivalences which sharpen the relationships between product and morphism categories on the one hand, and module categories on the other, which we first touched on in (1.1.12) and (1.1.14).

1.3.16 Theorem

Let R and S be rings. Then the direct product category $\mathcal{M}od_R \times \mathcal{M}od_S$ is equivalent to $\mathcal{M}od_{R\times S}$.

Proof

As we noted in (1.1.12), an object (M, N) in the product category can be regarded as an $R \times S$-module by the rule $(m, n) \cdot (r, s) = (mr, ns)$, and a morphism in $\mathcal{M}od_R \times \mathcal{M}od_S$ is an $R \times S$-module homomorphism.

On the other hand, any $R \times S$-module is isomorphic to one of the form $M \oplus N$, where M is an R-module and N is an S-module, and the direct sum is the direct sum of abelian groups; furthermore, this decomposition respects homomorphisms (see [BK: IRM] (2.6.6), (2.6.7)).

It is now evident that the equivalence is given by

$$(M, N) \mapsto M \oplus N \text{ and } (\alpha, \beta) \mapsto \alpha \oplus \beta. \qquad \square$$

Recall that $\mathcal{M}_R$ and $\mathcal{P}_R$ are respectively the categories of finitely generated right R-modules and finitely generated projective right R-modules. The components of a finitely generated or projective module will again be finitely generated or projective, as the case may be, since the equivalence given in the previous theorem evidently preserves surjections and direct sums (see Corollary 2.6.9 in [BK: IRM] for more detail). Thus the following result is immediate.

1.3.17 Corollary

There are equivalences of categories

$$\mathcal{M}_R \times \mathcal{M}_S \simeq \mathcal{M}_{R\times S}$$

and

$$\mathcal{P}_R \times \mathcal{P}_S \simeq \mathcal{P}_{R\times S}. \qquad \square$$

Next we describe the morphisms of $\mathcal{M}od_R$ in terms of modules over a triangular matrix ring.

1.3.18 Theorem

Let $T = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ *be the ring of upper triangular* 2×2 *matrices over a ring* R*. Then the categories* $\mathcal{M}\text{od}_T$ *and* $\mathcal{M}\text{or}(\mathcal{M}\text{od}_R)$ *are equivalent.*

Proof

Let L be a (right) T-module. Since the direct product $R \times R$ is the subring $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ of T, we can write $L = M \oplus N$, where both M and N are R-modules, as in (1.3.16). Using the action of T on $M \oplus N$, define $\alpha : M \to N$ by

$$(m, 0) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (0, \alpha m).$$

Then α is easily seen to be a homomorphism of R-modules, and the action of T is given by

$$(*) \qquad (m, n) \begin{pmatrix} r & u \\ 0 & s \end{pmatrix} = (mr, \alpha mu + ns).$$

Let L' be another T-module, let $L' = M' \oplus N'$ be its decomposition and α' the corresponding homomorphism. A T-module homomorphism $\lambda : L' \to L$ induces R-module homomorphisms $\mu : M' \to M$ and $\nu : N' \to N$.

Calculating $\lambda(m', 0) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $m' \in M'$, in two ways, we find that $\nu\alpha' = \alpha\mu$, which shows that we can define a functor $F : \mathcal{M}\text{od}_T \to \mathcal{M}\text{or}(\mathcal{M}\text{od}_R)$ by sending L to α.

Conversely, given an R-module homomorphism α from M to N, it is clear how to use the formula $(*)$ to turn $M \oplus N$ into a T-module L, thus defining a functor G in the reverse direction. Evidently, F and G are mutually inverse equivalences. □

Exercises

1.3.1 Let X and Y be sets regarded as discrete categories, and maps between them regarded as functors, as in Exercise 1.2.1.

Show that there is a natural transformation between two maps $f, g : X \to Y$ if and only if $f = g$.

1.3.2 **Yoneda's Lemma**

Let $F : \mathcal{C} \to \mathcal{S}\text{et}$ be a covariant functor, and let L be an object of $\mathcal{C}$ and ℓ an element of $F(L)$. For each object X of $\mathcal{C}$, define

$$\eta_X(\ell) : \text{Mor}_{\mathcal{C}}(L, X) \longrightarrow F(X)$$

by

$$\eta_X(\ell)(\alpha) = F(\alpha)(\ell).$$

Verify that

$$\eta(\ell) = \{\eta_X(\ell) \mid X \text{ an object of } \mathcal{C}\}$$

defines a natural transformation from $\mathrm{Mor}_{\mathcal{C}}(L,-)$ to $F(-)$.

Show that the map $\ell \mapsto \eta(\ell)$ defines a bijection from $F(L)$ to $\mathrm{Nat}(\mathrm{Mor}(L,-), F)$, with inverse given by the map which associates to a natural transformation $\zeta : \mathrm{Mor}_{\mathcal{C}}(L,-) \to F(-)$ the element $\zeta_L(id_L)$ of $F(L)$.

By taking $F(-) = \mathrm{Mor}(M,-)$, show that every natural transformation from $\mathrm{Mor}(L,-)$ to $\mathrm{Mor}(M,-)$ has the form μ^* for some $\mu : M \to L$.

Repeat the exercise for contravariant functors from $\mathcal{C}$ to $\mathcal{S}_{\mathcal{ET}}$.

Remark. A covariant functor $F : \mathcal{C} \to \mathcal{S}_{\mathcal{ET}}$ is said to be *representable* if it is naturally isomorphic to $\mathrm{Mor}(L,-)$ for some object L in $\mathcal{C}$. The analogous definition is made for contravariant functors.

1.3.3 **Isomorphisms of categories**

Isomorphisms of categories come in four varieties, since the mutually inverse functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ may be either both covariant or contravariant and either both cochiral or contrachiral.

Show that the identity functor, the opposite functor Op, the mirror functor Mir and the composite $\mathrm{Mir} \circ \mathrm{Op}$ provide examples of all four varieties.

Show also that an arbitrary isomorphism can be converted into a covariant, cochiral isomorphism by composition with one of the isomorphisms listed above.

Now let G and H be groups viewed as one-element categories $\mathcal{B}G$ and $\mathcal{B}H$ (Exercise 1.1.3), and let $F : \mathcal{B}G \to \mathcal{B}H$ be an isomorphism of categories. Show that if F is covariant then F corresponds to an isomorphism of groups in the usual sense, while if F is contravariant, F corresponds to an anti-isomorphism of groups.

1.3.4 We again regard a group G as a category, $\mathcal{B}G$ or $\mathcal{E}G$, in two ways, as in Exercise 1.1.3. Given groups G and H, describe any natural transformations and isomorphisms that there are between a pair of functors between two of the categories $\mathcal{B}G$, $\mathcal{B}H$, $\mathcal{E}G$ and $\mathcal{E}H$.

1.3.5 Let $\mathrm{Ab} : \mathcal{G}_{\mathcal{P}} \to \mathcal{A}_{\mathcal{B}}$ be the abelianization functor between the category $\mathcal{G}_{\mathcal{P}}$ of groups and the category $\mathcal{A}_{\mathcal{B}}$ of abelian groups (Exercise 1.2.3), and let $\Upsilon : \mathcal{A}_{\mathcal{B}} \to \mathcal{G}_{\mathcal{P}}$ be the inclusion functor. Show that Ab, Υ is an adjoint pair.

1.3.6 (a) Show that if F, G and F', G' are both adjoint pairs, then $F'F$, GG' is also an adjoint pair, whenever the composite functors are defined.

(b) Show that if F, G is an adjoint pair, then each of F and G determines the other to within natural isomorphism.

1.3.7 Show that if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are mutually inverse equivalences, then F, G is an adjoint pair.

1.3.8 Let $A : \mathcal{C}' \to \mathcal{C}$ and $B : \mathcal{D} \to \mathcal{D}''$ be (covariant) funetors between small categories. Define functors $[A, -] : [\mathcal{C}, \mathcal{D}] \to [\mathcal{C}', \mathcal{D}]$ and $[-, B] : [\mathcal{C}, \mathcal{D}] \to [\mathcal{C}, \mathcal{D}'']$, and verify that $[-,-]$ is a bifunctor from $\mathcal{C}_{\mathcal{AT}} \times \mathcal{C}_{\mathcal{AT}}$ to $\mathcal{C}_{\mathcal{AT}}$.

1.3.9 (a) In the notation of the preceding exercise (1.3.8) show that if $G : \mathcal{D}'' \to \mathcal{D}$ is right adjoint to $F : \mathcal{D} \to \mathcal{D}''$, then $[\mathcal{C}, F]$, $[\mathcal{C}, G]$ is an adjoint pair.

(b) Each object D in the category $\mathcal{D}$ defines a *constant functor* $\mathrm{Cnst}_D : \mathcal{C} \to \mathcal{D}$ sending each object of $\mathcal{C}$ to D and each morphism of $\mathcal{C}$ to id_D. Show that this assignment defines another constant functor $\mathrm{Cnst}_{\mathcal{D}} : \mathcal{D} \to [\mathcal{C}, \mathcal{D}]$ with the property that, for any functor $G : \mathcal{D}'' \to \mathcal{D}$, the square

$$\begin{array}{ccc} \mathcal{D}'' & \xrightarrow{\mathrm{Cnst}_{\mathcal{D}''}} & [\mathcal{C}, \mathcal{D}''] \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle [\mathcal{C}, G]} \\ \mathcal{D} & \xrightarrow{\mathrm{Cnst}_{\mathcal{D}}} & [\mathcal{C}, \mathcal{D}] \end{array}$$

commutes.

(c) Suppose that $L_{\mathcal{D}} : [\mathcal{C}, \mathcal{D}] \to \mathcal{C}$ is left adjoint to $\mathrm{Cnst}_{\mathcal{D}}$ with $L_{\mathcal{D}''}$ similarly left adjoint to $\mathrm{Cnst}_{\mathcal{D}''}$. By combining Exercise 1.3.6 with the above, show that if $F : \mathcal{D} \to \mathcal{D}''$ has a right adjoint, then $F \circ L_{\mathcal{D}}$ is naturally isomorphic to $L_{\mathcal{D}''} \circ [\mathcal{C}, F]$. In other words, the square obtained from (b) above by replacing the constant functors by their left adjoints also commutes, up to natural isomorphism.

Remark. When $L_{\mathcal{D}} : [\mathcal{C}, \mathcal{D}] \to \mathcal{C}$ is left adjoint to $\mathrm{Cnst}_{\mathcal{D}}$, then, for any functor $G : \mathcal{C} \to \mathcal{D}$, the object $L_{\mathcal{D}}(G)$ is called the *colimit* of G – see (5.1.16) and Exercise 5.1.7.

1.3.10 Let Ω be the *loop* category: Ω has one object $\bullet$, and one generating

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morphism $\alpha : \bullet \to \bullet$, so that $\mathrm{Mor}(\bullet,\bullet) = \{id, \alpha, \alpha^2, \ldots\}$. Show that $[\Omega, \mathcal{D}] = \mathcal{E}\textsc{nd}(\mathcal{D})$ for any category $\mathcal{D}$.

Find a functor $[1] \to \Omega$, where $[1]$ is the category $0 \to 1$, that gives rise to the inclusion of $\mathcal{E}\textsc{nd}(\mathcal{D})$ in $\mathcal{M}\textsc{or}(\mathcal{D})$.

Devise categories $\mathcal{C}$ such that $[\mathcal{C}, \mathcal{D}]$ is (i) $\mathcal{I}\textsc{so}\,\mathcal{D}$, (ii) $\mathcal{A}\textsc{ut}\,\mathcal{D}$.

1.3.11 Let $[1]^2 = [1] \times [1]$, where $[1]$ is the category $0 \to 1$, and let $\mathcal{D}$ be an arbitrary category. Show that $[1]^2$ is the 'generic' commutative square

$$\begin{array}{ccc} (0,1) & \to & (1,1) \\ \uparrow & \nearrow & \uparrow \\ (0,0) & \to & (1,0) \end{array}$$

and that $[[1]^2, \mathcal{D}]$ is the category of all commutative squares in $\mathcal{D}$.

Show also that $\left[[1]^2, \mathcal{D}\right] = \mathcal{M}\textsc{or}^2(\mathcal{D})$, that is, $\mathcal{M}\textsc{or}(\mathcal{M}\textsc{or}(\mathcal{D}))$.

Interpret $[1]^n$ and $[[1]^n, \mathcal{D}]$.

Is the category $[2]^2 = [2] \times [2]$ equivalent to $[4]$?

1.3.12 Show that $\mathcal{S}\textsc{tan}\mathcal{B}\textsc{ases}_R$ (1.3.5) is isomorphic to the category whose objects are the natural numbers and whose morphisms from n to m are the $m \times n$ matrices over R.

1.3.13 Let $\mathcal{S}\textsc{tan}\mathcal{B}\textsc{ases}_R$ and $\mathcal{B}\textsc{ases}_R$ be as in (1.3.5) and (1.3.12). Show that $\mathcal{S}\textsc{tan}\mathcal{B}\textsc{ases}_R$ is a skeletal subcategory of $\mathcal{B}\textsc{ases}_R$ if and only if the ring R has invariant basis number.

1.3.14 Given an arbitrary ordered set Λ, the *standard free* right R-module R^Λ is the module of 'column vectors' (r_λ) indexed by the members λ of Λ. The *standard basis* of R^Λ is $\{e_\lambda\}$, where the 'unit vector'e_λ has λ-entry 1 and all other entries 0.

Let $\mathcal{S}\textsc{tan}\mathcal{B}\textsc{ases}_R^\infty$ and $\mathcal{B}\textsc{ases}_R^\infty$ be as $\mathcal{S}\textsc{tan}\mathcal{B}\textsc{ases}_R$ and $\mathcal{B}\textsc{ases}_R$, but with infinite index sets also allowed. Extend the discussion which we gave in (1.3.5) to $\mathcal{S}\textsc{tan}\mathcal{B}\textsc{ases}_R^\infty$ and $\mathcal{B}\textsc{ases}_R^\infty$. (The definition and properties of matrices of homomorphisms with respect to infinite bases are fairly straightforward – see (2.2.13) of [BK: IRM].)

1.3. 5 Let ${}^*\mathcal{F}\textsc{ree}_R$ denote the category of based free (right) R-modules: an object is a pair (F, B) where F is free and B is a basis, and a morphism from (F', B') to (F, B) is an R-homomorphism from F' to F which restricts to an order-preserving map from B' to B. Thus ${}^*\mathcal{F}\textsc{ree}_R$ is the category of functors from the category $\mathcal{O}\textsc{rd}$ of ordered sets to $\mathcal{F}\textsc{ree}_R$. Let $\mathrm{Fr} : \mathcal{O}\textsc{rd} \to {}^*\mathcal{F}\textsc{ree}_R$ associate with Λ the standard free module R^Λ with basis $\{e_\lambda\}$ and let $S : {}^*\mathcal{F}\textsc{ree}_R \to \mathcal{O}\textsc{rd}$ send (F, B) to the ordered set that indexes B.

Show that ${}^*\mathcal{F}_{REE_R}$ and $\mathcal{O}_{RD}$ are equivalent categories, with $S \circ \mathrm{Fr} = \mathrm{Id}$ but $\mathrm{Fr} \circ S \neq \mathrm{Id}$.

Define a dense and faithful functor $I : \mathcal{O}_{RD} \to \mathcal{S}_{TAN}\mathcal{B}_{ASES}{}_R^{\infty}$.

1.3.16 Prove the following generalization of (1.3.18).

For any ring R, there is a natural isomorphism between the categories $[[n-1], \mathcal{M}_{OD_R}]$ and $\mathcal{M}_{OD_{T_n}}$, where T_n is the ring of upper triangular $n \times n$ matrices over R.

(More results in a similar vein are given by [Mitchell 1965], IX §7ff.)

1.3.17 Let Δ (denoted Δ^* in [May 1967]) be the category whose objects are the categories $[n]$, $n = 0, 1, 2, \ldots$ and whose morphisms are the functors among these categories. For want of a name, we may think of Δ as the *proto-simplicial category* (justification in (c) below).

(a) For $0 \leq i \leq n$, define $\delta_i : [n-1] \to [n]$ and $\sigma_i : [n+1] \to [n]$ by

	$j < i$	$j = i$	$j > i$	**Effect**
$\delta_i(j)$:	j	$j+1$	$j+1$	omit i
$\sigma_i(j)$:	j	j	$j-1$	repeat i

Show that any non-identity morphism $\mu : [n] \to [m]$ in Δ can be factorized uniquely in the following way. Let the objects of $[m]$ that are not in the image of μ be $i_1 > i_2 > \cdots > i_s$, and let $\mu(j) = \mu(j+1)$ precisely when j is one of $j_1 < j_2 < \cdots < j_t$. (Thus $n + s = m + t$.) Then

$$\mu = \delta_{i_1} \cdots \delta_{i_s} \sigma_{j_1} \cdots \sigma_{j_t}.$$

(b) Algebraic topologists define a *simplicial object* in a category $\mathcal{C}$ to be a contravariant functor from Δ to $\mathcal{C}$. Show that this amounts to specifying objects $S_0, S_1, S_2, \ldots$ in $\mathcal{C}$, and two kinds of morphisms, *face operators* $d_i : S_n \to S_{n-1}$ and *degeneracy operators* $s_i : S_n \to S_{n+1}$ $(0 \leq i \leq n)$, and commuting diagrams corresponding to relations

(i)

$$d_i d_j = d_{j-1} d_i \qquad \text{for } i < j$$

(ii)

$$s_i s_j = s_{j+1} s_i \qquad \text{for } i \leq j$$

(iii)

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ id & \text{for } i = j, j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases}$$

This apparatus is informally represented as

$$S_0 \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} S_1 \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} S_2 \cdots$$

where the face operators map from right to left, and the degeneracy operators from left to right. When $\mathcal{C} = \mathcal{S}_{\mathcal{ET}}$, we obtain a simplicial set, when $\mathcal{C} = \mathcal{G}_{\mathcal{P}}$ a simplicial group, etc.

(c) The terminology comes from the following situation. As in [Spanier 1966](3.1), a *simplicial complex* is a set V (the set of *vertices*) together with a distinguished set S (the set of *simplices*) of finite subsets of V, such that

(i) every singleton in V is in S, and
(ii) every nonempty subset of a simplex is a simplex.

Show how an ordering of the vertices gives rise to a simplicial set, and conversely, how a simplicial set describes an ordered simplicial complex, where the face and degeneracy operators are given by

$$d_i(v_0, \dots, v_n) = (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n),$$
$$s_i(v_0, \dots, v_n) = (v_0, \dots, v_i, v_i, v_{i+1}, \dots, v_n).$$

(See [May 1967] Ch. 1 for further discussion and topological applications. See also [Hovey 1999] and [Jardine 1996] for highly readable accounts of the relation to Quillen's closed model categories and K-theory.)

1.3.18 A non-natural isomorphism: duality

After so many examples of natural behaviour, the reader may begin to wonder if anything is unnatural. Here is an example which anticipates our general discussion of duality in (4.1.1).

Given a finite-dimensional vector space V over a field $\mathcal{K}$, the *dual* of V is $V^* = \mathrm{Hom}_{\mathcal{K}}(V, \mathcal{K})$. Verify that this is also a $\mathcal{K}$-space, with the addition and scalar multiplication :

$$(\alpha + \beta)(v) = \alpha(v) + \beta(v), \quad (k \cdot \alpha)(v) = k(\alpha(v)),$$

where $\alpha, \beta \in V^*$, $v \in V$, $k \in \mathcal{K}$.

Let $\{f_1, \dots, f_n\}$ be a basis of V. The *dual basis* $\{f_1^*, \dots, f_n^*\}$ of V^* is defined by

$$f_j^*(f_i) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

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Check that the dual basis is indeed a basis, and so there is an isomorphism $\eta_V : (V, \{f_j\}) \xrightarrow{\cong} (V^*, \{f_j^*\})$.

Show that this isomorphism cannot be natural as defined here, because the assignment $\Delta : \mathcal{B}\textsc{ases}_{\mathcal{K}} \to \mathcal{B}\textsc{ases}_{\mathcal{K}}$, $\Delta((V, \{f_j\})) = (V^*, \{f_j^*\})$, corresponds to a *contravariant* functor on $\mathcal{B}\textsc{ases}_{\mathcal{K}}$; it is a morphism functor $\mathrm{Mor}(-, \mathcal{K})$ as in (1.2.7).

One might try to rescue some naturality by hoping that squares of the form

$$\begin{array}{ccc} (V, \{f_j\}) & \xrightarrow{\eta_V} & (V^*, \{f_j^*\}) \\ \theta \big\downarrow & & \big\uparrow \theta^* \\ (W, \{g_j\}) & \xrightarrow{\eta_W} & (W^*, \{g_j^*\}) \end{array}$$

commute.

Take $(V, \{f_j\})$ to be $(\mathcal{K}^1, 1)$ and write $\theta(1) = t_1 f_1 + \cdots + t_n f_n$. Verify that $\theta^* \circ \eta_V \circ \theta(1) = (t_1^2 + \cdots + t_n^2)1^*$, and hence that such squares are not commutative in general.

If we go one step further and define $\Delta^2(V) = (V^*)^*$, the *double dual*, we obtain a covariant functor on the category of all finite-dimensional vector spaces over $\mathcal{K}$, (with morphisms the $\mathcal{K}$-linear transformations of such spaces).

Define $\epsilon_V : V \to V^{**}$ by

$$(\epsilon_V(v))(\alpha) = \alpha(v) \quad \text{for } v \in V,\ \alpha \in V^*.$$

Show that each ϵ_V is an injective linear transformation, and hence an isomorphism by dimension counting, and that the family $\{\epsilon_V\}$ is a natural isomorphism between the identity functor and Δ^2.

1.4 UNIVERSAL OBJECTS

An underlying principle in category theory is that an object should be considered in relation to the other objects in the category, rather than being viewed in isolation. This perspective leads to the idea of a universal object, which, informally, is an object that is defined by the requirement that it should have a specified relationship to the other objects in the category. When such a requirement has been determined, it can often be seen that apparently distinct kinds of object are defined by the same requirement, but in different categories.

An illustration will be provided by free objects. Although the free group on a set X and the free R-module on X have different appearances, both share the same universal property, but in the categories $\mathcal{G}_{\mathcal{P}}$ of groups and $\mathcal{M}_{\mathcal{OD}R}$ of R-modules respectively. We can therefore define a free object on X in any suitable category $\mathcal{C}$ as an object, in $\mathcal{C}$, with this property, and we can attempt to determine its structure. In such generality, free objects may not exist.

The description of an object in terms of a universal property also provides a means of transferring definitions from one category, in which it is easy to define a particular type of object, to another, where the definition is not immediate. For example, in the module category $\mathcal{M}_{\mathcal{OD}R}$, the kernel of a homomorphism is defined explicity in terms of the elements of a module. However, a kernel can also be described axiomatically as a universal object in a category associated with $\mathcal{M}_{\mathcal{OD}R}$, which enables us to extend the definition of a kernel to more general types of category, and to verify whether or not kernels exist in the new setting. This analysis will be very useful in the next chapter.

In fact, many of the definitions in the remainder of this text, and also in K-theory, are best treated as the specification for some universal object.

In this section, we explain the meanings of the terms 'universal object' (in (1.4.2) below), 'universal construction' (1.4.8) and 'universal property' (1.4.16), and present some important examples.

1.4.1 Initial objects

First, we present some basic definitions that underly the theory. An *initial object* of an (abstract) category $\mathcal{C}$ is an object I in $\mathcal{C}$ with the property that there is exactly one morphism

$$\iota_C : I \longrightarrow C$$

for each object C in $\mathcal{C}$; thus

$$\mathrm{Mor}(I, C) = \{\iota_C\} \text{ for each } C.$$

In the category $\mathcal{G}_{\mathcal{P}}$ of groups, the trivial group is initial, in $\mathcal{M}_{\mathcal{OD}R}$ the zero module is initial, and in the category $\mathcal{R}_{\mathcal{ING}}$ of rings (with identity), $\mathbb{Z}$ is initial. On the other hand, the infinite cyclic group Z is not initial in $\mathcal{G}_{\mathcal{P}}$, since there is always more than one homomorphism from Z to a nontrivial group.

An example of a category without an initial object is provided by $\mathcal{F}_{\mathcal{IELD}}$, the full subcategory of $\mathcal{R}_{\mathcal{ING}}$ whose objects are the fields. This follows from the observation that no field can be mapped homomorphically into both $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$.

An initial object need not be absolutely unique. For example, there are many manifestations of the zero module since the zero element of any module serves. However, an initial object is essentially unique.

To see this, we note that the morphism ι_I must be the identity morphism on I, and that if $\alpha : C \to C''$ is any morphism in $\mathcal{C}$, then $\alpha\iota_C = \iota_{C''}$. Thus, if I and I' are both initial objects of $\mathcal{C}$, then there are unique morphisms

$$\iota : I \longrightarrow I' \quad \text{and} \quad \iota' : I' \longrightarrow I,$$

which must satisfy the relations

$$\iota'\iota = id_I \quad \text{and} \quad \iota\iota' = id_{I'},$$

that is, ι and ι' are mutually inverse (and unique) isomorphisms.

This type of uniqueness is sometimes summarised in the phrase 'an initial object is *unique up to unique isomorphism*'. Because of it, one often speaks of 'the' initial object rather than 'an' initial object.

In many explicit situations, an initial object will appear to be absolutely unique since there is an obvious choice for one. An example is given by the cokernel $\operatorname{Cok}\alpha$ of a homomorphism $\alpha : M \to N$ between R-modules. This is almost invariably defined to be the set of cosets of N modulo $\operatorname{Im}(\alpha)$, given the appropriate module structure. However, it can be thought of as the initial object in the category of homomorphic images N'' of N such that the surjection $\pi : N \twoheadrightarrow N''$ composes with α to yield the zero map $0 = \pi\alpha$.

1.4.2 Universal objects

The philosophy underlying the term 'universal object' is as follows.

We wish to specify a type of object in a category $\mathcal{C}$ by its relationships to the totalility of objects of $\mathcal{C}$. Such relationships manifest themselves as a set of conditions which may or may not be satisfied by an object of $\mathcal{C}$. In turn, it is often possible to characterize those objects C of $\mathcal{C}$ which do satisfy the given conditions in terms of associated objects C' of a related or ancillary category $\mathcal{C}'$. If it happens that C' is an initial object in $\mathcal{C}'$, then the corresponding object C of $\mathcal{C}$ is called a universal object of $\mathcal{C}$, for the given conditions.

Since an initial object is unique up to unique isomorphism, the same will be true of a universal object. Thus one often refers to 'the' universal object with some specified properties. An example will illustrate these ideas better than formalism alone. We consider a characterization of free modules.

1.4.3 Free modules revisited

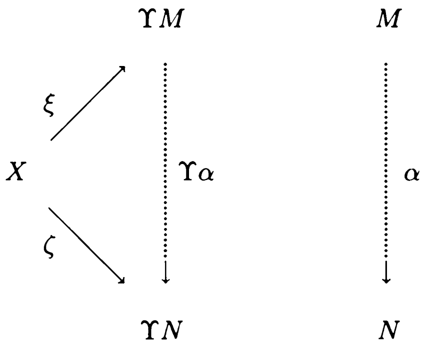
Let X be a set and let R be a ring, and consider the following condition on a pair consisting of a right R-module M together with a specified subset $\{m_x\}_{x\in X}$ of M.

Given any set $\{n_x \mid x \in X\}$ of elements n_x of any right R-module N, there exists a unique R-module homomorphism $\alpha : M \to N$ with $\alpha m_x = n_x$ for all x in X.

We show that the free right R-module $\mathrm{Fr}_R(X)$ on X (1.2.3) is the universal object in $\mathcal{M}\textsc{od}_R$ which satisfies this condition.

To construct the appropriate ancillary category, observe that specifying a set $\{n_x\}$ in N amounts to the same thing as defining a mapping of sets $\zeta : X \to N$. We therefore consider X as an object of the category $\mathcal{S}\textsc{et}$ of sets and their mappings, and use the forgetful functor $\Upsilon : \mathcal{M}\textsc{od}_R \to \mathcal{S}\textsc{et}$. If M satisfies our condition, then there must be a map $\xi : X \to M$ and a homomorphism $\alpha : M \to N$ with $\alpha\xi = \zeta$.

We therefore arrive at the category $X\backslash\Upsilon$ of (right) R-modules under X via Υ (1.2.9). The objects of this category are pairs (M, ξ) in which M is an R-module and $\xi : X \to \Upsilon M$ is a mapping (of sets), and a morphism $\alpha : (M, \xi) \to (N, \zeta)$ is given by a commutative triangle



where α is an R-module homomorphism.

The free module $\mathrm{Fr}_R(X)$, together with the map $\xi_0 : X \to \mathrm{Fr}_R(X)$ that simply identifies an element x of X as a free generator of $\mathrm{Fr}_R(X)$, as in (1.2.3), gives an initial element of $X\backslash\Upsilon$. Thus $\mathrm{Fr}_R(X)$, or more properly, the pair $(\mathrm{Fr}_R(X), \xi_0)$, is universal among R-modules which are Υ-under X.

Moreover, any free module F with a set of free generators $\{b_x\}$ labelled by X also gives an initial object (F, ξ) of $X\backslash\Upsilon$, with ξ sending x to b_x. (See [BK: IRM] (2.1.19) ff. for further details.)

Although the map ξ_0 is an essential part of the information, it is often omitted in practice, so that the free module $\mathrm{Fr}_R(X)$ is itself called the universal object. The utility of the notion of a universal object should be apparent. Replacing $\mathcal{M}\textsc{od}_R$ by $\mathcal{G}\textsc{p}$, we are led to recover the free group on X, and if instead we subsitute $\mathcal{A}_{\mathcal{B}}$, we obtain the free abelian group on X (see [Cohn 1982], §9). Indeed, for an arbitrary category $\mathcal{C}$, one has a specification for the 'free object in $\mathcal{C}$ on X', whenever this exists.

To describe a universal object in a category $\mathcal{C}$, how should we choose the ancillary category $\mathcal{C}'$ whose initial object defines the desired universal object in $\mathcal{C}$? The above example gives a clue. It suggests that $\mathcal{C}'$ is likely to be chosen as a suitable right-fibre category.

Our first result shows this idea to be formally correct in that an initial object must arise in a right-fibre category. Thus we define a *universal object* to be an initial object in a right-fibre category.

1.4.4 Lemma

If a category $\mathcal{D}$ has an initial object, then it is (isomorphic to) a right-fibre category.

Proof

Let $I \in \mathcal{D}$ be initial. Then $\mathcal{D}$ is isomorphic to the right-fibre category $I \backslash Id_{\mathcal{D}}$, via the correspondence which associates an object D of $\mathcal{D}$ with the object $I \to D$ of $I \backslash Id_{\mathcal{D}}$, and a morphism $D \to D'$ of $\mathcal{D}$ with the morphism

$$\begin{array}{ccccc} I & \longrightarrow & D & \qquad & D \\ & \searrow & \downarrow & & \downarrow \\ & & D' & & D' \end{array}$$

of $I \backslash Id_{\mathcal{D}}$. □

The converse to this lemma is false, as may be seen from Exercise 1.4.1.

In general, the right-fibre categories that we seek will arise from adjoint pairs of functors, as we now proceed to demonstrate.

We start by reminding ourselves of the definition of an adjoint pair (1.3.7). A pair $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ of (covariant) functors between the categories $\mathcal{C}$ and $\mathcal{D}$ is an adjoint pair if there is a natural isomorphism φ between the bifunctors $\mathrm{Mor}_{\mathcal{D}}(F(-), -)$ and $\mathrm{Mor}_{\mathcal{C}}(-, G(-))$.

To define such an isomorphism φ, we must exhibit a family of bijections $\varphi_{(C,D)} : \mathrm{Mor}_{\mathcal{D}}(F(C), D) \to \mathrm{Mor}_{\mathcal{C}}(C, G(D))$ which are natural in C and D. We argue in two stages. In the first, we fix C and allow D to vary. After a preliminary lemma to decide when $\varphi_{(C,D)}$ is natural in D, we show that $\varphi_{(C,D)}$ being a bijection for all D corresponds to the existence of an initial object in a suitable right-fibre category.

1.4.5 Lemma

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor, let C be an object of $\mathcal{C}$ and let $F(C)$ be an object of $\mathcal{D}$. Suppose that for each object D in $\mathcal{D}$ there is a mapping of sets

$$\varphi_{(C,D)} : \mathrm{Mor}_{\mathcal{C}}(F(C), D) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(D)).$$

Then the following are equivalent:

(a) *for all $g \in \mathrm{Mor}_{\mathcal{D}}(F(C), D)$,*

$$\varphi_{(C,D)}(g) = G(g) \circ \varphi_{(C,F(C))}(id_{F(C)});$$

(b) *φ is natural in D.*

Proof

(a) $\Rightarrow$ (b). To show that φ_D is natural in D, we have to verify the criterion of (1.3.2)(i), that for any morphism $\delta : D \to D'$ in $\mathcal{D}$, the diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{D}}(F(C), D) & \xrightarrow{\phi_{(C,D)}} & \mathrm{Mor}_{\mathcal{C}}(C, G(D)) \\ \Big\downarrow \delta_* & & \Big\downarrow G(\delta)_* \\ \mathrm{Mor}_{\mathcal{D}}(F(C), D') & \xrightarrow{\phi_{(C,D')}} & \mathrm{Mor}_{\mathcal{C}}(C, G(D')) \end{array}$$

is commutative. Take any $g : F(C) \to D$. Then, combining (a) with the definitions in (1.2.7), we have

$$\begin{aligned} G(\delta)_*(\varphi_{(C,D)}(g)) &= G(\delta) \circ G(g) \circ \varphi_{(C,F(C))}(id_{F(C)}) \\ &= G(\delta g) \circ \varphi_{(C,F(C))}(id_{F(C)}) \\ &= \varphi_{(C,D')}(\delta g) \\ &= \varphi_{(C,D')}(\delta_*(g)). \end{aligned}$$

(b) $\Rightarrow$ (a). In the above diagram, we set $D = F(C)$ and $\delta = g : F(C) \to D$. Then commutativity yields

$$\begin{aligned} \varphi_{(C,D)}(g) &= \varphi_{(C,D)}(g \circ id_{F(C)}) \\ &= \varphi_{(C,D)}(g_*(id_{F(C)})) \\ &= G(g)_*(\varphi_{(C,F(C))}(id_{F(C)})). \end{aligned}$$

□

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1.4.6 Lemma

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor, let C be an object of $\mathcal{C}$ and let $F(C)$ be an object of $\mathcal{D}$, with $u : C \to G(F(C))$ a morphism in $\mathcal{C}$. Then the following assertions are equivalent.

(i) *$(F(C), u)$ is an initial object in the right-fibre category $C\backslash G$.*

(ii) *There is a natural isomorphism of functors*

$$\varphi_{(C,-)} : \mathrm{Mor}_{\mathcal{D}}(F(C), -) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(-))$$

such that $u = \varphi_{(C,F(C))}(id_{F(C)})$.

Proof

We first use the preceding lemma to make explicit the effect of the natural transformation φ occurring in (ii) on any $g \in \mathrm{Mor}_{\mathcal{D}}(F(C), D)$. On the one hand, if (ii) holds, then by the lemma,

$$\varphi_{(C,D)}(g) = G(g) \circ \varphi_{(C,F(C))}(id_{F(C)}) = G(g) \circ u.$$

On the other hand, if we start with u, then for each D in $\mathcal{D}$ we may define a mapping

$$\varphi_{(C,D)} : \mathrm{Mor}_{\mathcal{D}}(F(C), D) \longrightarrow \mathrm{Mor}_{\mathcal{C}}(C, G(D))$$

by the equation

$$\varphi_{(C,D)}(g) = G(g) \circ u.$$

Since then

$$\varphi_{(C,F(C))}(id_{F(C)}) = G(id_{F(C)}) \circ u = u,$$

it follows from the lemma that this mapping is a natural transformation with respect to D. So in either event, it suffices to consider the natural transformation $\varphi_{(C,-)}$ defined by

$$\varphi_{(C,D)}(g) = G(g) \circ u.$$

Then it is easy to see that this natural transformation is an isomorphism if and only if (i) holds, since both properties correspond to the existence of a unique morphism g from $(F(C), u)$ to (D, f) for each object (D, f) of $C\backslash G$, as described by the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\;u\;} & G(F(C)) \\
 & {\scriptstyle f}\searrow & \Big\downarrow{\scriptstyle G(g)} \\
 & & G(D)
\end{array}
\qquad\qquad
\begin{array}{c}
F(C) \\
\Big\downarrow{\scriptstyle \exists!\, g} \\
D
\end{array}
$$

(Here the triangle commutes, and the exclamation mark expresses uniqueness rather than astonishment.) □

We now extend the preceding lemma by showing that the existence of an initial object in $C\backslash G$ for every object C can be neatly reformulated in the language of adjoint functors.

1.4.7 Theorem

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then the following assertions are equivalent.

(i) *For any object C of $\mathcal{C}$ there exists an object $F(C)$ of $\mathcal{D}$ and a morphism $u_C : C \to G(F(C))$ in $\mathcal{C}$ such that $(F(C), u_C)$ is an initial object in the right-fibre category $C\backslash G$.*

(ii) *G has a left adjoint $F : \mathcal{C} \to \mathcal{D}$.*

Proof

(ii) ⇒ (i) follows immediately from the above lemma.

(i) ⇒ (ii): As the notation suggests, the effect of the functor F on an object C of $\mathcal{C}$ is to send it to $F(C)$. We must specify the effect of F on morphisms and verify functoriality.

Let $\gamma : C \to C'$ be a morphism in $\mathcal{C}$. The composite $u_{C'} \circ \gamma$ gives an object $(F(C'), u_{C'} \circ \gamma)$ of the right-fibre category $C\backslash G$, so we may define $F(\gamma) : F(C) \to F(C')$ as the unique morphism from $(F(C), u_C)$ to $(F(C'), u_{C'} \circ \gamma)$ in $C\backslash G$; this definition can be encapsulated by the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\;u_C\;} & G(F(C)) \\
\Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle G(F(\gamma))} \\
C' & \xrightarrow{\;u_{C'}\;} & G(F(C'))
\end{array}
\qquad\qquad
\begin{array}{c}
F(C) \\
\Big\downarrow{\scriptstyle F(\gamma)} \\
F(C')
\end{array}
$$

in which the square is commutative.

Appealing to uniqueness again, we see that $F(id_C) = id_{F(C)}$ and that $F(\gamma'\gamma) = F(\gamma')F(\gamma)$ for a morphism $\gamma' : C' \to C''$. Thus F is a functor.

By the preceding lemma, the map $\varphi_{(C,D)}$ is a bijection for each pair C, D, and φ is natural in D. The remaining point to check is that it is also natural in C.

By (1.3.2)(i), this amounts to checking that, for any morphism $\gamma : C \to C'$ in $\mathcal{C}$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{D}}(F(C), D) & \xrightarrow{\phi_{(C,D)}} & \mathrm{Mor}_{\mathcal{C}}(C, G(D)) \\ \uparrow F(\gamma)^* & & \uparrow \gamma^* \\ \mathrm{Mor}_{\mathcal{D}}(F(C'), D) & \xrightarrow{\phi_{(C',D)}} & \mathrm{Mor}_{\mathcal{C}}(C', G(D)) \end{array}$$

(Note that the vertical arrows point upwards.)

Take any $h \in \mathrm{Mor}_{\mathcal{D}}(F(C'), D)$. Then

$$\begin{aligned} \varphi_{(C,D)}(F(\gamma)^*(h)) &= \varphi_{(C,D)}(h \circ F(\gamma)) \\ &= G(h \circ F(\gamma)) \circ u_C \\ &= G(h) \circ GF(\gamma) \circ u_C, \end{aligned}$$

while

$$\begin{aligned} \gamma^*(\varphi_{(C',D)}(h)) &= \gamma^*(G(h) \circ u_{C'}) \\ &= G(h) \circ u_{C'} \circ \gamma. \end{aligned}$$

However, $u_{C'} \circ \gamma = GF(\gamma) \circ u_C$ by construction; so the result follows. □

1.4.8 Universal constructions

In the light of this theorem we may now make the formal definition that a *universal construction* is a functor which has a right adjoint.

We have already noted in (1.3.8) above that the forgetful functor $\Upsilon : \mathcal{M}\text{od}_R \to \mathcal{S}\text{et}$ has as its left adjoint the functor $\mathrm{Fr}_R : \mathcal{S}\text{et} \to \mathcal{M}\text{od}_R$ which associates to each set X the free right R-module $\mathrm{Fr}_R(X)$ on X. Similarly, the free group and free abelian group constructions defined on $\mathcal{S}\text{et}$ are examples of universal constructions whose right adjoints are the forgetful functors from $\mathcal{G}\text{p}$ and $\mathcal{A}\text{b}$ to $\mathcal{S}\text{et}$.

An extensive catalogue of examples may be found in [Mac Lane 1971] and [Herrlich & Strecker 1979].

1.4.9 Terminal objects

Before we give more examples of universal objects, we mention another manner in which they can arise.

A *terminal object* in a category $\mathcal{C}$ is an object T such that $\mathrm{Mor}(C, T)$ contains exactly one morphism $\tau_C : C \to T$ for each C in $\mathcal{C}$. As with initial objects, a terminal object is unique to within unique isomorphism, if one exists at all.

It sometimes happens that the description of a universal object which satisfies some set of conditions is more naturally given in terms of a terminal object of a convenient ancillary category, rather than an initial object as in (1.4.2) above. We therefore extend the definition of a universal object to include those arising in this way. (Some authors prefer to use the term *couniversal object* for such objects.) Notice that the object T in $\mathcal{C}$ is terminal if and only if the corresponding object T^{op} is initial in the opposite category $\mathcal{C}^{\mathrm{op}}$ (1.1.6), so it is possible to avoid the use of terminal objects by changing the ancillary category. However, it would be rather artificial to do this.

1.4.10 The direct sum revisited

Our next examples of universal objects arise by generalizing the definitions of direct sums and products of modules to arbitrary categories. We draw on some well-known basic definitions and results that are discussed in detail in section 2.1 of [BK: IRM].

The most elementary definition of the direct sum $M = M' \oplus M''$ of right R-modules M' and M'' is in terms of ordered pairs:

$$M = \{(m', m'') \mid m' \in M', \;\; m'' \in M''\}$$

with the expected componentwise addition and scalar multiplication. However, this definition does not lend itself to generalization, so we must recast it in terms of homomorphisms. When $M = M' \oplus M''$, there are homomorphisms

$$\sigma' : M' \longrightarrow M \quad \text{and} \quad \sigma'' : M'' \longrightarrow M$$

given by inclusion on the first and second summands respectively, and homomorphisms

$$\pi' : M \longrightarrow M' \quad \text{and} \quad \pi'' : M \longrightarrow M''$$

given by the corresponding projections.

These homomorphisms satisfy the relations

$$\pi'\sigma' = id_{M'}, \;\; \pi''\sigma'' = id_{M''} \;\text{ and }\; \sigma'\pi' + \sigma''\pi'' = id_M.$$

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Conversely, the existence of such a set of homomorphisms for a module M shows that it is isomorphic to the direct sum of M' and M'' (Proposition 2.1.7 of [BK: IRM]).

A further characterization is in terms of the existence of split short exact sequences [BK: IRM] (2.4.5).

In a category where we can speak about sums of maps or about exact sequences, we can investigate whether there are pairs of morphisms analogous to the projections π', π'' and the inclusions σ', σ''. We pursue this line of attack in the next chapter.

In the most general setting, however, sums of maps and exact sequences are not available to us. It is then more productive to focus on the pair of projections, or alternatively the pair of inclusion maps, from the standpoint of universal objects. The former approach gives rise to the general definition of the product of two objects, while the latter leads to the coproduct. These two generalizations of 'direct sum' need not coincide.

1.4.11 The product

Let $\mathcal{C}$ be any category and let C' and C'' be any two objects of $\mathcal{C}$. Define a category $\mathcal{P}\mathcal{RD}(C', C'')$ as follows. The objects of $\mathcal{P}\mathcal{RD}(C', C'')$ are all the triples (A, α', α'') where $\alpha' : A \to C'$ and $\alpha'' : A \to C''$ are morphisms in $\mathcal{C}$, and a morphism $\xi : (A, \alpha', \alpha'') \to (B, \beta', \beta'')$ is given by a morphism $\xi : A \to B$ in $\mathcal{C}$ so that $\beta'\xi = \alpha'$ and $\beta''\xi = \alpha''$. Thus a morphism is given by a commutative diagram

$$\begin{array}{ccccc} C' & \xleftarrow{\alpha'} & A & \xrightarrow{\alpha''} & C'' \\ \| & & \downarrow{\scriptstyle \xi} & & \| \\ C' & \xleftarrow{\beta'} & B & \xrightarrow{\beta''} & C'' \end{array}$$

A *product* of C' and C'' is a terminal object $(C' \prod C'', \pi', \pi'')$ in the category $\mathcal{P}\mathcal{RD}(C', C'')$, if such exists. It is usual in everyday language to refer to the corresponding universal object $C' \prod C''$ of $\mathcal{C}$ as the product, suppressing the morphisms π' and π''.

In $\mathcal{M}\mathcal{OD}_R$, the direct sum $M' \oplus M''$ is the product in this sense, while in the category $\mathcal{G}\mathcal{P}$ of groups, the product of G' and G'' is the usual cartesian product $G' \times G''$ with componentwise multiplication.

1.4.12 The coproduct

The analogous definition in terms of inclusions leads to the coproduct. Again, let C' and C'' be two fixed objects of a category $\mathcal{C}$. The category $\mathcal{C}_{OPRD}(C', C'')$ has as objects triples (A, α', α'') where now $\alpha' : C' \to A$ and $\alpha'' : C'' \to A$, and a morphism ξ from (A, α', α'') to (B, β', β'') is given by a morphism $\xi : A \to B$ with $\xi\alpha' = \beta'$ and $\xi\alpha'' = \beta''$. A *coproduct* of C' and C'' is an initial object $(C' \coprod C'', \sigma', \sigma'')$ in this category, or alternatively, the corresponding universal object $C' \coprod C''$ in $\mathcal{C}$.

In $\mathcal{M}_{OD_R}$, the direct sum $M' \oplus M''$ is the coproduct of the modules M' and M'' as well as being their product. On the other hand, the coproduct in the category of groups is the *free product* $G' * G''$ of G' and G'', which assertion is easily seen from the explicit description of the free product. The elements of $G' * G''$ are formal products $h_1 \cdots h_k$ with each h_i in either G' or G'', of arbitrary length $k \geq 1$; the product is given by

$$(h_1 \cdots h_k)(h'_1 \cdots h'_{k'}) = h_1 \cdots h_k h'_1 \cdots h'_{k'}$$

and the inverse by

$$(h_1 \cdots h_k)^{-1} = h_k^{-1} \cdots h_1^{-1}.$$

A detailed account of the construction of the free product can be found in, for example, [Scott 1964], Chapter 8.

Underlying the coincidence of the product and coproduct in $\mathcal{M}_{OD_R}$ is the fact that $\operatorname{Hom}(M, N)$ is an abelian group for any two modules M, N. We look at this property in more detail in the next chapter.

1.4.13 Arbitrary products and coproducts

We next outline briefly the definition of products and coproducts for an arbitrary set of objects.

Let Λ be any ordered set and let $C(\Lambda) = \{C_\lambda \mid \lambda \in \lambda\}$ be a set of objects of some category $\mathcal{C}$ which are indexed by Λ. To define the product, we introduce the category $\mathcal{P}_{RD}(C(\Lambda))$ whose objects are sets $\{\alpha_\lambda : A \to C_\lambda\}$ of morphisms in $\mathcal{C}$, a morphism

$$\xi : \{\alpha_\lambda : A \longrightarrow C_\lambda\} \longrightarrow \{\beta_\lambda : B \longrightarrow C_\lambda\}$$

being given by a morphism $\xi : A \to B$ with $\alpha_\lambda = \beta_\lambda \xi$ for all λ.

Then the *product* $\prod_\Lambda C_\lambda$ is the universal object in $\mathcal{C}$ corresponding to a terminal object in $\mathcal{P}_{RD}(C(\Lambda))$.

Dually, we define the category $\mathcal{C}_{OPRD}(C(\Lambda))$ to have objects $\{\alpha_\lambda : C_\lambda \to A\}$

and evident morphisms, and the *coproduct* $\coprod_\Lambda C_\lambda$ to be the universal object of $\mathcal{C}$ given by an initial object in $\mathcal{C}_{OPRD}(C(\Lambda))$.

When the index set Λ is infinite, the product and coproduct differ even in the category $\mathcal{M}_{OD_R}$. Using the definitions ([BK: IRM] (2.1.11)), the reader should have no difficulty verifying that the product of a set of modules $\{M_\lambda\}$ is the direct product $\prod_\Lambda M_\lambda$, while the coproduct is the direct sum $\bigoplus_\Lambda M_\lambda$. (However, the direct sum and product coincide for any finite index set.)

1.4.14 Zero objects

An object 0 of a category $\mathcal{C}$ which is both initial and terminal is said to be a *zero object* of $\mathcal{C}$. In this case, the unique morphisms $0 \to C$ and $C \to 0$ are both denoted 0, so that both $\mathrm{Mor}_\mathcal{C}(0, C) = \{0\}$ and $\mathrm{Mor}_\mathcal{C}(C, 0) = \{0\}$. Clearly, a zero object is unique up to unique isomorphism if it exists.

The zero module is the zero object in the category $\mathcal{M}_{OD_R}$ and the trivial group is the zero object in $\mathcal{G}_{P}$. On the other hand, $\mathcal{R}_{ING}$ has no zero object, since its initial and terminal objects $\mathbb{Z}$ and 0 are different.

When $\mathcal{C}$ has a zero object 0, we can define the zero morphism between any two objects C and D to be the composite

$$0 : C \longrightarrow 0 \longrightarrow D.$$

This morphism is easily seen to be independent of the choice of zero object, and has the property that

$$0\beta = 0 : B \longrightarrow D \quad \text{and} \quad \delta 0 = 0 : C \longrightarrow E$$

for any morphisms $\beta : B \to C$ and $\delta : D \to E$.

1.4.15 Kernels and cokernels

The definition of a kernel and cokernel of a morphism can be extended to any category with a zero object, although at this level of abstraction, a given morphism need have neither a kernel nor cokernel.

Let $\gamma : C \to D$ be a morphism in such a category $\mathcal{C}$, and define a new category $\mathcal{K}_{ER}(\gamma)$ as follows. The objects of $\mathcal{K}_{ER}(\gamma)$ are all the pairs (B, β) where B is an object of $\mathcal{C}$ and $\beta : B \to C$ is a morphism with $\gamma\beta = 0$. A morphism $\xi : (B', \beta') \to (B, \beta)$ is a morphism $\xi : B' \to B$ in $\mathcal{C}$ such that $\beta\xi = \beta'$. The morphisms can be described by commutative diagrams

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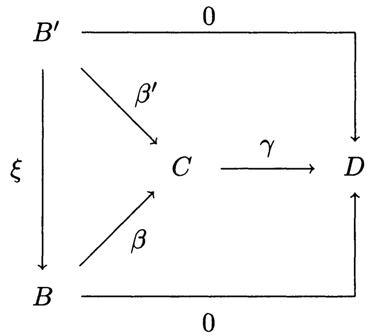


Evidently, $\mathcal{K}er(\gamma)$ is a subcategory of the left-fibre category $Id_{\mathcal{C}}/C$ (see Exercise 1.4.4 below).

A *kernel* of γ is a pair $(\operatorname{Ker}\gamma, \kappa)$ which is terminal in $\mathcal{K}er(\gamma)$, if such an object exists. There is some redundancy in this notation, since $\operatorname{Ker}\gamma$ is the domain of κ. Thus one may validly refer to the morphism κ as the kernel of γ. Alternatively, given the object $\operatorname{Ker}\gamma$, there is often a canonical choice of κ (most commonly, κ is taken to be an inclusion map, which is usually possible when $\mathcal{C}$ is a subcategory of $\mathcal{S}et$). In this case, it is the object $\operatorname{Ker}\gamma$ that is called the kernel of γ.

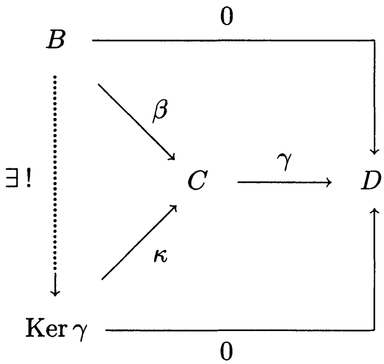
To define the cokernel of γ, we introduce a category $\mathcal{C}ok(\gamma)$ with objects (E, ϵ) where $\epsilon\gamma = 0$, and morphisms $\nu : (E, \epsilon) \to (E'', \epsilon'')$ given by morphisms $\nu : E \to E''$ with $\nu\epsilon = \epsilon''$. A *cokernel* of γ is then an initial object $(\operatorname{Cok}\gamma, \chi)$ in this category, if there is one.

It is easy to check that in $\mathcal{M}od_R$, the usual kernel and cokernel remain such in this abstract sense. Because the maps κ and χ are then always taken to be inclusion and the standard surjection respectively, it is usual to omit them.

Further discussion of the circumstances in which categories may fail to contain kernels and cokernels is given in the next chapter. An immediate (but contrived) example is provided by $\mathcal{I}nf\mathcal{A}b$, the full subcategory of $\mathcal{A}b$ whose objects are the infinite abelian groups. Since $\mathcal{A}b$ is $\mathcal{M}od_{\mathbb{Z}}$ under another name, it has kernels and cokernels as expected. Take A to be the multiplicative group of nonzero real numbers, and let σ be the endomorphism $\sigma(x) = x^2$ of A. Then σ has neither kernel nor cokernel in $\mathcal{I}nf\mathcal{A}b$.

1.4.16 Universal properties

At this point we introduce a third term commonly found in the literature, along with 'universal object' and 'universal construction'. A *universal property* is the property that characterizes a universal object; it is frequently described by means of a diagram. Such a diagram typically highlights the morphism whose existence and uniqueness determines the universal object. For example, one may say that the kernel of a morphism $\gamma : C \to D$ enjoys the universal property



that makes the above diagram commutative. Here, the existence and uniqueness of the vertical arrow from B to $\operatorname{Ker}\gamma$ which makes the above diagram commutative expresses the fact that the pair $(\operatorname{Ker}\gamma, \kappa)$ is terminal in the category $\mathcal{K}_{\mathcal{ER}}(\gamma)$.

1.4.17 Pleasure versus guilt: the universal dilemma

Being natural, functors have their emotional side too. Typically, as we have just seen, a universal property is pleasantly depicted by a commuting diagram (leading to the existence of a unique morphism). It is said that the universal object 'enjoys' the universal property. The existence of an adjoint functor, on the other hand, often involves a tortuous check; a functor is said to 'admit' an adjoint. See Exercise 1.4.8 for an example of this phenomenon.

1.4.18 Kernels of natural transformations

We can also define kernels, cokernels, and so on, in the functor category $[\mathcal{C}, \mathcal{D}]$ provided they exist in $\mathcal{D}$. Suppose, for example, that any morphism α in $\mathcal{D}$ has a kernel $(\operatorname{Ker}\alpha, \kappa(\alpha))$. We construct for each $\eta \in \operatorname{Nat}(F, G)$ a kernel $(\operatorname{Ker}\eta, \kappa)$ as follows.

The object $\operatorname{Ker}\eta$ in $[\mathcal{C}, \mathcal{D}]$ is to be a functor from $\mathcal{C}$ to $\mathcal{D}$. For each object C of $\mathcal{C}$ we take the evident definition

$$(\operatorname{Ker}\eta)(C) = \operatorname{Ker}\eta_C.$$

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(Here, $\operatorname{Ker}\eta_C$ is any fixed choice from among the mutually isomorphic objects in $\mathcal{D}$ which represent the kernel of η_C.)

To define $(\operatorname{Ker}\eta)(\alpha)$ for a morphism $\alpha : C' \to C$, we first recall that the diagram

$$\begin{array}{ccc} FC' & \xrightarrow{\eta_{C'}} & GC' \\ {\scriptstyle F\alpha}\big\downarrow & & \big\downarrow{\scriptstyle G\alpha} \\ FC & \xrightarrow{\eta_C} & GC \end{array}$$

is commutative. Thus the definition of a kernel as a terminal object (1.4.15) shows that there is an induced morphism

$$(\operatorname{Ker}\eta)(\alpha) : (\operatorname{Ker}\eta)(C') \longrightarrow (\operatorname{Ker}\eta)(C)$$

which fits into the commutative diagram

$$\begin{array}{ccccc} (\operatorname{Ker}\eta)(C') & \xrightarrow{\kappa(\eta_{C'})} & FC' & \xrightarrow{\eta_{C'}} & GC' \\ {\scriptstyle (\operatorname{Ker}\eta)(\alpha)}\big\downarrow & & {\scriptstyle F\alpha}\big\downarrow & & \big\downarrow{\scriptstyle G\alpha} \\ (\operatorname{Ker}\eta)(C) & \xrightarrow{\kappa(\eta_C)} & FC & \xrightarrow{\eta_C} & GC \end{array}$$

It is now easy to confirm that, with the above choice of $(\operatorname{Ker}\eta)(\alpha)$, $\operatorname{Ker}\eta$ is indeed a functor.

The natural transformation

$$\kappa : \operatorname{Ker}\eta \longrightarrow F$$

is given by $\kappa_C = \kappa(\eta_C) : (\operatorname{Ker}\eta)(C) \to F(C)$. A routine verification shows that we have a kernel for η, as claimed.

1.4.19 Some history

A nice description of the origin of category theory is given in Chapter 1, §5E of [Dieudonné 1989], which we summarize. In the early 1940s, Eilenberg and Mac Lane were seeking to make precise the idea of a 'natural' transformation between various constructions in algebraic topology. Typically, such constructions start with a topological space and yield a group (fundamental group, homology group, etc.). The authors sought a framework for saying that a map of topological spaces gives rise to a homomorphism of groups in a well-regulated manner. In [Eilenberg & Mac Lane 1942], they achieved this aim by introducing the notion of a 'functor', as they called a function F that not only operates on a set C (with extra structure) to give a new set $F(C)$, but also on a (structure-preserving) function $\gamma : C \to D$ to give a function $F(\gamma) : F(C) \to F(D)$. Then the idea of a natural transformation can be made precise, as in (1.3).

Subsequently, in [Eilenberg & Mac Lane 1945], the concept of a category was introduced to formalize the viewpoint that objects that share a common structure should not be considered in isolation, but together with the mappings that respect this structure. Once this level of abstraction had been attained, it became possible to introduce categories, such as opposite categories, morphism categories, etc., whose objects are no longer 'sets-with-structure'. In turn, the freedom to construct categories led to the characterization of universal constructions as initial or terminal objects in a suitably manufactured category. The notion of 'universality' first appears in [Samuel 1948], while the description of universal constructions in terms of adjoint functors (1.4.8) is due to [Kan 1958].

Exercises

1.4.1 Let $\mathcal{D}$ be the category with one object X and one morphism f other than the identity id_X, subject to $f \circ f = f$, say. Let $\mathcal{C}$ be the subcategory obtained by removing the morphism f, and let $G : \mathcal{D} \to \mathcal{C}$ be the constant functor (in fact, the only possible functor). Show that the right-fibre category $X \backslash G$ is isomorphic to $\mathcal{D}$. Evidently $\mathcal{D}$ lacks an initial object, so the same is true of $X \backslash G$.

The reader is invited to construct other right-fibre categories which lack an initial object. However, initial objects are surprisingly common in any choices of right-fibre category that one is likely to make.

1.4.2 Show that if (K, κ) is the kernel of a morphism γ in a category $\mathcal{C}$, then $(K^{\mathrm{op}}, \kappa^{\mathrm{op}})$ is the cokernel of γ^{op} in $\mathcal{C}^{\mathrm{op}}$ (and conversely).

Show also that if C, D are objects of $\mathcal{C}$ with product $C \prod D$ in $\mathcal{C}$, then $(C \prod D)^{\mathrm{op}} = C^{\mathrm{op}} \coprod D^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$ (and conversely).

1.4.3 Let R and S be rings. Show that the direct product $R \times S$ is a product in $\mathcal{R}_{\mathcal{ING}}$ but not a coproduct.

1.4.4 Sometimes one has to be quite ingenious to present a universal construction as an initial or terminal object in a right- or left-fibre category. To discuss kernels and cokernels in this light, let $\gamma : C \to D$ be a morphism in a category $\mathcal{C}$ which has zero object 0. As noted, the category $\mathcal{K}_{\mathcal{ER}}\,\gamma$ is a subcategory of the left-fibre category $Id_{\mathcal{C}}/C = \mathcal{C}/C$.

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Define a functor $\gamma_* : \mathcal{C}/C \to \mathcal{C}/D$ by $\gamma_*(B,\beta) = (B,\gamma\beta)$ and, for any morphism $\xi : (B',\beta') \to (B,\beta)$ in $\mathcal{C}/C$ (that is, $\beta\xi = \beta'$),

$$\gamma_*\xi = \xi : (B',\gamma\beta') \longrightarrow (B,\gamma\beta).$$

Show that the left-fibre category $\gamma_*/(0 \to D)$, where $(0 \to D)$ is the initial object of the category $\mathcal{C}/D$, has as objects commuting squares of the form

$$\begin{array}{ccc} B & \longrightarrow & C \\ \Big\downarrow & & \Big\downarrow \gamma \\ 0 & \longrightarrow & D \end{array}$$

Deduce that $\mathcal{K}_{\mathcal{ER}}\,\gamma$ is isomorphic to $\gamma_*/(0 \to D)$.

Provide the dual description of $\mathcal{C}_{\mathcal{OK}}\,\gamma$.

1.4.5 What is the universal property enjoyed by the abelianization G_{ab} of a group G?

1.4.6 Let $\gamma : G \to F$ be a homomorphism of groups. Show that the usual definition of $\operatorname{Ker}\gamma$ does give a kernel in the category $\mathcal{G}_{\mathcal{P}}$.

Suppose that H is a normal subgroup of G, and let η be the inclusion homomorphism. Show that $\operatorname{Cok}\eta = G/H$.

Let H be an arbitrary subgroup of G. Then the *normal closure* H^G of H is the smallest normal subgroup of G which contains H. Verify that the inclusion homomorphism $\eta : H \to G$ has $\operatorname{Cok}\eta = G/H^G$.

Deduce that for an arbitrary homomorphism $\eta : H \to G$ in $\mathcal{G}_{\mathcal{P}}$, $\operatorname{Cok}\eta = G/(\operatorname{Im}\eta)^G$.

1.4.7 In the category $\mathcal{R}_{\mathcal{NG}}$ of nonunital rings (that is, 'rings' which need not have an identity element and homomorphisms which need not preserve it), show that the zero ring 0 is a zero object.

Let $\theta : R \to S$ be a homomorphism of nonunital rings. Show that the usual kernel $\operatorname{Ker}\theta$ is a kernel in $\mathcal{R}_{\mathcal{NG}}$, and that $\operatorname{Ker}\theta$ is an ideal of R (where we extend the definition of ideals to nonunital rings in the obvious way).

Show that if I is an ideal of S, then there is a residue nonunital ring R/I which is a cokernel for the inclusion homomorphism of I in R.

Given a sub-nonunital ring S' of S, let $I(S')$ be the smallest ideal of S which contains S'. Show that $\operatorname{Cok}\theta = S/I(\operatorname{Im}\theta)$.

1.4.8 Consider the functor from $\mathcal{R}_{\mathcal{NG}}$ to $\mathcal{R}_{\mathcal{ING}}$ which takes a nonunital ring R to its enveloping ring $\overline{R}$ (1.3.2)(iv). Show that its right adjoint is the forgetful functor from $\mathcal{R}_{ING}$ to $\mathcal{R}_{NG}$ which simply ignores the multiplicative identity of a ring.

To describe the associated universal property, let $f : R \to S$ be a nonunital ring homomorphism. Show that if S is a ring, then f factors uniquely through the standard embedding of R in $\overline{R}$. Diagrammatically, we have:

$$\begin{array}{ccc} R & & \\ \downarrow & \searrow^{f} & \\ \overline{R} & \xrightarrow[\exists !]{} & S \end{array}$$

1.4.9 Sets

Show that the empty set $\emptyset$ is an initial object in $\mathcal{S}_{ET}$ (note that $\mathrm{Map}(\emptyset, X)$ contains $|X|^0$ members since $0 = |\emptyset|$). Show also that $\mathcal{S}_{ET}$ has no terminal object.

Prove that the product (in $\mathcal{S}_{ET}$) of a set $\{X_\lambda \mid \lambda \in \Lambda\}$ of sets is the cartesian product $\prod_\Lambda X_\lambda$.

Define the *disjoint union* $\bigsqcup_\Lambda X_\lambda$ to be the union $\bigcup_\Lambda (X_\lambda, \lambda)$, where (X_λ, λ) is the set of pairs (x_λ, λ) with $x_\lambda \in X_\lambda$. (The purpose of this construction is to remove any 'accidental' overlap between the sets X_λ; note that $(X_\lambda, \lambda) \cap (X_\mu, \mu) = \emptyset$ for $\lambda \neq \mu$, even if X_λ and X_μ have common members.)

Show that the coproduct of the set of sets $\{X_\lambda\}$ is $\bigsqcup_\Lambda X_\lambda$.

1.4.10 Push-outs

This and the following exercises indicate how the definitions of push-outs and pull-backs can be generalized from modules to arbitrary categories. The constructions for modules are given in detail in [BK: IRM] (2.4.8)ff.

Given a diagram

$$\begin{array}{ccc} L' & \xrightarrow{\mu} & L \\ \Big\downarrow{\scriptstyle \phi} & & \\ M' & & \end{array}$$

of right R-modules, the push-out (of M' and L over L') is

$$M' \oplus_{L'} L = (M' \oplus L)/\{(\phi\ell', -m\ell') \mid \ell' \in L'\}.$$

Furthermore, there are homomorphisms $\overline{\mu} : M' \to M' \oplus_{L'} L$ and $\overline{\phi} : L \to M' \oplus_{L'} L$ which give a commutative diagram

$$\begin{array}{ccc} L' & \xrightarrow{\mu} & L \\ \downarrow \phi & & \downarrow \overline{\phi} \\ M' & \xrightarrow{\overline{\mu}} & M' \oplus_{L'} L \end{array}$$

and if

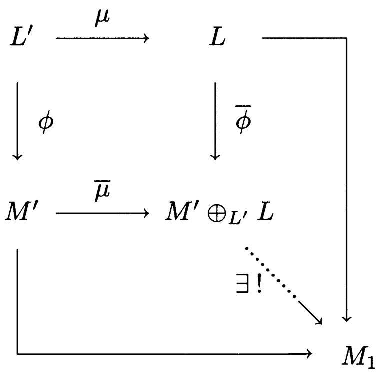
$$\begin{array}{ccc} L' & \xrightarrow{\mu} & L \\ \downarrow \phi & & \downarrow \phi_1 \\ M' & \xrightarrow{\mu_1} & M_1 \end{array}$$

is a commutative diagram, then there is a unique homomorphism

$$\xi : M' \oplus_{L'} L \longrightarrow M_1$$

with $\xi\overline{\mu} = \mu_1$ and $\xi\overline{\phi} = \phi_1$.

Thus the universal property may be described by the diagram



Show that the push-out arises through an initial object in an appropriate category, and hence extend the definition of a push-out to an arbitrary category $\mathcal{C}$.

Prove further that a push-out in $\mathcal{C}$ may be regarded as a coproduct in a suitable right-fibre category, and conversely, that if $\mathcal{C}$ has an initial object, then a coproduct is a special case of a push-out.

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1.4.11 **Pull-backs**

Suppose that, in a category $\mathcal{C}$, we are given a diagram of the form

$$\begin{array}{ccc} & & L'' \\ & & \downarrow \phi \\ M & \xrightarrow{\beta} & M'' \end{array}$$

where β and θ are morphisms.

Arguing by duality, define the *pull-back* $M \times_{M''} L''$ of M and L'' over M'' in terms of a universal property.

Confirm that in the category of right R-modules, the pull-back is given explicitly (as in (2.4.8) of [BK: IRM]) by the formula

$$M \times_{M''} L'' = \{(m, \ell'') \in M \oplus L'' \mid \beta m = \theta \ell''\}.$$

Show also that if $\mathcal{C}$ has a terminal object, then a product is a special case of a pull-back.

1.4.12 Determine the push-outs and pull-backs in the category $\mathcal{S}\mathrm{ET}$ (see Exercise 1.4.9 above).

1.4.13 Let $\beta : M \to M''$ and $\delta : M'' \to N''$ be morphisms in a category with a zero object, and let $\alpha : N' \to M''$ be the kernel of δ. By comparing universal properties, show that, whenever the pull-back exists, the canonical morphism $\overline{\alpha} : M \times_{M''} N' \to M$ is the kernel of $\delta\beta : M \to N''$. (See Exercise 2.4.7 of [BK: IRM] for a discussion of this situation in $\mathcal{M}\mathrm{OD}_R$.)

1.4.14 For this, one needs some routine facts about products of groups. These can be found in [Scott 1964] Chapter 8, for example.

Show that the pull-back in $\mathcal{G}\mathrm{P}$ is the restricted direct product and that the push-out is the amalgamated free product.

Remark. There are corresponding but more complicated results for rings, where it is usually preferable to work in a category of *A-algebras*: for an arbitrary commutative ring A, an A-algebra is a ring R that has A contained in its centre. Some details can be found in [Rowen 1988], §§1.4, 1.9, and [Eisenbud 1995] A6.3.

1.4.15 Let $\mathcal{C}\mathrm{AT}$ be a category of small categories. Show that for categories $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{C}\mathrm{AT}$, their direct product $\mathcal{C} \times \mathcal{D}$ is a product in $\mathcal{C}\mathrm{AT}$ in the sense of (1.4.11).