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## ABSTRACT

The idea that the cohomology of finite groups might be fruitfully approached via the cohomology of ambient semisimple algebraic groups was first shown to be viable in the papers [E. Cline, B. Parshall, and L. Scott, *Cohomology of finite groups of Lie type, I*, Publ. Math. Inst. Hautes Études Sci. **45** (1975), 169–191] and [E. Cline, B. Parshall, L. Scott and W. van der Kallen, *Rational and generic cohomology*, Invent. Math. **39** (1977), 143–163]. The second paper introduced, through a limiting process, the notion of generic cohomology, as an intermediary between finite Chevalley group and algebraic group cohomology. The present paper shows that, for irreducible modules as coefficients, the limits can be eliminated in all but finitely many cases. These exceptional cases depend only on the root system and cohomological degree. In fact, we show that, for sufficiently large  $r$ , depending only on the root system and  $m$ , and not on the prime  $p$  or the irreducible module  $L$ , there are isomorphisms  $H^m(G(p^r), L) \cong H^m(G(p^r), L') \cong H_{\text{gen}}^m(G, L') \cong H^m(G, L')$ , where the subscript ‘gen’ refers to generic cohomology and  $L'$  is a constructibly determined irreducible ‘shift’ of the (arbitrary) irreducible module  $L$  for the finite Chevalley group  $G(p^r)$ . By a famous theorem of Steinberg, both  $L$  and  $L'$  extend to irreducible modules for the ambient algebraic group  $G$  with  $p^r$ -restricted highest weights. This leads to the notion of a module or weight being ‘shifted  $m$ -generic’, and thus to the title of this paper. Our approach is based on questions raised by the third author in [D. I. Stewart, *The second cohomology of simple  $SL_3$ -modules*, Comm. Algebra **40** (2012), 4702–4716], which we answer here in the cohomology cases. We obtain many additional results, often with formulations in the more general context of  $\text{Ext}_{G(p^r)}^m$  with irreducible coefficients.

## 1. Introduction

Let  $G$  be a simply connected, semisimple algebraic group defined and split over the prime field  $\mathbb{F}_p$  of positive characteristic  $p$ . Write  $k = \bar{\mathbb{F}}_p$ . For a power  $q = p^r$ , let  $G(q)$  be the subgroup of  $\mathbb{F}_q$ -rational points in  $G$ . Thus,  $G(q)$  is a finite Chevalley group. Let  $M$  be a finite-dimensional rational  $G$ -module and let  $m$  be a nonnegative integer. In [CPSvdK77], the first two authors of this paper, together with Cline and van der Kallen, defined the notion of the generic  $m$ -cohomology

$$H_{\text{gen}}^m(G, M) := \varprojlim_q H^m(G(q), M)$$

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of  $M$ . The limit is, in fact, a *stable* limit for any given  $M$ . Moreover,  $H_{\text{gen}}^m(G, M) \cong H^m(G, M^{[e_0]})$ , where  $M^{[e_0]}$  denotes the twist of  $M$  through some  $e_0$ th power of the Frobenius endomorphism of  $G$ . Although the nonnegative integer  $e_0$  may be chosen independently of  $p$  and  $M$ , it can also be chosen as a function  $e_0(M)$  of  $M$ . Unfortunately, given a rational  $G$ -module  $M$  for which one wants to compute  $H^m(G(q), M)$ , it is frequently necessary to take  $e_0(M) > 0$ . This problem and related computational issues have been noted by others; see [FFSS99, p. 664] and [UGA12, §1]. Worse, it may be necessary to enlarge  $q$  in order to obtain  $H^m(G(q), M) \cong H_{\text{gen}}^m(G, M)$ . The problem is exacerbated if one is interested in calculations for an infinite family of modules  $M$ , such as the irreducible  $G$ -modules. By a famous result of Steinberg, all irreducible  $kG(q)$ -modules are, up to isomorphism, the restrictions to  $G(q)$  of the irreducible rational  $G$ -modules whose highest weights are  $q$ -restricted.

We propose here a remedy to this situation. Observe that, for any  $q$ -restricted dominant weight  $\lambda$  and nonnegative integer  $e$ , there is a unique  $q$ -restricted dominant weight  $\lambda'$  with  $L(\lambda)^{[e]}|_{G(q)} = L(\lambda')|_{G(q)}$ . Write  $\lambda' = \lambda^{[e]_q}$  and  $L(\lambda') = L(\lambda)^{[e]_q}$ . We shall refer to any weight  $\lambda'$  of this form as a  *$q$ -shift* of  $\lambda$ . The main result, Theorem 5.8, in this paper shows that, for  $r \gg 0$ ,<sup>1</sup> and any  $q$ -restricted dominant weight  $\lambda$ ,

$$H^m(G(q), L(\lambda)) \cong H_{\text{gen}}^m(G, L(\lambda')) \cong H^m(G, L(\lambda')), \tag{1.0.1}$$

for some  $q$ -restricted weight  $\lambda' = \lambda^{[e]_q}$  with  $e = e(\lambda) = e(\lambda, q) \geq 0$ . Similar results hold for  $\text{Ext}_{G(q)}^m(L(\mu), L(\lambda))$  with  $\lambda, \mu$  both  $q$ -restricted, though with some conditions on  $\mu$ . The first isomorphism in (1.0.1) may be viewed as saying that  $L(\lambda)$  is ‘shifted  $m$ -generic at  $q$ ’; see the end of this introduction.<sup>2</sup> The map  $\lambda \mapsto \lambda^{[e]_q}$  defines an action of the cyclic group  $\mathbb{Z}/r\mathbb{Z}$  on the set  $X_r^+$  of  $q$ -restricted weights, and  $\lambda'$  in (1.0.1) is a ‘distinguished’ member in the orbit of  $\lambda$  under this action, chosen to optimize the positions of zero terms in its  $p$ -adic expansion.

The origin of these results goes back to  $\text{Ext}^m$ -questions raised by the third author in [Ste12a, §3], where the  $q$ -shift  $\lambda^{[e]_q}$  of  $\lambda$  was denoted  $\lambda^{\{e\}}$ , and  $L(\lambda^{\{e\}})$  was called a  $q$ -wrap of  $L(\lambda)$ . While raised for general  $m \geq 0$ , these questions arose in part from observations for  $m = 1, 2$ , namely, from noting a parallel between the 2-cohomology result [Ste12a, Theorem 2] and a 1-cohomology result in [BNP06, Theorem 5.5], which also had an  $\text{Ext}^1$ -analog [BNP06, Theorem 5.6]. The conclusions of all these results involve what we now call  $q$ -shifted weights.<sup>3</sup> Essentially, our main Theorem 5.8 provides a strong answer to [Ste12a, Question 3.8] in the cohomology cases, in addition to interpreting it in terms of generic cohomology. Also, Theorem 6.2(c) proves a similar result for  $\text{Ext}_{G(q)}^m(L(\mu), L(\lambda))$  when  $p$  is sufficiently large, and with no requirement on  $r$ , but with  $\lambda$  and  $\mu$  required to have a zero digit in common (i.e.,  $\lambda_i = \mu_i = 0$  for some  $i < r$ , using the terminology below). Remark 5.9 gives an example showing this result is near best possible, especially when  $\lambda = \mu$ , and that the original [Ste12a, Question 3.8] must be reformulated. Such a reformulation is given in Question 5.10.

Our investigation yields many other useful results. We mention a few. First, any dominant weight  $\lambda$  has a  $p$ -adic expansion  $\lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + \dots$ , where each  $\lambda_i$  is  $p$ -restricted. We call the pairs  $(i, \lambda_i)$  *digits* of  $\lambda$ , and we say a digit is 0 if  $\lambda_i = 0$ . Theorem 5.4 states that, given  $m \geq 0$ , there is an integer  $d$ , depending only on the root system  $\Phi$  and  $m$ , such that, for any

<sup>1</sup> The lower bound on  $r$  here depends only on the root system  $\Phi$  of  $G$  and the cohomological degree  $m$ , and not on  $p$  or  $\lambda$ . Moreover, this bound can be recursively determined.

<sup>2</sup> We ask in Question 5.10 below if the  $\text{Ext}$ -analog of the first isomorphism holds for all  $q$ -restricted  $\lambda$  and  $\mu$ , though we know conditions on  $\mu$  are needed for the second.

<sup>3</sup> As far as we know, [BNP06, Theorem 5.6] is the first use of  $q$ -shifted weights in a general homological theorem. However, this shifting (or wrapping) behavior for  $\text{SL}_2$  had been observed much earlier: see [AJL83, Corollary 4.5].

prime  $p$ , any power  $q = p^r$ , and any pair of  $q$ -restricted weights  $\lambda, \mu$ , if  $\text{Ext}_{G(q)}^m(L(\mu), L(\lambda)) \neq 0$ , then  $\lambda$  and  $\mu$  differ in at most  $d$  of their respective digits. Thus, in the cohomology case, if  $H^m(G(q), L(\lambda)) \neq 0$ , then  $\lambda$  has at most  $d$  nonzero digits. Versions of these results hold for both rational  $G$ -cohomology and  $\text{Ext}^m$ -groups; see Theorem 5.2. These digit-bounding results were inspired by questions in [Ste12a, Question 3.10], which we answer completely.

Second, combining the main Theorem 5.8 with the large prime cohomology results [BNP01, Theorem 7.5] gives a new proof<sup>4</sup> that there is a bound on  $\dim H^m(G(q), L(\lambda))$ , for  $q$ -restricted  $\lambda$ , depending only on  $\Phi$  and  $m$ , and not on  $p$  or  $r$ . In fact, after throwing away finitely many values of  $q$ , Theorem 5.12 shows that  $\dim H^m(G(q), L(\lambda))$  is bounded by the maximum dimension of the spaces  $H^m(G, L(\mu))$ , with  $p$  and  $\mu \in X^+$  allowed to vary (with only  $m$  and  $\Phi$  fixed). The latter maximum has been shown to be finite in [PS11, Theorem 7.1]. Indeed, apart from finitely many exceptional  $q$ , the finite group cohomology  $H^m(G(q), L(\lambda))$  identifies with a rational cohomology group  $H^m(G, L(\mu))$ , for an explicitly determined dominant weight  $\mu$  (which depends on  $\lambda$ ).

Though the main focus of this paper is on results which hold for all primes  $p$ , we collect several results in § 6, most formulated in the  $\text{Ext}_{G(q)}^m$ -context, which are valid in the special case when  $p$  is modestly large. One such result is Theorem 6.2(c) discussed above. This theorem, given in a ‘shifted generic’ framework, leads also to a fairly definitive treatment of generic cohomology for large primes in Theorem 6.5 and the Appendix A.

A key ingredient in this work is the elegant filtration, due to Bendel, Nakano and Pillen, of the induced module  $\mathcal{G}_r(k) := \text{ind}_{G(q)}^G k$ ; see [BNP11] and the other references at the start of § 4. This result is, in our view, the centerpiece of a large collection of results and ideas of these authors, focused on using the induction functor  $\text{ind}_{G(q)}^G$  in concert with truncation to smaller categories of rational  $G$ -modules. The filtration of  $\mathcal{G}_r(k)$  is described in Theorem 4.2 below, and we derive some consequences of it in § 4.

Also, the specific theorems and ideas establishing generic cohomology, as originally formulated in [CPSvdK77], play an important role in § 5, both directly and as a background motivation for exploring digit bounding.

Finally, to explain the title of this paper, a finite-dimensional, rational  $G$ -module  $M$  may be called ‘ $m$ -generic at  $q$ ’ if  $H^m(G(q), M) \cong H_{\text{gen}}^m(G, M)$ .<sup>5</sup> A natural generalization of this notion is to say that  $M$  is ‘shifted’  $m$ -generic at  $q$  if there exists a module  $M'$  which is  $m$ -generic at  $q$  and such that  $M'|_{G(q)} \cong M^{[e]}|_{G(q)}$  for some  $e \geq 0$ . Thus,  $H^m(G(q), M) \cong H^m(G(q), M') \cong H_{\text{gen}}^m(G, M')$ . Our paper shows that many modules may be fruitfully regarded as shifted  $m$ -generic at  $q$ , when it is unreasonable or false that they are  $m$ -generic at  $q$ . The digit-bounding results discussed above, which mesh especially well with the generic cohomology theory, provide the main tool for finding such modules in nontrivial cases, and this is the strategy for the proof of Theorem 5.8. In fact, Theorem 5.8 shows that often one can obtain the additional isomorphism  $H_{\text{gen}}^m(G, M') \cong H^m(G, M')$ , an attractive property for computations.

## 2. Some preliminaries

Fix an irreducible root system  $\Phi$  with positive (respectively, simple) roots  $\Phi^+$  (respectively,  $\Pi$ ) selected.<sup>6</sup> Let  $\alpha_0 \in \Phi^+$  be the maximal short root, and let  $h = (\rho, \alpha_0^\vee) + 1$  be the Coxeter number

<sup>4</sup> The first proof that such a bound exists is in a joint paper [BNPPSS12] of the present three authors together with Bendel, Nakano and Pillen.

<sup>5</sup> In practice, this often happens when the stable limit defining the generic cohomology has already occurred at  $q$ ; however, we do not make this part of the definition.

<sup>6</sup> The assumption that  $\Phi$  is irreducible is largely a convenience. The reader can easily extend to the general case, i.e., when the group  $G$  below is only assumed to be semisimple.

of  $\Phi$  (where  $\rho$  is the half sum of the positive roots). Write  $X$  for the full weight lattice of  $\Phi$ , and let  $X^+ \subset X$  be the set of dominant weights determined by  $\Pi$ .

Now fix a prime  $p$ . For a positive integer  $b$ , let  $X_b^+ := \{\lambda \in X^+ \mid (\lambda, \alpha^\vee) < p^b, \forall \alpha \in \Pi\}$  be the set of  $p^b$ -restricted dominant weights. At times it is useful to regard the 0 weight as (the only)  $p^0$ -restricted dominant weight.

Let  $G$  be a simple, simply connected algebraic group, defined and split over a prime field  $\mathbb{F}_p$  and having root system  $\Phi$ . Fix a maximal split torus  $T$ , and let  $B \supset T$  be the Borel subgroup determined the negative roots  $-\Phi^+$ . For  $\lambda \in X^+$ ,  $L(\lambda)$  denotes the irreducible rational  $G$ -module of highest weight  $\lambda$ . If  $F : G \rightarrow G$  is the Frobenius morphism, then, for any positive integer  $b$ , let  $G_b = \text{Ker}(F^b)$  be the (scheme theoretic) kernel of  $F^b$ . Thus,  $G_b$  is a normal, closed (infinitesimal) subgroup of  $G$ . Similar notations are used for other closed subgroups of  $G$ .

The representation and cohomology theory for linear algebraic groups (especially semisimple groups and their important subgroups) is extensively developed in Jantzen’s book [Jan03], with which we assume the reader is familiar. We generally follow his notation (with some small modifications). The reader should keep in mind that  $\text{Ext}_G^m(L(\lambda), L(\mu)) \cong \text{Ext}_G^m(L(\mu), L(\lambda))$  and a similar statement holds for  $G(q)$ . Often we write the  $L(\mu)$  on the left, because  $\mu$  sometimes plays a special role (with restrictions of some kind), and taking  $\mu = 0$  gives  $H^m(G, L(\lambda))$ . But we are not always consistent, as in some places where it is more convenient to have  $L(\mu)$  on the right. If  $M$  is a rational  $G$ -module and  $b$  is a nonnegative integer, write  $M^{[b]}$  for the rational  $G$ -module obtained by making  $g \in G$  act through  $F^b(g)$  on  $M$ . If  $M$  already has the form  $M \cong N^{[r]}$  for some  $r \geq b$ , write  $M^{[-b]} := N^{[r-b]}$ .<sup>7</sup>

Let  $\text{ind}_B^G$  be the induction functor from the category of rational  $B$ -modules to rational  $G$ -modules. (See §4 for a brief discussion of induction in general.) Given  $\lambda \in X$ , we denote the corresponding one-dimensional rational  $B$ -module also by  $\lambda$ , and write  $H^0(\lambda)$  for  $\text{ind}_B^G \lambda$ . Then  $H^0(\lambda) \neq 0$  if and only if  $\lambda \in X^+$ ; when  $\lambda \in X^+$ ,  $H^0(\lambda)$  has irreducible socle  $L(\lambda)$  of highest weight  $\lambda$ , and formal character  $\text{ch } H^0(\lambda)$  given by Weyl’s character formula at the dominant weight  $\lambda$ . In most circumstances, especially when regarding  $H^0(\lambda)$  as a co-standard (i.e., a dual Weyl) module in the highest weight category of rational  $G$ -modules, we denote  $H^0(\lambda)$  by  $\nabla(\lambda)$ . Given  $\lambda \in X$ , let  $\lambda^* := -w_0(\lambda)$ , where  $w_0$  is the longest element in the Weyl group  $W$  of  $\Phi$ . If  $\lambda \in X^+$ , then  $\lambda^* \in X^+$  is just the image of  $\lambda$  under the opposition involution. For  $\lambda \in X^+$ , put  $\Delta(\lambda) = \nabla(\lambda^*)^*$ , the dual of  $\nabla(\lambda^*)$ . In other words,  $\Delta(\lambda)$  is the Weyl module for  $G$  of highest weight  $\lambda$ . Of course,  $L(\lambda)^* = L(\lambda^*)$ .

For  $i \geq 0$ , let  $R^i \text{ind}_B^G$  be the  $i$ th derived functor of  $\text{ind}_B^G$ . Then  $R^i \text{ind}_B^G = 0$  for  $i > |\Phi^+|$ .

We will need another notion of the magnitude of a weight. If  $b$  is a nonnegative integer,  $\lambda \in X$  is called *b-small* if  $|(\lambda, \alpha^\vee)| \leq b$  for all  $\alpha \in \Phi^+$ . If  $\lambda \in X^+$ ,  $\lambda$  is *b-small* if and only if  $(\lambda, \alpha_0^\vee) \leq b$ . We say a (rational)  $G$ -module is *b-small* provided all of its weights are *b-small*. Equivalently, it is *b-small* provided its maximal weights (in the dominance order) are *b-small*. In particular,

<sup>7</sup> For more discussion of the Frobenius twist of a representation, see [Jan03, I.9]. Our notation  $V^{[r]}$  agrees with that in [Jan03], which also discusses another Frobenius twist  $V^{(r)}$  defined by modifying the action of the base field on the vector space  $V$ . When  $V$  is defined over the prime field  $\mathbb{F}_p$  (in the sense of algebraic groups, using a prime field comodule structure for a prime field coordinate algebra of  $G$ ), the two twisted modules are isomorphic. The use of the modules  $V^{(r)}$  leads to semilinear maps, which can be avoided using the modules  $V^{[r]}$ . (In this way, the semilinear maps in [CPSvdK77] can be avoided.) The ‘twisting’ maps  $H^\bullet(G, V) \rightarrow H^\bullet(G, V^{[r]})$ , used below in §5, are just the inflation maps through  $F^r$ ; see also [CPS83, §1], where the inflation map is shown to agree with the abutment map in the Hochschild–Serre spectral sequence  $E_2^{s,t} = H^s(G/G_r, H^t(G_r, V^{[r]})) \Rightarrow H^{s+t}(G, V^{[r]})$ . In particular, these maps are linear. We thank the referee for suggesting that the authors clarify this point.

if  $\lambda \in X^+$  is  $b$ -small, then any highest weight module  $M$  with highest weight  $\lambda$ , e.g.,  $L(\lambda)$ ,  $\nabla(\lambda)$ , or  $\Delta(\lambda)$ , is also  $b$ -small. We make some elementary remarks about small-ness.

LEMMA 2.1. *Let  $\nu$  be any dominant weight and let  $b, b', r, u$  be nonnegative integers.*

(a) *If  $b > 0$ , assume  $u \geq [\log_p b] + 1$ , where  $[ \ ]$  denotes the greatest integer function. Then  $b \leq p^u - 1$ , and, if  $\nu$  is  $b$ -small,  $\nu$  is  $p^u$ -restricted.*

(b) *If  $\nu$  is  $p^r$ -restricted, then  $\nu$  is  $(h - 1)(p^r - 1)$ -small.*

(c) *Let  $M$  and  $N$  be two highest weight modules for  $G$  with highest weights  $\nu, \mu$ . If  $\mu$  and  $\nu$  are  $b$ -small and  $b'$ -small, respectively, the tensor product  $M \otimes N$  is  $(b + b')$ -small.*

(d) *If  $\lambda, \mu \in X_b^+$  are both  $p^b$ -restricted, then all composition factors of  $L(\lambda) \otimes L(\mu)$  are  $p^{b + [\log_p(h-1)] + 2}$ -restricted. If, in fact,  $p \geq 2h - 2$  all the composition factors of  $L(\lambda) \otimes L(\mu)$  are  $p^{b+1}$ -restricted.*

*Proof.* First we prove property (a). The case  $b = 0$  is clear, so assume  $b > 0$ . If  $b \geq p^u$ , then  $\log_p b \geq [\log_p b] + 1$ , which is false. Thus,  $b \leq p^u - 1$ . If  $\nu$  is  $b$ -small, then  $(\nu, \alpha^\vee) \leq b$  for each  $\alpha \in \Phi^+$ , so  $\nu$  is  $p^u$ -restricted. Hence, property (a) holds.

For property (b) we note  $(\nu, \alpha_0^\vee) \leq ((p^r - 1)\rho, \alpha_0^\vee) \leq (p^r - 1)(h - 1)$ . This proves property (b) since  $\nu \in X^+$  is dominant.

For property (c), the highest weight of  $M \otimes N$  is  $(b + b')$ -small. Since any other weight of  $M \otimes N$  is obtained by subtracting positive roots, the statement follows.

To prove property (d), note that property (b) implies that  $\lambda$  and  $\mu$  are  $(h - 1)(p^b - 1)$ -small. Thus, by property (c), all composition factors of  $L(\lambda) \otimes L(\mu)$  are  $2(h - 1)(p^b - 1)$ -small. Then, by property (a), all composition factors of  $L(\lambda) \otimes L(\mu)$  are  $p^e$ -restricted, where

$$\begin{aligned} e &= [\log_p(2(h - 1)(p^b - 1))] + 1 \\ &\leq [\log_p 2 + \log_p(p^b - 1)] + [\log_p(h - 1)] + 2 \\ &\leq b + [\log_p(h - 1)] + 2. \end{aligned}$$

The case  $p \geq 2h - 2$  follows similarly. □

### 3. Bounding weights

Let  $U = R_u(B)$  be the unipotent radical of  $B$ , and let  $u$  be the Lie algebra of  $U$ .

LEMMA 3.1. *For any nonnegative integer  $m$ , the  $T$ -weights in the ordinary cohomology space  $H^m(u, k)$  are  $3m$ -small (and they are sums of positive roots).*

*Proof.* The  $T$ -weights in  $H^m(u, k)$  are included among the  $T$ -weights of the exterior power  $\bigwedge^m(u^*)$  appearing in the Koszul complex computing  $H^\bullet(u, k)$ . Hence, they are sums of  $m$  positive roots. Since each positive root is 3-small, these weights are  $3m$ -small. □

Recall that, for  $r \geq 1$ ,  $U_r$  is the Frobenius kernel of  $F^r|_U$ .

LEMMA 3.2. *For any nonnegative integer  $m$ , the  $T$ -weights of  $H^m(U_1, k)$  are  $3mp$ -small.*

*Proof.* By [Jan03, I.9.20] and [FP86, (1.2)(b)] there are spectral sequences

$$\begin{cases} p = 2 : E_2^{i,j} := S^i(u^*)^{[1]} \otimes H^j(u, k) \implies H^{i+j}(U_1, k), \\ p \neq 2 : E_2^{2i,j} := S^i(u^*)^{[1]} \otimes H^j(u, k) \implies H^{2i+j}(U_1, k). \end{cases}$$



Suppose that  $p = 2$ . A weight  $\nu$  in  $H^m(U_1, k)$  is a weight in  $E_2^{i,j}$  for some  $i + j = m$ . Using Lemma 3.1, the largest value of  $(\nu, \alpha^\vee)$  clearly occurs when  $i = m$  and  $j = 0$ . Since the weights of  $S^m(u^*)^{[1]}$  are given as a sum  $\sum_{k=1}^m p\alpha_k$ , the weight  $\nu$  is  $3mp$ -small. Similarly, when  $p > 2$ , the weight  $\mu$  is  $3(m/2)p$ -small, so certainly  $3mp$ -small also.  $\square$

LEMMA 3.3. *For any  $r > 0$  and nonnegative integer  $m$ , the  $T$ -weights of  $H^m(U_r, k)$  are  $3mp^r$ -small.*

*Proof.* The case  $r = 1$  follows from the previous lemma. We prove the result by induction using the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{i,j} := H^i(U_r/U_1, H^j(U_1, k)) \implies H^{i+j}(U_r, k).$$

The  $E_2^{i,j}$ -term has the same weights for  $T$  as the  $T$ -module

$$H^i(U_r/U_1, k) \otimes H^j(U_1, k) \cong H^i(U_{r-1}, k)^{[1]} \otimes H^j(U_1, k).$$

By induction, the weights on the left-hand tensor factor are  $p(3ip^{r-1}) = 3ip^r$ -small. On the right-hand side, the weights are  $3jp$ -small. Adding these together, the worst case occurs for  $i = m$ , and the lemma follows.  $\square$

The following is immediate.

COROLLARY 3.4. *Suppose the weight  $\lambda$  is  $b$ -small. Then the weights of  $H^m(B_r, \lambda) \cong (H^m(U_r, k) \otimes \lambda)^{T_r}$  are  $(3mp^r + b)$ -small. Moreover, the weights of  $H^m(B_r, \lambda)^{[-r]}$  are  $(3m + [b/p^r])$ -small, where  $[ \ ]$  denotes the greatest integer function.*

THEOREM 3.5. *Let  $m$  be a nonnegative integer, and let  $r, b$  be positive integers. Let  $M$  be a  $b$ -small  $G$ -module. Then the  $G$ -module  $H^m(G_r, M)^{[-r]}$  is  $(3m + [b/p^r])$ -small.*

*In particular, if  $M$  is  $p^r$ -restricted, then  $H^m(G_r, M)^{[-r]}$  is  $(3m + [(h - 1)(p^r - 1)/p^r]) \leq (3m + h - 2)$ -small.*

*Proof.* We will show the statement holds when  $M$  is an induced module with highest  $b$ -small weight  $\lambda$ ; thence we deduce that the statement holds for  $L(\lambda)$ , so the statement follows for all  $M$ , since it holds for its composition factors.

By [Jan03, II.12.2], there is a first quadrant spectral sequence

$$E_2^{i,j} := R^i \operatorname{ind}_B^G H^j(B_r, \lambda)^{[-r]} \implies H^{i+j}(G_r, H^0(\lambda))^{[-r]}.$$

Any weight in  $H^m(G_r, H^0(\lambda))^{[-r]}$  is a weight of  $E_2^{i,j}$  for some  $i, j$  with  $i + j = m$ , hence a weight of  $H^i(\mu)$  for some  $\mu \in H^j(B_r, \lambda)^{[-r]}$ . So it suffices to show that any weight of  $H^i(\mu)$  for  $\mu \in H^j(B_r, \lambda)^{[-r]}$  is  $(3m + [b/p^r])$ -small.

By Corollary 3.4, a weight  $\mu$  of  $H^j(B_r, \lambda)^{[-r]}$  is  $(3j + [b/p^r])$ -small; hence it is also  $b' := (3m + [b/p^r])$ -small. Choose  $w \in W$  so that  $w \cdot \mu \in X^+ - \rho$ . If  $w \cdot \mu$  is not in  $X^+$  then  $R^\bullet \operatorname{Ind}_B^G \mu = 0$ . Hence we may assume that  $w \cdot \mu \in X^+$ . Now,  $w \cdot \mu = w(\mu) + w\rho - \rho \leq w(\mu)$ . Since  $w \cdot \mu \in X^+$ ,  $w \cdot \mu$  is  $b'$ -small if and only if  $(w \cdot \mu, \alpha_0^\vee) \leq b'$ . But

$$(w \cdot \mu, \alpha_0^\vee) \leq (w(\mu), \alpha_0^\vee) = (\mu, w^{-1}\alpha_0^\vee) \leq b'.$$

Now if  $L(\nu)$  is a composition factor of  $R^i \operatorname{ind}_B^G \mu$ , the strong linkage principle [Jan03, II.6.13] implies  $\nu \uparrow w \cdot \mu$  and is, in particular,  $b'$ -small.

Thus we have proved the statement in the case  $M = H^0(\lambda)$ .

For the general case, we apply induction on  $m$ . We have a short exact sequence  $0 \rightarrow L(\lambda) \rightarrow H^0(\lambda) \rightarrow N \rightarrow 0$  where the  $G$ -module  $N$  has composition factors whose high weights are less

than  $\lambda$  in the dominance order and are therefore  $b$ -small. Associated to this sequence is a long exact sequence of which part is

$$H^{m-1}(G_r, N)^{[-r]} \rightarrow H^m(G_r, L(\lambda))^{[-r]} \rightarrow H^m(G_r, H^0(\lambda))^{[-r]}$$

so that any  $G$ -composition factor of the middle term must be a  $G$ -composition factor of one of the outer terms. Now, the composition factors of the rightmost term are  $(3m + [b/p^r])$ -small by the discussion above. Since  $N$  has composition factors with high weights less than  $\lambda$  in the dominance order, these weights are  $(3(m - 1) + [b/p^r])$ -small by induction, and are, in particular,  $(3m + [b/p^r])$ -small. Thus the weights of the middle term are also  $(3m + [b/p^r])$ -small.

This proves the statement in the case  $M = L(\lambda)$ . The case for all  $b$ -small modules  $M$  now follows since it is true for each of its composition factors.

For the last statement, we use Lemma 2.1(b). □

The following corollary for general  $m$  follows from the previous theorem. For  $m = 1$ , it is proved in [BNP04b, Proposition 5.2].

**COROLLARY 3.6.** *Let  $\lambda, \mu \in X_r^+$ . For any  $r \geq 1$ , the weights of  $\text{Ext}_{G_r}^m(L(\lambda), L(\mu))^{[-r]}$  are  $(3m + 2h - 3)$ -small. If  $m = 1$ , the weights of  $\text{Ext}_{G_r}^1(L(\lambda), L(\mu))^{[-r]}$  are  $(h - 1)$ -small.*

*Remarks 3.7.* (a) The result [BNP04b, Proposition 5.2] quoted above also gives, for  $m > 1$ , an integer  $b$  such that the weights in  $\text{Ext}_{G_r}^m(L(\lambda), L(\mu))^{[-r]}$  are  $b$ -small. However,  $b$  is multiplicative in  $h$  and  $m$ , and so is weaker than Corollary 3.6 for large  $m$  and  $h$  (for instance if  $m, h \geq 4$ ). For  $p > h$  and  $m \geq 2$  note that our bound coincides with that given in [BNP04b, Proposition 5.3] using the improvements from remark (c) below.

(b) It is interesting to ask when  $\text{Ext}_{G_r}^m(L(\lambda), L(\mu))^{[-r]}$  for  $\lambda, \mu \in X_r^+$  has a good filtration. Even for  $r = 1$ , there are examples due to Sin (for instance, see [Sin94, Proposition 4.11]) showing that for small  $p$  this question can have a negative answer. Obviously, if  $p \geq 3m + 3h - 4$  (or  $p \geq 2h - 2$  in case  $m = 1$ ), then  $\text{Ext}_{G_r}^m(L(\lambda), L(\mu))^{[-r]}$  has highest weights in the lowest  $p$ -alcove, so it trivially has a good filtration.<sup>8</sup>

(c) The reader may check that many of the results in this section can be improved under certain mild conditions. For instance, if  $\Phi$  is not of type  $G_2$ , its roots are all 2-small. In this instance, wherever we have ‘ $3m$ ’ it can be replaced with ‘ $2m$ ’. In addition, if  $p > 2$ , the last sentence of the proof of Lemma 3.2 shows that one can replace  $m$  with  $[m/2]$ . The same statement follows for most formulas in the remainder of the paper; however, we will not elaborate further in individual cases.

#### 4. Relating $G(q)$ -cohomology to $G$ -cohomology

Inspired by [BNP01, BNP02, BNP04a, BNP06, BNP11], Theorem 4.4 establishes an important procedure for describing  $G(q)$ -cohomology in terms of  $G$ -cohomology. This result will be used in § 5 to prove the digit-bounding results mentioned in the Introduction.

Before stating the theorem, we review some elementary results. The coordinate algebra  $k[G]$  of  $G$  is a left  $G$ -module ( $f \mapsto g \cdot f, x \mapsto f(xg), x, g \in G, f \in k[G]$ ) and a right  $G$ -module ( $f \mapsto f \cdot g, x \mapsto f(gx), x, g \in G, f \in k[G]$ ). Given a closed subgroup  $H$  of  $G$  and a rational  $H$ -module, the induced module  $\text{ind}_H^G(V) := \text{Map}_H(G, V)$  consists of all morphisms  $f : G \rightarrow V$

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<sup>8</sup> The first two authors have proved in [PS13, Theorem 5.3] that a good filtration of  $\text{Ext}_{G_r}^m(L(\lambda), L(\mu))^{[-r]}$  always exists for restricted regular weights when  $r = 1, p \geq 2h - 2$  and the Lusztig character formula holds for all irreducible modules with restricted highest weights.



(i.e., morphisms of the algebraic variety  $G$  into the underlying variety of a finite-dimensional subspace of  $V$ ), which are  $H$ -equivariant in the sense that  $f(h \cdot g) = h \cdot f(g)$  for all  $g, h \in G$ . If  $x \in G$  and  $f \in \text{ind}_H^G V$ , then  $x \cdot f \in \text{ind}_H^G V$ , making  $\text{ind}_H^G V$  into a rational  $G$ -module (characterized by the property that  $\text{ind}_H^G$  is the right adjoint of the restriction functor  $\text{res}_H^G : G\text{-mod} \rightarrow H\text{-mod}$ ). If  $G/H$  is an affine variety (e.g., if  $H$  is a finite subgroup), then  $\text{ind}_H^G$  is an exact functor [CPS77, Theorem 4.3], which formally takes injective  $H$ -modules to injective  $G$ -modules. Thus,  $H^\bullet(H, V) \cong H^\bullet(G, \text{ind}_H^G V)$  for any rational  $H$ -module.

Let  $q = p^r$  for some prime integer  $p$  and positive integer  $r$ , and let  $m$  be a fixed nonnegative integer which will serve as the cohomological degree. As in §2, let  $G$  be the simple, simply connected algebraic group defined and split over  $\mathbb{F}_p$  with root system  $\Phi$ .

LEMMA 4.1 [Kop84]. *Consider the coordinate algebra  $k[G]$  of  $G$  as a rational  $(G \times G)$ -module with action*

$$((g, h) \cdot f)(x) = f(h^{-1}xg), \quad g, h, x \in G, f \in k[G].$$

Then  $k[G]$  has an increasing  $G \times G$ -stable filtration  $0 \subset \mathfrak{F}'_0 \subset \mathfrak{F}'_1 \subset \dots$  in which, for  $i \geq 1$ ,  $\mathfrak{F}'_i/\mathfrak{F}'_{i-1} \cong \nabla(\gamma_i) \otimes \nabla(\gamma_i^*)$ ,  $\gamma_i \in X^+$ , and  $\bigcup_i \mathfrak{F}'_i = k[G]$ . Each dominant weight  $\gamma \in X^+$  appears exactly once in the sequence  $\gamma_0, \gamma_1, \gamma_2, \dots$ .

THEOREM 4.2 [BNP11, Proposition 2.2 and Proof]. (a) *The induced module  $\text{ind}_{G(q)}^G(k)$  is isomorphic to the pull-back of the  $G \times G$ -module  $k[G]$  above through the map  $G \rightarrow G \times G$ ,  $g \mapsto (g, F^r(g))$ .*

(b) *In this way,  $\text{ind}_{G(q)}^G k$  inherits an increasing  $G$ -stable filtration  $0 \subset \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \dots$  with  $\bigcup \mathfrak{F}_i = \text{ind}_{G(q)}^G k$ , in which, for  $i \geq 1$ ,  $\mathfrak{F}_i/\mathfrak{F}_{i-1} \cong \nabla(\gamma_i) \otimes \nabla(\gamma_i^*)^{[r]}$ . Moreover, each dominant weight  $\gamma \in X^+$  appears exactly once in the sequence  $\gamma_0, \gamma_1, \dots$ .*

Following [BNP11], put  $\mathcal{G}_r(k) := \text{ind}_{G(q)}^G k$ , with  $q = p^r$ . The filtration  $\mathfrak{F}_\bullet$  of the rational  $G$ -module  $\mathcal{G}_r(k)$  arises from the increasing  $G \times G$ -module filtration  $\mathfrak{F}'_\bullet$  of  $k[G]$  with sections  $\nabla(\gamma) \otimes \nabla(\gamma^*)$ . Since these latter modules are all co-standard modules for  $G \times G$ , their order in  $\mathfrak{F}'_\bullet$  can be manipulated, using the fact that

$$\text{Ext}_{G \times G}^1(\nabla(\gamma) \otimes \nabla(\gamma^*), \nabla(\mu) \otimes \nabla(\mu^*)) = 0 \quad \text{unless } \mu < \gamma \text{ (and } \mu^* < \gamma^*). \tag{4.2.1}$$

Thus, for any nonnegative integer  $b$ , there is a (finite-dimensional)  $G$ -submodule  $\mathcal{G}_{r,b}(k)$  of  $\mathcal{G}_r(k)$  which has an increasing  $G$ -stable filtration with sections precisely the  $\nabla(\gamma) \otimes \nabla(\gamma^*)^{[r]}$  with  $(\gamma, \alpha_0^\vee) \leq b$ , and with each such  $\gamma$  appearing with multiplicity 1. Such a submodule may be constructed from a corresponding (unique)  $G \times G$ -submodule of  $k[G]$  with corresponding sections  $\nabla(\gamma) \otimes \nabla(\gamma^*)$ . With this construction, the quotient  $\mathcal{G}_r(k)/\mathcal{G}_{r,b}(k)$  has a  $G$ -stable filtration with sections  $\nabla(\gamma) \otimes \nabla(\gamma^*)^{[r]}$ ,  $(\gamma, \alpha_0^\vee) > b$ . More precisely, using (4.2.1) again and setting  $\mathcal{G}_{r,-1}(k) = 0$ , we have the following result.

LEMMA 4.3. *For  $b \geq 0$ ,*

$$\mathcal{G}_{r,b}(k)/\mathcal{G}_{r,b-1}(k) \cong \bigoplus_{\lambda \in X^+, (\lambda, \alpha_0^\vee) = b} \nabla(\lambda) \otimes \nabla(\lambda^*)^{[r]}.$$

Also,  $\bigcup_{b \geq 0} \mathcal{G}_{r,b}(k) = \mathcal{G}_r(k)$ .

We will usually abbreviate  $\mathcal{G}_r(k)$  to  $\mathcal{G}_r$  and  $\mathcal{G}_{r,b}(k)$  to  $\mathcal{G}_{r,b}$  for  $b \geq -1$ . We remark that  $\mathcal{G}_r(k)$  is in some sense already an abbreviation, since it depends on the characteristic  $p$  of  $k$ .

For  $\lambda, \mu \in X_r^+$ , consider  $\text{Ext}_{G(q)}^m(L(\lambda), L(\mu))$ . Because the induction functor  $\text{ind}_{G(q)}^G$  is exact from the category of  $kG(q)$ -modules to the category of rational  $G$ -modules, then

$$\text{Ext}_{G(q)}^m(L(\lambda) \otimes L(\mu^*), k) \cong \text{Ext}_G^m(L(\lambda) \otimes L(\mu^*), \mathcal{G}_r), \tag{4.3.1}$$

where  $q$  is  $p^r$ .

The following result provides an extension beyond the  $m = 1$  case treated in [BNP02, Theorem 2.2]. A similar result in the cohomology case is given with a bound on  $p$  (namely,  $p \geq (2m + 1)(h - 1)$ ) for any  $m$  by [BNP01, Proof of Corollary 7.4]. Our result, for  $\text{Ext}^m$ , does not require any condition on  $p$ .

**THEOREM 4.4.** *Let  $b \geq 6m + 6h - 8$ , independently of  $p$  and  $r$ , or, more generally,*

$$b \geq b(\Phi, m, p^r) := \left\lceil \frac{3m + 3h - 4}{1 - 1/p^r} \right\rceil,$$

when  $p$  and  $r$  are given. Then, for any  $\lambda, \mu \in X_r^+$ , we have

$$\text{Ext}_{G(q)}^m(L(\lambda), L(\mu)) \cong \text{Ext}_G^m(L(\lambda), L(\mu) \otimes \mathcal{G}_{r,b}). \tag{4.4.1}$$

(Recall  $q = p^r$ .) In addition,

$$\text{Ext}_G^n(L(\lambda), L(\mu) \otimes \nabla(\nu) \otimes \nabla(\nu^*)^{[r]}) = 0, \quad \forall n \leq m, \forall \nu \in X^+ \text{ satisfying } (\nu, \alpha_0^\vee) > b. \tag{4.4.2}$$

*Proof.* It suffices to show that

$$b \geq b(\Phi, m, p^r) \implies \text{Ext}_G^n(L(\lambda), L(\mu) \otimes \mathcal{G}_r/\mathcal{G}_{r,b}) = 0, \quad \forall n \leq m. \tag{4.4.3}$$

Suppose that (4.4.3) fails. Then for some  $\nu$  with  $(\nu, \alpha_0^\vee) > b$  and some nonnegative integer  $n \leq m$ , we must have  $\text{Ext}_G^n(L(\lambda), L(\mu) \otimes \nabla(\nu) \otimes \nabla(\nu^*)^{[r]}) \neq 0$ . (That is, (4.4.2) fails.) For some composition factor  $L(\xi) \cong L(\xi_0) \otimes L(\xi')^{[r]}$  ( $\xi_0 \in X_r^+$ ,  $\xi' \in X^+$ ) of  $\nabla(\nu)$ , we obtain

$$\text{Ext}_G^n(\Delta(\nu)^{[r]} \otimes L(\lambda), L(\mu) \otimes L(\xi_0) \otimes L(\xi')^{[r]}) \neq 0 \tag{4.4.4}$$

by rearranging terms. Here we use the fact that  $\nabla(\nu^*)^* \cong \Delta(\nu)$ , the Weyl module of highest weight  $\nu$ . To compute the left-hand side of (4.4.4) we use a Lyndon–Hochschild–Serre spectral sequence involving the normal subgroup  $G_r$ . The  $E_2$ -page is given by

$$E_2^{i,j} = \text{Ext}_G^i(\Delta(\nu), \text{Ext}_{G_r}^j(L(\lambda) \otimes L(\mu^*), L(\xi_0))^{[-r]} \otimes L(\xi')), \quad i + j = n. \tag{4.4.5}$$

Using Lemma 2.1(b)(c), we see that the module  $L(\lambda^*) \otimes L(\mu) \otimes L(\xi_0)$  is  $3(h - 1)(p^r - 1)$ -small; thus by Theorem 3.5 we have  $\text{Ext}_{G_r}^j(L(\lambda) \otimes L(\mu^*), L(\xi_0))^{[-r]} = \text{Ext}_{G_r}^j(k, L(\lambda^*) \otimes L(\mu) \otimes L(\xi_0))^{[-r]}$  is  $(3j + 3h - 4)$ -small, and, in particular, is  $(3m + 3h - 4)$ -small.

Put  $x = (\nu, \alpha_0^\vee) > b$ . Then, as  $L(\xi)$  is a composition factor of  $\nabla(\nu)$ ,  $\xi = \xi_0 + p^r \xi'$  is  $x$ -small. Clearly  $p^r \xi'$  is  $x$ -small also, and thus  $\xi'$  is  $[x/p^r]$ -small. So, the composition factors of  $\text{Ext}_{G_r}^j(L(\lambda) \otimes L(\mu^*), L(\xi_0))^{[-r]} \otimes L(\xi')$  are  $([x/p^r] + 3m + 3h - 4)$ -small. Recall the fact, for general  $\nu, \omega \in X^+$ , that  $\text{Ext}_G^\bullet(\Delta(\nu), L(\omega)) \neq 0$  implies that  $\nu \leq \omega$ . For  $\nu$  as above and  $L(\omega)$  a composition factor of  $\text{Ext}_{G_r}^j(L(\lambda) \otimes L(\mu^*), L(\xi_0))^{[-r]} \otimes L(\xi')$ , it follows that  $\nu \leq \omega$ . Hence,  $x = (\nu, \alpha_0^\vee) \leq (\omega, \alpha_0^\vee) \leq 3m + 3h - 4 + [x/p^r]$ . Rearranging this gives

$$x \leq \left\lceil \frac{3m + 3h - 4}{1 - 1/p^r} \right\rceil \leq b,$$

which is a contradiction. This proves (4.4.1) and (4.4.2).

For the remaining part of the theorem, just note that the smallest value of  $p^r$  is 2. Hence, the largest value of  $v(\Phi, m, p^r)$  is  $6m + 6h - 8$ . □

5. Digits and cohomology

Any  $\lambda \in X^+$  has a  $p$ -adic expansion  $\lambda = \lambda_0 + p\lambda_1 + \dots + p^r\lambda_r + \dots$  where each  $\lambda_i$  is  $p$ -restricted. We refer to each pair  $(i, \lambda_i)$  as a digit of  $\lambda$ . We say the  $i$ th digit of  $\lambda$  is 0 if  $\lambda_i = 0$ . Clearly  $\lambda$  has finitely many nonzero digits. Let also  $\mu \in X^+$ . We say  $\lambda$  and  $\mu$  agree on their  $i$ th digit whenever  $\lambda_i = \mu_i$ . We say  $\lambda$  and  $\mu$  differ on  $n$ -digits if  $|\{i : \lambda_i \neq \mu_i\}| = n$ .

Theorem 5.2 below requires the following result.

LEMMA 5.1 [BNP06, Proposition 3.1]. *Let  $\lambda, \mu \in X_r^+$  and  $M$  a finite-dimensional rational  $G$ -module whose (dominant) weights are  $(p^r - 1)$ -small. Then*

$$\text{Hom}_{G_r}(L(\lambda), L(\mu) \otimes M) = \text{Hom}_G(L(\lambda), L(\mu) \otimes M).$$

Hence, the left-hand side has trivial  $G$ -structure.

We can now prove the following ‘digit bounding’ theorem. It both answers the open question [Ste12a, Question 3.10] in a strong way, and paves the way for the rest of this section.

THEOREM 5.2. *Given an irreducible root system  $\Phi$  and a nonnegative integer  $m$ , there is an integer  $\delta = \delta(\Phi, m)$ , so that if  $\lambda, \mu, \nu \in X^+$ , and  $\nu$  is  $(3m + 2h - 2)$ -small, then we have the following, where  $\phi = \lceil \log_p(3m + 2h - 2) \rceil + 1$ .*

(1) *The condition*

$$\text{Ext}_G^n(L(\lambda), L(\nu) \otimes L(\mu)) \neq 0$$

*for some  $n \leq m$  implies  $\lambda$  and  $\mu$  differ in at most  $\delta$  digits.*

(2) *Also,*

$$\text{Ext}_G^m(L(\lambda), L(\mu)) \neq 0$$

*implies  $\lambda$  and  $\mu$  differ in at most  $\delta - \phi$  digits.*

*Proof.* We prove both statements together by induction on  $m$ . Set  $b := 3m + 2h - 2$  and  $u := \phi$ . Thus,  $u = \lceil \log_p b \rceil + 1$ . Let  $\lambda^\circ := \lambda_0 + \dots + p^{u-1}\lambda_{u-1}$ , so that  $\lambda = \lambda^\circ + p^u\lambda'$ , for  $\lambda' \in X^+$ ; write  $\mu$  similarly. By Lemma 2.1, the  $b$ -small weight  $\nu$  is  $p^u$ -restricted. In fact,  $b \leq p^u - 1$ .

The case  $m = 0$  follows easily from Lemma 5.1, with  $\delta(\Phi, 0) := \phi$ , as shown below.

For statement (1), we have

$$H := \text{Hom}_G(L(\lambda), L(\mu) \otimes L(\nu)) = \text{Hom}_G(L(\lambda'), \text{Hom}_{G_u}(L(\lambda^\circ), L(\mu^\circ) \otimes L(\nu))^{[-u]} \otimes L(\mu'))$$

with  $H$  assumed to be nonzero. As  $\nu$  is  $(p^u - 1)$ -small, Lemma 5.1 implies that the module  $\text{Hom}_{G_u}(L(\lambda^\circ), L(\mu^\circ) \otimes L(\nu)) \cong \text{Hom}_G(L(\lambda^\circ), L(\mu^\circ) \otimes L(\nu))$  and so has a trivial  $G$ -structure. Thus,

$$H = \text{Hom}_G(L(\lambda'), L(\mu')) \otimes \text{Hom}_G(L(\lambda^\circ), L(\mu^\circ) \otimes L(\nu)).$$

If this expression is nonzero, then  $\lambda' = \mu'$  and  $\lambda^\circ, \mu^\circ$  can differ in at most all of their  $u = \phi$  places, so that  $\lambda$  and  $\mu$  can differ in at most  $\phi$  places. Statement (2) is trivial. This completes the  $m = 0$  case.

Assume we have found  $\delta(\Phi, i)$  for all  $i < m$  such that the theorem holds when  $i$  plays the role of  $m$  and  $\delta(\Phi, i)$  plays the role of  $\delta(\Phi, m)$ . We claim that the theorem holds at  $m$  if we set  $\delta = \delta(\Phi, m) := 2\phi + \max_{i < m} \delta(\Phi, i)$ .

Suppose otherwise. Let  $b = (3m + 2h - 2)$ . Then either statement (1) fails with  $n = m$ , namely,

$$\begin{cases} \text{Ext}_G^m(L(\lambda), L(\nu) \otimes L(\mu)) \neq 0 & \text{for some } \lambda, \mu, \nu \in X^+, \\ \text{with } \nu \text{ } b\text{-small and } \lambda \text{ and } \mu \text{ differing in more than } \delta\text{-digits,} \end{cases} \tag{5.2.1}$$

or statement (2) fails, namely,

$$\text{Ext}_G^m(L(\lambda), L(\mu)) \neq 0, \quad \lambda \text{ and } \mu \text{ differing in more than } \delta - \phi \text{ digits.} \tag{5.2.2}$$

Let  $\lambda, \mu, \nu \in X_s^+$  be a such a counterexample with  $s$  minimal (where, in (5.2.2), we take  $\nu = 0$ , and, in (5.2.1),  $\nu$  is  $b$ -small). We continue to write  $\lambda = \lambda^\circ + p^u \lambda'$ , where  $\lambda^\circ \in X_u^+$ ,  $\lambda' \in X^+$ . We use a similar notation for  $\mu$ .

We investigate  $\lambda$  and  $\mu$  using the Lyndon–Hochschild–Serre spectral sequence for the normal (infinitesimal) group  $G_u \triangleleft G$ . First, suppose that

$$E_2^{m-i,i} := \text{Ext}_G^{m-i}(L(\lambda'), \text{Ext}_{G_u}^i(L(\lambda^\circ), L(\nu) \otimes L(\mu^\circ))^{[-u]} \otimes L(\mu')) \neq 0,$$

for some positive integer  $0 < i \leq m$ . Then  $\text{Ext}_G^{m-i}(L(\lambda'), L(\tau) \otimes L(\mu')) \neq 0$  for some composition factor  $L(\tau)$  of  $\text{Ext}_{G_u}^i(L(\lambda^\circ), L(\nu) \otimes L(\mu^\circ))^{[-u]}$ . By Lemma 2.1(b) and (c), all composition factors of  $L(\nu) \otimes L(\mu^\circ) \otimes L(\lambda^\circ)^*$  are  $(p^u - 1)(2h - 2) + b \leq (p^u - 1)(2h - 1)$ -small. (Note that  $\nu$  is  $(p^u - 1)$ -small.) Thus, by Theorem 3.5,  $\tau$  is  $(3i + 2h - 2) \leq (3(m - 1) + 2h - 2)$ -small.

By induction,  $\mu'$  differs from  $\lambda'$  in at most  $\delta(\Phi, m - i)$  digits. (Apply statement (1) with  $n = m - i$  and  $m - 1$  playing the role of  $m$ .) So the number of digits where  $\lambda$  differs from  $\mu$  is at most  $\phi + \delta(\Phi, m - i) \leq \delta - \phi$  digits. This is a contradiction to (5.2.1) or to (5.2.2) in the  $\nu = 0$  case, as  $\lambda$  was assumed to differ from  $\mu$  by more than  $\delta - \phi$  digits. Hence, we may assume that the terms  $E_2^{m-i,i} = 0$ , for all positive integers  $i \leq m$ .

By assumption,  $\text{Ext}_G^m(L(\lambda), L(\nu) \otimes L(\mu)) \neq 0$ , so,

$$E_2^{m,0} \cong \text{Ext}_G^m(L(\lambda'), \text{Hom}_{G_r}(L(\lambda^\circ), L(\nu) \otimes L(\mu^\circ))^{[-r]} \otimes L(\mu')) \neq 0.$$

Now, by Lemma 5.1,

$$\text{Hom}_{G_r}(L(\lambda^\circ), L(\nu) \otimes L(\mu^\circ)) \cong \text{Hom}_G(L(\lambda^\circ), L(\nu) \otimes L(\mu^\circ))$$

has trivial  $G$ -structure, and

$$E_2^{m,0} \cong \text{Ext}_G^m(L(\lambda'), L(\mu')^{\oplus t}) \cong (\text{Ext}_G^m(L(\lambda'), L(\mu')))^{\oplus t} \neq 0,$$

where  $t = \dim \text{Hom}_G(L(\lambda^\circ), L(\nu) \otimes L(\mu^\circ))$  and  $\lambda', \mu' \in X_{s-u}^+$ .

By minimality of  $s$  we have that  $\lambda'$  and  $\mu'$  differ in at most  $\delta - \phi$  places. But  $\mu^\circ$  and  $\lambda^\circ$  differ in at most all their  $u = \phi$  digits. So  $\lambda$  and  $\mu$  differ in at most  $\delta$  places.

This is a contradiction when  $\nu \neq 0$ . So we may assume  $\nu = 0$  and

$$E_2^{m,0} \cong \text{Ext}_G^m(L(\lambda'), \text{Hom}_{G_u}(L(\lambda^\circ), L(\mu^\circ))^{[-u]} \otimes L(\mu')) \neq 0.$$

The nonvanishing forces  $\lambda^\circ = \mu^\circ$ . Since  $u = \phi > 0$ , we have, by minimality of  $s$ , that

$$\text{Ext}_G^m(L(\lambda'), L(\mu')) \neq 0 \implies \lambda' \text{ and } \mu' \text{ differ in at most } \delta - u \text{ places.}$$

But as  $\lambda$  and  $\mu$  agree on their first  $u$  places, so  $\lambda$  and  $\mu$  differ in at most  $\delta - u = \delta - \phi$  places.

This is a contradiction, and completes the proof of the theorem.  $\square$

*Remark 5.3.* The proof of the theorem implies that if  $\text{Ext}_G^1(L(\lambda), L(\mu)) \neq 0$ , then  $\lambda$  and  $\mu$  differ in at most  $2 + 2[\log_p(h - 1)]$  digits; indeed, following the proof carefully, one sees these digits can be found in a substring of length  $2 + 2[\log_p(h - 1)]$ .

Let  $r = [\log_p(h - 1)] + 1$ . Write  $\lambda = \lambda^\circ + p^r \lambda' = \lambda^\circ + p^r \lambda'^\circ + p^{2r} \lambda''$ , with  $\lambda^\circ, \lambda'^\circ \in X_r^+$ ,  $\lambda', \lambda'' \in X_r^+$ , and take a similar expression for  $\mu$ . If  $\text{Ext}_G^1(L(\lambda), L(\mu)) \neq 0$  then either

$$\text{Ext}_G^1(L(\lambda'), \text{Hom}_{G_r}(L(\lambda^\circ), L(\mu^\circ))^{[-r]} \otimes L(\mu')) \neq 0,$$

which implies  $\lambda^\circ = \mu^\circ$  (and we are done by induction, say on the maximum number of digits of  $\lambda$  and  $\mu$ ), or the space  $\text{Hom}_G(L(\lambda'), \text{Ext}_{G_r}^1(L(\lambda^\circ), L(\mu^\circ))^{[-r]} \otimes L(\mu')) \neq 0$ . Now the weights

of  $\text{Ext}_{G_r}^1(L(\lambda^\circ), L(\mu^\circ))^{[-r]}$  are  $(h - 1)$ -small by Corollary 3.6. By Lemma 2.1(a),  $h - 1 \leq p^r - 1$ . By Lemma 5.1, the  $G$ -structure on  $\text{Hom}_{G_r}(L(\lambda^\circ), L(\tau) \otimes L(\mu^\circ))$  is trivial for a composition factor  $L(\tau)$  of  $\text{Ext}_{G_r}^1(L(\lambda_0), L(\mu_0))^{[-r]}$ . Hence, we must have  $\text{Hom}_G(L(\lambda''), L(\mu'')) \neq 0$ , and we can identify  $\lambda''$  and  $\mu''$ . Thus,  $\lambda$  and  $\mu$  differ in their first  $2r$  digits, as required.

**THEOREM 5.4.** *For every  $m \geq 0$  and irreducible root system  $\Phi$ , choose any nonnegative integer  $m'$  so that  $3m' + 2h + 2 \geq 6m + 6h - 8$ . Recall from Theorem 5.2 the integer  $\delta = \delta(\Phi, m)$  and let  $d = d(m) = d(\Phi, m)$  be an integer  $\geq \delta(\Phi, m')$ . Then for all prime powers  $q = p^r$ , and all  $\lambda, \mu \in X_r^+$ ,*

$$\text{Ext}_{G(q)}^m(L(\lambda), L(\mu)) \neq 0$$

*implies that  $\lambda$  and  $\mu$  differ in at most  $d$  digits.*

*Proof.* Let  $b = 6m + 6h - 8$ . By Theorem 4.4,

$$E = \text{Ext}_{G(q)}^m(L(\lambda), L(\mu)) \cong \text{Ext}_G^m(L(\lambda), L(\mu) \otimes \mathcal{G}_{r,b}).$$

Also,  $\mathcal{G}_{r,b}$  has composition factors  $L(\zeta) \otimes L(\zeta')^{[u]}$  with  $\zeta, \zeta'$  in the set  $\Xi$  of  $b$ -small weights.

Then we have  $\text{Ext}_G^m(L(\lambda) \otimes L(\zeta')^{[r]}, L(\mu) \otimes L(\zeta)) \neq 0$  for some  $\zeta, \zeta' \in \Xi$ . By our choice of  $m'$ , we have  $m' \geq m$ , and the weight  $\zeta$  is  $(3m' + 2h - 2)$ -small. The result now follows from Theorem 5.2. □

Next, define some notation in order to quote the main result of [CPSvdK77].

- (1) Let  $t$  be the torsion exponent of the index of connection  $[X : \mathbb{Z}\Phi]$ .
- (2) For a weight  $\lambda$ , define  $\bar{\lambda} = t\lambda$ . Also let  $t(\lambda)$  be the order of  $\lambda$  in the abelian group  $X/\Phi$ . Let  $t_p(\lambda)$  be the  $p$ -part of  $t(\lambda)$ . Of course one has  $t_p(\lambda) \leq t(\lambda) \leq [X : \mathbb{Z}\Phi]$ .
- (3) Let  $c(\mu)$  for  $\mu$  in the root lattice be the maximal coefficient in an expression of  $\mu$  as a sum of simple roots; for  $\mu = 2\rho$  these values can be read off from [Bou82, Planches I–IX].
- (4) Let  $c = c(\tilde{\alpha})$  where  $\tilde{\alpha}$  is the highest long root. One can also find the value of  $c$  from [Bou82, Planches I–IX].
- (5) For any  $r \in \mathbb{Z}$  and any prime  $p$ , define  $e(r) = [(r - 1)/(p - 1)]$ ; when  $r$  is at least 1, clearly  $e(r) \leq r - 1$ .
- (6) For any  $r \in \mathbb{Z}$  and any prime  $p$ , let  $f(r) = [\log_p(|r| + 1)] + 2$ ; clearly  $f(r) \leq [\log_2(|r| + 1)] + 2$ .

Then the main result of [CPSvdK77] is as follows.

**THEOREM 5.5** [CPSvdK77, 6.6]. *Let  $V$  be a finite-dimensional rational  $G$ -module and let  $m$  be a nonnegative integer. Let  $e, f$  be nonnegative integers with  $e \geq e(ctm)$ ,  $f \geq f(c(\bar{\lambda}))$  for every weight  $\lambda$  of  $T$  in  $V$ . If  $p \neq 2$ , assume also  $e \geq e(ct_p(\lambda)(m - 1)) + 1$ .*

*Then, for  $q = p^{e+f}$ , the restriction map  $\text{H}^n(G, V^{[e]}) \rightarrow \text{H}^n(G(q), V)$  is an isomorphism for  $n \leq m$  and an injection for  $n = m + 1$ .*

We alert the reader that this result will be applied by first determining  $e_0, f_0$  satisfying the inequalities required of  $e$  and  $f$  above, respectively, and then checking  $e \geq e_0$  and  $f \geq f_0$ , respectively, for actual values of  $e$  and  $f$  which arise in our applications.

*Remarks 5.6.* (a) As pointed out in [CPSvdK77, Remark 6.7(c)], it is not necessary to check the numerical conditions in the theorem for each weight  $\lambda$  of  $V$ . It is sufficient to check these conditions for the highest weights of the composition factors of  $V$ .

(b) Let  $e, f$  be as in Theorem 5.5. For any  $e' \geq e$ , twisting induces an isomorphism  $H^m(G, V^{[e]}) \xrightarrow{\sim} H^m(G, V^{[e']})$ . In fact, the theorem implies there are isomorphisms

$$H^m(G(p^{e'+f}), V^{[e]}) \xleftarrow{\sim} H^m(G, V^{[e]}) \xrightarrow{\sim} H^m(G(p^{e+f}), V^{[e]}),$$

where on the left-hand side we write  $e' + f = e + f'$ ,  $f' = f + e' - e$ . Now consider the following diagram.

$$\begin{array}{ccc} H^m(G, V^{[e']}) & \xrightarrow{\sim} & H^m(G(p^{e'+f}), V^{[e']}) \\ \uparrow \text{twisting} & & \\ H^m(G, V^{[e]}) & \xrightarrow{\sim} & H^m(G(p^{e+f}), V^{[e]}) \end{array}$$

The left-hand vertical map is an injection [CPSvdK77], so since the two cohomology groups on the right-hand side have the same dimension (notice there is an isomorphism  $H^m(G(q), V) \cong H^m(G(q), V^{[e]})$  for any  $e \geq 0$ , since  $F^e$  induces an automorphism of  $G(q)$ ), it must be an isomorphism. Compare [CPSvdK77, Corollary 3.8].

In particular,  $H^m(G, V^{[e]}) \cong H_{\text{gen}}^m(G, V)$  (essentially, by the definition of generic cohomology).

LEMMA 5.7. (a) If  $\lambda \in X_r^+$ , then  $f(c(\bar{\lambda})) \leq r + [\log_p(tc(2\rho)/2)] + 2$ . In particular, setting  $f_0 = f_0(\Phi) = \log_2(tc(2\rho)/2) + 2$ , we have  $f(c(\bar{\lambda})) \leq r + f_0$ .

(b) Set  $e_0 = e_0(\Phi, m) = ctm$ . Then, for  $\lambda \in X^+$ ,  $e_0 \geq e(ctm)$  and, for all  $p \neq 2$ ,  $e_0 \geq e(ct_p(\lambda)(m - 1)) + 1$ .

Proof. (a) Note that  $c(\bar{\lambda}) = c(t\lambda) \leq c(t(p^r - 1)\rho) = t(p^r - 1)c(2\rho)/2$  and  $c(\bar{\lambda}) + 1 \leq tp^r c(2\rho)/2$ . Thus,  $f(c(\bar{\lambda})) = [\log_p(c(\bar{\lambda}) + 1)] + 2 \leq r + [\log_p(tc(2\rho)/2)] + 2$ . The second statement is clear.

(b) Note that  $t_p(\lambda) \leq t$ . We leave the details to the reader. □

As an overview for the proof of the following theorem, which becomes quite technical, let us outline the basic strategy. We show that if there is a nontrivial  $m$ -extension between two  $L(\lambda)$  and  $L(\mu)$  which are  $q = p^r$ -restricted, one can insist that  $r$  is so big that Theorem 5.4 implies that the digits of the weights of the two irreducible modules must agree on a large contiguous string of zero digits. Since the cohomology for a finite Chevalley group is insensitive to twisting (as noted above), one can replace the modules with Frobenius twists. The resulting modules are still simple; so, wrapping the resulting non- $p^r$ -restricted factors to the beginning, we may assume they are  $p^r$ -restricted high weights  $\lambda'$  and  $\mu'$ , respectively. In particular we can arrange that a large string of zero digits occurs at the end of  $\lambda'$  and  $\mu'$ . This forces  $\lambda'$  and  $\mu'$  to be bounded away from  $q$ . The result is that we can apply Theorem 5.5 above.

We recall some notation from the Introduction. Let  $q = p^r$  be a  $p$ -power. For  $e \geq 0$  and  $\lambda \in X$ , there is a unique  $\lambda' \in X_r^+$  such that  $\lambda'|_{T(q)} = p^e \lambda|_{T(q)}$ . Denote  $\lambda'$  by  $\lambda^{[e]_q}$ .

THEOREM 5.8. Let  $\Phi$  be an irreducible root system and let  $m \geq 0$  be given.

(a) There exists a nonnegative integer  $r_0 = r_0(\Phi, m)$  such that, for all  $r \geq r_0$  and  $q = p^r$  for any prime  $p$ , if  $\lambda \in X_r^+$ , then, for some  $e \geq 0$ , there are isomorphisms

$$H^n(G(q), L(\lambda)) \cong H(G(q), L(\lambda)^{[e]}) \cong H^n(G, L(\lambda')), \quad n \leq m,$$

where  $\lambda' = \lambda^{[e]_q}$ . In addition, these isomorphisms can be factored as

$$H^n(G(q), L(\lambda)) \cong H^n(G(q), L(\lambda')) \cong H_{\text{gen}}^n(G, L(\lambda')) \cong H^n(G, L(\lambda')).$$



Also, for any  $\ell \geq 0$ , the restriction maps,

$$H^n(G(p^{r+\ell}), L(\lambda')) \rightarrow H^n(G(q), L(\lambda')), \quad n \leq m,$$

are isomorphisms.

(b) More generally, given a nonnegative integer  $\epsilon$ , there is a nonnegative integer  $r_0 = r_0(\Phi, m, \epsilon) \geq \epsilon$  such that, for all  $r \geq r_0$ , if  $\lambda \in X_r^+$  and  $\mu \in X_\epsilon^+$ , there exists an  $e \geq 0$  and an isomorphism

$$\text{Ext}_{G(q)}^n(L(\mu), L(\lambda)) \cong \text{Ext}_{G(q)}^n(L(\mu'), L(\lambda')), \quad n \leq m,$$

where  $\lambda' = \lambda^{[e]_q}$ ,  $\mu' = \mu^{[e]_q}$ . In addition,

$$\begin{aligned} \text{Ext}_{G(q)}^n(L(\mu), L(\lambda)) &\cong \text{Ext}_{G(q)}^n(L(\mu)^{[e]}, L(\lambda)^{[e]}) \cong \text{Ext}_{G(q)}^n(L(\mu'), L(\lambda')) \\ &\cong \text{Ext}_{G, \text{gen}}^n(L(\mu'), L(\lambda')) \cong \text{Ext}_G^n(L(\mu'), L(\lambda')) \end{aligned}$$

for all  $n \leq m$ . Also, for any  $\ell \geq 0$ , the restriction map

$$\text{Ext}_{G(p^{r+\ell})}^n(L(\mu'), L(\lambda')) \rightarrow \text{Ext}_{G(q)}^n(L(\mu'), L(\lambda'))$$

is an isomorphism.

*Proof.* Clearly, part (a) is a special case of part (b), so it suffices to prove part (b).

By Theorem 5.4, there is a nonnegative integer  $d = d(\Phi, n)$  so that, given  $\lambda, \mu \in X_r^+$ , then  $\lambda$  and  $\mu$  differ in at most  $d$  digits if  $\text{Ext}_{G(q)}^n(L(\mu), L(\lambda)) \neq 0$ . We have  $d(\Phi, n) \geq \delta(\Phi, n)$ , where the latter integer satisfies Theorem 5.2, so that  $\lambda$  and  $\mu$  differ in at most  $d$  digits if  $\text{Ext}_G^n(L(\mu), L(\lambda)) \neq 0$ . (For convenience, we are quoting Theorems 5.4 and 5.2 with  $\lambda$  and  $\mu$  in reverse order. This does not cause any problem.) Of course, we may also take  $d \geq \epsilon$ . Equally, it is clear that for  $e' > 0$ ,  $\lambda$  and  $\mu$  differ in at most  $d$  digits if  $\text{Ext}_G^n(L(\mu)^{[e']}, L(\lambda)^{[e']}) \neq 0$ . Therefore, if  $\lambda$  and  $\mu$  differ in more than  $d = d(\Phi, n)$  digits, the claims of part (b) hold with  $\lambda' = \lambda$ ,  $\mu' = \mu$ , and  $e = 0$  since all the relevant cohomology groups vanish.

In the same spirit, if  $\text{Ext}_{G(q')}^n(L(\mu), L(\lambda)) \neq 0$  for some larger power  $q'$  of  $p$ , then  $\lambda$  and  $\mu$  differ in at most  $d$  digits. So if  $\lambda$  and  $\mu$  differ in more than  $d$  digits, the relevant Ext-groups all vanish, so that the isomorphisms of part (b) hold with  $\lambda = \lambda'$ ,  $\mu = \mu'$  and  $e = 0$ .

Put  $d' = \max_{n \leq m} d(\Phi, n)$ . By the discussion above, we can assume that  $\lambda$  and  $\mu$  differ by at most  $d'$  digits. Recall also the constants  $e_0 = e_0(\Phi, m)$ ,  $f_0 = f_0(\Phi)$  from Lemma 5.7. Set

$$r_0 := r_0(\Phi, m, \epsilon) = (d' + 1)(e_0 + f_0 + g + 1) + \epsilon - 1,$$

where  $g = g(\Phi) := \lceil \log_2(h - 1) \rceil + 2 \geq \lceil \log_p 2(h - 1) \rceil + 1$ . We claim that part (b) holds for  $r_0$ . The hypothesis of part (b) guarantees that  $\mu \in X_\epsilon^+$ , and  $r \geq r_0$ . Observe  $r_0 \geq \epsilon$ , so that  $\mu \in X_\epsilon^+$  is  $p^r$ -restricted. Also,  $\lambda \in X_r^+$  under the hypothesis of part (b). In particular, every composition factor  $L(\tau)$  of  $L(\mu^*) \otimes L(\lambda)$  is  $p^{r+g}$ -restricted by Lemma 2.1.

By the ‘digits’ of  $\lambda$  and  $\mu$ , we will mean just the first  $r$  digits, the remainder being zero. By hypothesis,  $\mu \in X_\epsilon^+$ , so all of its digits after the first  $\epsilon$  digits are zero digits. We claim that  $\lambda$  and  $\mu$  have a common string of at least  $(r - \epsilon + 1)/(d' + 1) - 1$  zero digits. To see this, let  $x$  denote the longest string (common to both  $\lambda$  and  $\mu$ ) of zero digits after the first  $\epsilon$  digits. Our claim is equivalent (by arithmetic) to the assertion that

$$(x + 1)(d' + 1) \geq r - \epsilon + 1. \tag{5.8.1}$$

To see (5.8.1), call a digit position which is not zero for one of  $\lambda$  or  $\mu$  an exceptional position. Thus, every digit position past the first  $\epsilon$  positions is either one of the (at most)  $d'$  exceptional digits, or occurs in a common string in  $\lambda$  and  $\mu$  of at most  $x$  zero digits, either right after an exceptional position, or right before the first exceptional position (after the  $\epsilon - 1$  position).

So, after the first  $\epsilon - 1$  positions, there are at most  $d' + 1$  strings of common zero digits, each of length at most  $x$ . Hence,  $(d' + 1)x + d' \geq r - \epsilon$ . That is,  $(d' + 1)(x + 1) \geq r - \epsilon + 1$ . This proves the inequality (5.8.1) and, thus, the claim.

Also,  $x \geq (r - \epsilon + 1)/(d' + 1) - 1 \geq f_0 + e_0 + g_0$  since, by hypothesis,  $r \geq r_0 = (d' + 1)(f_0 + e_0 + g + 1) + \epsilon - 1$ .

We can take Frobenius twists  $L(\lambda)^{[s]}$  and  $L(\mu)^{[s]}$ , with  $s$  a nonnegative integer, so that, up to the  $r$ th digit, the last  $e_0 + f_0 + g$  digits of  $\lambda^\circ := \lambda^{[s]_q}$  and  $\mu^\circ := \mu^{[s]_q}$  are zero. In particular,  $\lambda^\circ$  and  $\mu^\circ$  both belong to  $X_{r-e_0-f_0-g}^+$ . We are going to use for  $e = e(\lambda, \mu)$  in part (b) the integer  $s + e_0$ . We have

$$\text{Ext}_{G(q)}^n(L(\mu^\circ), L(\lambda^\circ)) \cong H^n(G(q), L(\mu^\circ)^* \otimes L(\lambda^\circ)),$$

and, by Lemma 2.1 again, the composition factors of  $M = L(\mu^\circ)^* \otimes L(\lambda^\circ)$  are in  $X_{r-e_0-f_0-g+g}^+ = X_{r-e_0-f_0}^+$ . Let  $L(\nu)$  be a composition factor of  $M$ . Then at least the last  $e_0 + f_0$  digits of  $\nu$  are zero. Now, using Lemma 5.7, the weights of  $M$  satisfy the hypotheses of Theorem 5.5, and, thus, we have  $H^n(G(q), M) \cong H^n(G, M^{[e]})$  for all  $n \leq m$ . The same isomorphism holds if  $q$  is replaced by any larger power of  $p$ , so  $H^n(G, M^{[e_0]}) \cong H_{\text{gen}}^n(G, M^{[e_0]})$ .

Observe that  $L(\lambda^{[e]_q}) = L(\lambda^\circ)^{[e_0]}$ , with a similar equation using  $\mu$ . From the definition of  $M$  above,

$$H^n(G, M^{[e_0]}) \cong \text{Ext}_G^n(L(\lambda^\circ)^{[e_0]}, L(\mu^\circ)^{[e_0]}) \cong \text{Ext}_G^n(L(\lambda^{[e]_q}), L(\mu^{[e]_q})), \quad \forall n, 0 \leq n \leq m,$$

and similar isomorphisms hold for  $G(q)$ -cohomology and  $\text{Ext}_{G(q)}^n$ -groups. We now have most of the isomorphisms needed in part (b), with the remaining ones obtained from group automorphisms on  $G(q)$ .

This completes the proof. □

*Remark 5.9.* The result [BNP06, Theorem 5.6] shows that when  $m = 1$ ,  $r \geq 3$ ,  $p^r \geq h$ , then, with  $e = \lfloor (r - 1)/2 \rfloor$ , we have  $\text{Ext}_{G(q)}^1(L(\lambda), L(\mu)) \cong \text{Ext}_G^1(L(\lambda)^{[e]_q}, L(\mu)^{[e]_q}) = \text{Ext}_G^1(L(\lambda^{[e]_q}), L(\mu^{[e]_q}))$ .

It is tempting to think, as suggested in [Ste12a, Question 3.8], that one might have similar behavior for higher values of  $m$  for some integer  $e \geq 0$  under reasonable conditions. Unfortunately, for  $p$  sufficiently large, this is never true.

In [Ste12b, Theorem 1] the third author gives an example, for any integer  $s \geq 1$ , of a module  $L_s$  for  $\text{SL}_2$  over  $\mathbb{F}_p$  with  $p > 2$  with the property that the dimension of  $\text{Ext}_G^2(L_s, L_s) = s$ . Specifically,  $L_s$  is the module  $L(1) \otimes L(1)^{[1]} \otimes \dots \otimes L(1)^{[s]}$ . So  $L_{r-1}$  is  $q = p^r$ -restricted; it is self-dual, since this is true for all simple  $\text{SL}_2$ -modules; it also has the property that  $L_{r-1}^{[e]_q} = L_{r-1}$  for any  $e \in \mathbb{N}$ , i.e., all its  $q$ -shifts are isomorphic. Note that as  $L(2)$  is isomorphic to the adjoint module for  $p > 2$ , we have  $\dim H^2(G(q), L(2)^{[i]}) \geq 1$  for any  $i \geq 0$ . Set  $D = \dim \text{Ext}_{G(q)}^2(L_{r-1}, L_{r-1})$ . We show  $D > \dim \text{Ext}_G^2(L_{r-1}, L_{r-1}) = r - 1$ .

We have

$$\begin{aligned} D &= \dim \text{Ext}_{G(q)}^2(k, L_{r-1} \otimes L_{r-1}) \\ &= \dim H^2(G(q), L_{r-1} \otimes L_{r-1}) \\ &= \dim H^2(G(q), (L(1) \otimes L(1)) \otimes (L(1) \otimes L(1))^{[1]} \otimes \dots \otimes (L(1) \otimes L(1))^{[r-1]}) \\ &= \dim H^2(G(q), (L(2) \oplus L(0)) \otimes (L(2) \oplus L(0))^{[1]} \otimes \dots \otimes (L(2) \oplus L(0))^{[r-1]}) \\ &\geq \dim H^2(G(q), L(2) \oplus L(2)^{[1]} \oplus \dots \oplus L(2)^{[r-1]}) \\ &\geq r > r - 1, \end{aligned}$$

as required.

Indeed, [Ste12b, Remark 1.2] gives a recipe for cooking up such examples for simple algebraic groups having any root system: one simply requires  $p$  large compared to  $h$ .

Essentially the problem as found above can be described by saying that  $\text{Ext}_G^2(L_{r-1}, L_{r-1})$  is not rationally stable. One does indeed have  $\dim \text{Ext}_G^2(L_{r-1}^{[1]}, L_{r-1}^{[1]}) = D$ .

Motivated by the above example, we ask the following question, a modification of [Ste12a, Question 3.8].

*Question 5.10.* Let  $e_0 = e_0(\Phi, m) := ctm$ . Does there exist a constant  $r_0 = r_0(\Phi, m)$ , such that for all  $r \geq r_0$ , the following holds?

For  $q = p^r$ , if  $\lambda, \mu \in X_r^+$ , then there exists a nonnegative integer  $e = e(\lambda, \mu)$  such that

$$\begin{aligned} \text{Ext}_{G(q)}^n(L(\lambda), L(\mu)) &\cong \text{Ext}_{G(q)}^n(L(\lambda)^{[e]}, L(\mu)^{[e]}) \cong \text{Ext}_{G(q)}^n(L(\lambda^{[e]_q}), L(\mu^{[e]_q})) \\ &\cong \text{Ext}_{G, \text{gen}}^n(L(\lambda^{[e]_q}), L(\mu^{[e]_q})) \cong \text{Ext}_G^n(L(\lambda^{[e]_q})^{[e_0]}, L(\mu^{[e]_q})^{[e_0]}) \end{aligned}$$

for  $n \leq m$ .

*Remark 5.11.* We make the simple observation that in the theorems above one needs the weights  $\lambda$  and  $\mu$  to be  $p^r$ -restricted; hence, these weights determine simple modules for  $G(q)$ . For instance, with the notation of the previous remark, again with  $G = \text{SL}_2$  and  $p > 2$ , we have  $H^0(G, L_{2n-1}) = 0$ , but

$$H^0(G(p), L_{2n-1}) \cong H^0(G(p), (L(2) \oplus L(0))^{\otimes n}) \neq 0,$$

for any positive integer  $n$ , with a similar phenomenon occurring for larger values of  $q$ . So even the 0-degree cohomology of  $G$  will not agree with that of  $G(q)$  on general simple  $G$ -modules.

Using results of [BNP01], we draw the following striking corollary from the main result of this section (and paper). Let  $W_p = W \ltimes p\mathbb{Z}\Phi$  be the affine Weyl group and  $\widetilde{W}_p = W \ltimes p\mathbb{Z}X$  be the extended affine Weyl group for  $G$ . Both groups act on  $X$  by the ‘dot’ action:  $w \cdot \lambda = w(\lambda + \rho) - \rho$ .

**THEOREM 5.12.** *For a given nonnegative integer  $m$  and irreducible root system  $\Phi$ , there is, for all but finitely many prime powers  $q = p^r$ , an isomorphism*

$$H^m(G(q), L(\mu)) \cong H^m(G, L(\mu')), \quad \mu \in X_r^+, \tag{5.12.1}$$

for some constructively given dominant weight  $\mu'$ . If  $r$  is sufficiently large, we can take  $\mu' \in X_r^+$  to be a  $q$ -shift of  $\mu$ . If  $p$  is sufficiently large, and if  $\mu$  is  $\widetilde{W}_p$ -conjugate to 0, then we can take  $\mu' = \mu$ .

In particular, there is a bound  $C = C(\Phi, m)$  such that  $\dim H^m(G(q), L(\mu)) \leq C$  for all values of  $q$  and  $q$ -restricted weights  $\mu$ .

*Proof.* We first prove the assertions in the first paragraph. It suffices to treat the case  $m > 0$ . If  $r$  is sufficiently large, then (5.12.1) holds by Theorem 5.8 for some  $q$ -shift  $\mu' \in X_r^+$  of  $\mu$ . On the other hand, suppose that  $p \geq (4m + 1)(h - 1)$ . We can assume that  $\mu$  is  $p$ -regular; otherwise [BNP01, Corollary 7.4] tells us that  $H^m(G(q), L(\mu)) \cong H^m(G, L(\mu)) = 0$ , and there is nothing to prove.

Suppose  $\mu = u \cdot \nu$  for some  $\nu \in X_r^+$  satisfying  $(\nu, \alpha_0^\vee) \leq 2m(h - 1)$ , and some  $u \in \widetilde{W}_p$ . Then [BNP01, Theorem 7.5] states that

$$H^m(G(q), L(\mu)) \cong H^m(G, L(u \cdot 0 + p^r \nu)),$$

so that (5.12.1) holds in this case. (Because of the size of  $p$ , if  $\mu \in \widetilde{W}_p \cdot 0$ , then  $\mu = 0$ .) If  $\mu$  does not have the form  $\mu = u \cdot \nu$  as above, set  $\mu' = \mu$ . The first paragraph of the proof of [BNP01, Theorem 7.5] shows that  $H^m(G(q), L(\mu)) = 0$ . Also,  $H^m(G, L(\mu')) = H^m(G, L(\mu)) = 0$ , by the linkage principle, since  $\mu$  is not  $W_p$ -linked to 0.

It remains to prove the statement in the second paragraph. We have just established that there is a number  $q_0$  such that for all prime powers  $q = p^r \geq q_0$ , for any  $\mu \in X_r^+$ , there exists a  $\mu' \in X^+$  such that  $H^m(G(q), L(\mu)) \cong H^m(G, L(\mu'))$ . By [PS11, Theorem 7.1], the numbers  $\dim H^m(G, L(\mu'))$  are bounded by a constant  $c = c(\Phi, m)$  depending only on  $m$  and  $\Phi$ . Let  $c' = \max\{\dim H^m(G(q), L(\mu))\}$ , the maximum taken over all prime powers  $q = p^r < q_0$  and weights  $\mu \in X_r^+$ ; clearly  $c'$  is finite. Then  $\dim H^m(G(q), L(\mu)) \leq \max\{c', c\}$ .  $\square$

The explicit bounds exist for  $r$  to be sufficiently large; see Theorem 5.8. The explicit bound on  $p$  in the proof can be improved using Theorem 6.2(c). The constructive dependence of  $\mu'$  on  $\mu$  merely involves the combinatorics of weights and roots.

We can give the following corollary, addressed more thoroughly in §6 below; see Theorem 6.2(c) and Theorem 6.5.

**COROLLARY 5.13.** *If  $p$  is sufficiently large, depending on  $\Phi$  and the nonnegative integer  $m$ , every weight of the form  $p\tau$ ,  $\tau \in X^+$ , is  $m$ -generic at  $q$ , where  $q$  is any power of  $p$  for which  $p\tau$  is  $q$ -restricted. In addition, if  $\mu \in X^+$  is  $q$ -restricted, and has a zero digit in its  $p$ -adic expansion  $\mu = \mu_0 + p\mu_1 + \dots + p^{r-1}\mu_{r-1}$  ( $p^r = q$ ), then  $\mu$  is shifted  $m$ -generic at  $q$ . Moreover, in the first case,*

$$H^m(G(q), L(p\tau)) \cong H^m(G(q), L(\tau)) \cong H_{\text{gen}}^m(G, L(\tau)) \cong H_{\text{gen}}^m(G, L(p\tau)),$$

and, in the second case, if  $\mu_{r-e} = 0$ , for  $0 < e \leq r$ , then we may take  $\mu' = \mu^{[e]_q}$  with

$$H^m(G(q), L(\mu')) \cong H^m(G(q), L(\mu)) \cong H_{\text{gen}}^m(G, L(\mu)) \cong H_{\text{gen}}^m(G, L(\mu')).$$

*Proof.* For the first part, observe that  $p\tau$  is  $\widetilde{W}_p$ -conjugate to 0. So the first part follows from Theorem 5.12. For the second part, simply observe that if  $\mu$  has a zero digit (among its first  $r$  digits), there is some  $q$ -shift  $\mu'$  of  $\mu$  with  $\mu' = p\tau$  for some dominant  $\tau$ .  $\square$

### 6. Large prime results

In this section, we give some ‘large prime’ results. Much work has been done on this topic; see [BNP01, BNP02], as well as earlier papers [And84, FP83].

The following result for  $p \geq 3h - 3$  is given in [BNP02, Corollary 2.4].

**THEOREM 6.1.** *Assume  $p \geq h$  and let  $\lambda, \mu \in X_r^+$ .*

(a) *Suppose  $\text{Ext}_G^1(L(\lambda), L(\mu)) \neq 0$ . Then  $\lambda$  and  $\mu$  differ in at most two digits, which must be adjacent.*

(b) *Let  $q = p^r$ . If  $\text{Ext}_{G(q)}^1(L(\lambda), L(\mu)) \neq 0$ , then  $\lambda$  and  $\mu$  differ in at most two digits, which must be either adjacent, or the first and the last digits.*

*Proof.* For part (a), we just need to apply Remark 5.3 since  $2 + [\log_p(h - 1)] = 2$  in this case. Part (b) follows from part (a) and [BNP06, Theorem 5.6]. (Note that both part (a) and part (b) are trivial unless  $r \geq 3$ .)  $\square$

The following result combines  $q$ -shifting with the themes of [BNP01, Theorems. 7.5, 7.6]. There is some overlap with the latter theorem, which is an analogue for  $G(q)$  and all  $m$  of

Andersen’s well-known  $m = 1$  formula for  $\text{Ext}_G^1$  (see [And84, Theorem 5.6]). However, we make no assumption, unlike [BNP01, Theorem 7.6], that  $\mu$  is sufficiently far from the walls of its alcoves. For the  $m = 1$  case and  $r \geq 3$ , [BNP02, Theorem 5.6] gives a much sharper formula, as noted in Remark 5.9. See the discussion above Theorem 6.5 for the precise definition of a simultaneous  $q$ -shift which is used in part (b).

THEOREM 6.2. (a) Assume that  $p \geq 6m + 7h - 9$ . Then for  $q = p^r$  (any  $r$ ), and  $\lambda, \mu \in X_r^+$ ,

$$\text{Ext}_{G(q)}^m(L(\lambda), L(\mu)) \cong \bigoplus_{\nu} \text{Ext}_G^m(L(\lambda) \otimes L(\nu), L(\mu) \otimes L(\nu)^{[r]}), \tag{6.2.1}$$

where  $\nu \in X^+$  runs over the dominant weights in the closure of the lowest  $p$ -alcove.

(b) Assume that  $p > 12m + 13h - 16$ , and  $\lambda, \mu \in X_r^+$  with  $\lambda$  having a zero digit. Then  $\lambda, \mu$  can be replaced, maintaining the dimension of the left-hand side of (6.2.1), by suitable simultaneous  $q$ -shifts  $\lambda', \mu'$  so that the sum on the right-hand side of (6.2.1) collapses to a single summand,

$$\text{Ext}_{G(q)}^m(L(\lambda), L(\mu)) \cong \text{Ext}_{G(q)}^m(L(\lambda'), L(\mu')) \cong \text{Ext}_G^m(L(\lambda') \otimes L(\nu), L(\mu') \otimes L(\nu^*)^{[r]}) \tag{6.2.2}$$

for some (constructively determined)  $\nu$  in the lowest  $p$ -alcove. Also,  $\lambda'$  can be chosen to be any  $q$ -shift whose first digit is zero (possibly with different weights  $\nu$  for different choices of  $\lambda'$ ). In addition,  $L(\lambda') \otimes L(\nu)$  and  $L(\mu') \otimes L(\nu^*)^{[r]}$  are irreducible in this case.

(c) Moreover, again with  $p > 12m + 13h - 16$  as in part (b), assume that  $\lambda, \mu \in X_r^+$  have a common 0 digit (among the first  $r$  digits, with  $q = p^r$ ). Then simultaneous  $q$ -shifts  $\lambda', \mu'$  may be chosen so that

$$\text{Ext}_{G(q)}^m(L(\lambda), L(\mu)) \cong \text{Ext}_{G(q)}^m(L(\lambda'), L(\mu')) \cong \text{Ext}_{G,\text{gen}}^m(L(\lambda'), L(\mu')) \cong \text{Ext}_G^m(L(\lambda'), L(\mu')). \tag{6.2.3}$$

These isomorphisms all hold, for any simultaneous  $q$ -shifts  $\lambda', \mu'$  of  $\lambda, \mu$  for which  $\lambda', \mu'$  both have a zero first digit.

*Proof.* We first prove part (a). Let  $p \geq 6m + 7h - 9$ . Put  $b := 6m + 6h - 8$ . Then if  $\nabla(\nu) \otimes \nabla(\nu^*)^{[r]}$  is a section in  $\mathcal{G}_{r,b}$ , we have

$$(\nu + \rho, \alpha_0^\vee) \leq b + (\rho, \alpha_0^\vee) = b + h - 1 = 6m + 7h - 9 \leq p. \tag{6.2.4}$$

Therefore,  $\mathcal{G}_{r,b}$  is completely reducible as a rational  $G$ -module with summands  $L(\nu) \otimes L(\nu^*)^{[r]}$ , in which  $\nu \in X^+$  is in the closure of the lowest  $p$ -alcove. (For larger  $p$ ,  $\nu$  is actually in the interior of the lowest  $p$ -alcove. In particular, this applies to parts (b) and (c) of the theorem. Of course,  $\nabla(\nu) = L(\nu)$  for  $\nu$  in the closure of the lowest  $p$ -alcove.) For  $p > 6m + 7h - 9$ , there are dominant weights  $\nu$  in the closure of the lowest  $p$ -alcove which do not satisfy  $(\nu, \alpha_0^\vee) \leq b$ . For such  $\nu$ , (4.4.2) gives that  $\text{Ext}_G^m(L(\lambda), L(\mu) \otimes L(\nu) \otimes L(\nu^*)^{[r]}) = 0$ . Thus, part (a) follows.

To prove part (b), assume that  $p > 12m + 13h - 16 = 2b + h$ . Choose  $e$  with  $0 \leq e < r$ , so that  $\lambda' := \lambda^{[e]_q}$  has its first digit equal to zero. Put  $\mu' := \mu^{[e]_q}$ . Then the left-hand isomorphism in (6.2.2) holds. In addition, the isomorphism (6.2.1) holds.

There is an expression like (6.2.1) with  $\lambda, \mu$  replaced by  $\lambda', \mu'$ . If one of the terms indexed by  $\nu$  on the right-hand side of (6.2.1) (for  $\lambda', \mu'$ ) is nonzero, then  $\mu'$  is  $\widetilde{W}_p$ -conjugate to  $\nu$ . To see this, first note that  $\lambda' = p\lambda^\dagger$  for some  $\lambda^\dagger \in X^+$ , so that  $L(\lambda') \otimes L(\nu)$  is irreducible by the Steinberg tensor product theorem. Similarly,  $L(\mu) \otimes L(\nu^*)^{[r]}$  is irreducible since  $\mu \in X_r^+$ . If  $\text{Ext}_G^m(L(\lambda') \otimes L(\nu), L(\mu') \otimes L(\nu^*)^{[r]}) \neq 0$ , then  $\nu + p\lambda^\dagger$  is  $W_p$ -conjugate to  $\mu' + p^r\nu^*$ . Therefore,  $\nu$  and  $\mu'$  are  $\widetilde{W}_p$ -conjugate. Now it is only necessary to show that any two dominant weights that are  $b$ -small and  $\widetilde{W}_p$ -conjugate are equal. Briefly, suppose that  $\nu, \nu'$  are  $\widetilde{W}_p$ -dot conjugate dominant

weights in the lowest  $p$ -alcove. Write  $\nu' = w \cdot \nu + p\tau$ , for  $w \in W$ ,  $\tau \in X$ . If  $\tau = 0$ , then  $\nu = \nu'$  because both weights are dominant. Hence,  $\tau \neq 0$ . Thus, for any  $\alpha \in \Pi$ ,

$$|(p\tau, \alpha^\vee)| = |(\nu' - w \cdot \nu, \alpha^\vee)| \leq |(\nu' + \rho, \alpha^\vee)| + |(w(\nu + \rho), \alpha^\vee)| \leq 2b + h$$

by the first part of (6.2.4). But  $2b + h < p$ , so  $|(\tau, \alpha^\vee)| < 1$ . It follows that  $\tau = 0$ , an evident contradiction. This completes the proof of part (b).<sup>9</sup> (Apart from the use of  $q$ -shifts and different numerical bounds, the argument in this paragraph parallels that of [BNP01, Theorem 7.5].)

Finally, to prove part (c), we continue the proof of part (b), taking  $\mu'$  to also have a zero first digit. Put  $\mu' = p\mu^\dagger$ , for some  $\mu^\dagger \in X^+$ . So  $\mu'$ , and thus  $\nu$  above, must be  $\widetilde{W}_p$ -conjugate to 0. Therefore  $\nu = 0$ , giving all the isomorphisms in (6.2.3) except the ones involving  $\text{Ext}_{G,\text{gen}}^m$ . We observe that the  $\widetilde{W}_p$ -conjugacy of  $\nu$  to 0 still holds passing from  $r$  to  $r + e_0$  for any integer  $e_0 \geq 0$ . Thus,  $\text{Ext}_{G(q)}^m(L(\lambda'), L(\mu')) \cong \text{Ext}_G^m(L(\lambda'), L(\mu'))$  and  $\text{Ext}_{G(p^{r+e_0})}^m(L(\lambda'), L(\mu')) \cong \text{Ext}_G^m(L(\lambda'), L(\mu'))$ , so

$$\text{Ext}_G^m(L(\lambda'), L(\mu')) \cong \text{Ext}_{G,\text{gen}}^m(L(\lambda'), L(\mu')).$$

This proves part (c). □

*Remarks 6.3.* (a) In the situation of Theorem 6.2(b), suppose it is  $\lambda$  itself that has first digit 0, and put  $\lambda' = \lambda$  and  $\mu' = \mu$ . Assume that  $\text{Ext}_G^m(L(\lambda), L(\mu)) \neq 0$ . Then the conclusion of Theorem 6.2(c) holds. To see this, note that the nonvanishing implies  $\lambda$  and  $\mu$  are  $W_p$ -conjugate, and thus  $\widetilde{W}_p$ -conjugate. Now, we can use the proofs of Theorem 6.2(b) and (c).

(b) The requirements in Theorem 6.2(b) and (c) on the existence of a simultaneous zero cannot be dropped. An example is provided in Remark 5.9.

(c) The sum on the right-hand side of (6.2.1) always involves a bounded number of summands, independent of  $p$ , namely, those in which  $(\nu, \alpha_0^\vee) \leq 6m + 6h - 8$ .

Taking  $\nu = 0$  in (6.2.1) yields the following result.

**COROLLARY 6.4.** *If  $p \geq 6m + 7h - 9$ , then restriction  $\text{Ext}_G^m(L(\lambda), L(\mu)) \rightarrow \text{Ext}_{G(q)}^m(L(\lambda), L(\mu))$  is an injection for every  $\lambda, \mu \in X_r^+$ .*

Suppose that  $\lambda$  and  $\mu$  are  $q$ -restricted weights. Let us call a pair  $(\lambda', \mu')$  a  $q$ -shift of  $(\lambda, \mu)$  if it is obtained by a simultaneous  $q$ -shift  $\lambda' = \lambda^{[e]_q}$ ,  $\mu' = \mu^{[e]_q}$ . Also, we say the pair  $(\lambda, \mu)$  of  $q$ -restricted weights is  $m$ -generic at  $q$  if

$$\text{Ext}_{G(q)}^m(L(\lambda), L(\mu)) \cong \text{Ext}_G^m(L(\lambda), L(\mu)).$$

Similarly, we say the pair  $(\lambda, \mu)$  is shifted  $m$ -generic if  $(\lambda', \mu')$  is  $m$ -generic at  $q$  for a  $q$ -shift  $(\lambda', \mu')$  of  $(\lambda, \mu)$ . Theorem 6.2(c) asserts that for  $p$  large, pairs  $(\lambda, \mu)$  are shifted  $m$ -generic at  $q$ . We improve this in the case that a zero digit occurs as the last digit. At the same time, we give an improvement, in the large prime case, to the limiting procedure of [CPSvdK77] in the result below. Theorem 6.2(a) is already implicit in [CPSvdK77] with a different bound, but Theorem 6.2(b) and (c) are new and at least theoretically interesting, since they give the best possible value for the increase required in  $q$  to obtain stability; see the examples in Remark 6.6 which follows.

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<sup>9</sup> Conceptually, the lowest  $p$ -alcove  $C := \{\lambda \in \mathbb{R} \otimes X \mid 0 \leq (\lambda + \rho, \alpha_0^\vee) < p\}$  is the union of closed simplices conjugate under the finite subgroup  $N$  of  $\widetilde{W}_p$  stabilizing  $C$ . The group  $N$  acts transitively and regularly on the interiors of these simplices. With our assumptions on the sizes of the various  $(\nu, \alpha_0^\vee)$ , the relevant dominant  $\nu$  all belong to the interior of the ‘lowest’ simplex, the one containing 0.



THEOREM 6.5. Assume that  $p > 12m + 13h - 16$ , and let  $\lambda, \mu \in X_r^+$ . Let  $q = p^r$ .

(a) The map

$$\text{Ext}_G^m(L(\lambda)^{[1]}, L(\mu)^{[1]}) \rightarrow \text{Ext}_G^m(L(\lambda)^{[e]}, L(\mu)^{[e]})$$

is an isomorphism for every  $e \geq 1$ . In particular,

$$\text{Ext}_{G,\text{gen}}^m(L(\lambda), L(\mu)) \cong \text{Ext}_G^m(L(\lambda)^{[1]}, L(\mu)^{[1]}).$$

(b) Put  $q' = p^{r+1}$ . Then the  $q'$ -shifts  $\lambda' = \lambda^{[1]_{q'}}$ , and  $\mu' = \mu^{[1]_{q'}}$  satisfy

$$\text{Ext}_{G(q')}^m(L(\lambda), L(\mu)) \cong \text{Ext}_{G(q')}^m(L(\lambda'), L(\mu')) \cong \text{Ext}_{G,\text{gen}}^m(L(\lambda'), L(\mu')) \cong \text{Ext}_G^m(L(\lambda'), L(\mu')).$$

In addition, for any  $s \geq 1$ , the map

$$\text{Ext}_{G(p^{r+s})}^m(L(\lambda'), L(\mu')) \rightarrow \text{Ext}_{G(q')}^m(L(\lambda'), L(\mu'))$$

is an isomorphism as is

$$\text{Ext}_{G(p^{r+s})}^m(L(\lambda), L(\mu)) \rightarrow \text{Ext}_{G(q')}^m(L(\lambda), L(\mu)).$$

In particular, the pair  $(\lambda, \mu)$  is always  $m$ -generic at  $q' = p^{r+1}$ .

(c) Let  $M, N$  be finite-dimensional rational  $G$ -modules whose composition factors are all  $p^r$ -restricted. Then, if  $s \geq 1$ , the natural restriction map

$$\text{Ext}_{G,\text{gen}}^n(M, N) \rightarrow \text{Ext}_{G(p^{r+s})}^n(M, N)$$

is an isomorphism for  $n \leq m$  and an injection for  $n = m + 1$ .

*Proof.* We begin by remarking that part (c) follows from parts (a) and (b): note that Corollary 6.4 applies with  $m$  replaced by  $m + 1$ , checking the required condition on  $p$ . Applied to  $(p\lambda, p\mu)$  and assuming part (a), Corollary 6.4 gives an injection

$$\text{Ext}_{G,\text{gen}}^{m+1}(L(p\lambda), L(p\mu)) \rightarrow \text{Ext}_{G(q')}^{m+1}(L(p\lambda), L(p\mu)).$$

It follows that there is a corresponding injection with  $(p\lambda, p\mu)$  replaced by  $(\lambda, \mu)$ . Now part (c) follows from this latter injection and the last isomorphism in part (b), valid also with  $m$  replaced by any smaller integer. (This is a well-known five-lemma argument needing only the injectivity for the degree  $m + 1$ -maps.)

So it remains to prove parts (a) and (b). The first display in part (b) follows from Theorem 6.2(c). We get a similar string of isomorphisms with  $q'$  replaced by  $p^{r+s}$ . Note that  $\lambda^{[1]_{p^{r+s}}} = \lambda^{[1]_{q'}} = \lambda'$ , with a similar equation for  $\mu'$ . (We use here the fact that  $\lambda, \mu$  are  $q$ -restricted.) Now consider the following commutative diagram, where the vertical maps are restriction maps.

$$\begin{array}{ccccc} \text{Ext}_{G(p^{r+s})}^m(L(\lambda), L(\mu)) & \xrightarrow{\sim} & \text{Ext}_{G(p^{r+s})}^m(L(\lambda'), L(\mu')) & \xrightarrow{\sim} & \text{Ext}_{G,\text{gen}}^m(L(\lambda), L(\mu)) \\ \downarrow & & \downarrow & & \parallel \\ \text{Ext}_{G(q')}^m(L(\lambda), L(\mu)) & \xrightarrow{\sim} & \text{Ext}_{G(q')}^m(L(\lambda'), L(\mu')) & \xrightarrow{\sim} & \text{Ext}_{G,\text{gen}}^m(L(\lambda), L(\mu)) \end{array}$$

It follows easily that two vertical maps (on the left) are isomorphisms, as required. This proves part (b).

To prove part (a), note that  $\dim \text{Ext}_G^m(L(\lambda)^{[1]}, L(\mu)^{[1]}) = \dim \text{Ext}_{G,\text{gen}}^m(L(\lambda'), L(\mu'))$  (by the first isomorphism in part (b)), which equals  $\dim \text{Ext}_{G,\text{gen}}^m(L(\lambda), L(\mu))$  by definition. However, the map

$$\text{Ext}_G^m(L(\lambda)^{[1]}, L(\mu)^{[1]}) \rightarrow \text{Ext}_{G,\text{gen}}^m(L(\lambda), L(\mu))$$

is injective by [CPS83]. So, by dimension considerations, it must be an isomorphism. Part (a) follows easily.  $\square$

*Remark 6.6.* For an example of a pair  $(\lambda, \mu)$  of  $q$ -restricted weights that is not  $m$ -generic or shifted  $m$ -generic at  $q$  with  $m = 2$ , see Remark 5.9. Even when  $p$  is large, there are examples for fixed  $\Phi$  (of type  $A_1$ ) and fixed  $m (=2)$  for arbitrarily large  $r$ . Of course, these examples have no zero digits in common.

### Appendix A. Large prime generic cohomology

In this brief appendix, we consider the large prime generic cohomology results of [FP86, § 3] from the point of view of Theorem 6.5 and the other results of this paper.

Suppose  $q = p$  is prime and let  $\mu \in X_1^+$ . Assume  $p > 12m + 13h - 16 = 2b + h$ , as in Theorem 6.2(b). Taking  $\lambda = 0$ , we thus get

$$H^m(G(p), L(\mu)) \cong \text{Ext}_G^m(L(\nu), L(\mu) \otimes L(\nu^*)^{[1]}), \tag{A.1}$$

for some  $b$ -small  $\nu \in X^+$ .

Necessarily  $\mu + p\nu^*$  lies in the Jantzen region. If  $\mu + p\nu^* \notin W_p \cdot \nu$ , then (A.1) vanishes. Otherwise, take  $p$  larger, if necessary, so that the Lusztig character formula holds for  $G$ . Then the dimension of (A.1) equals the coefficient of a Kazhdan–Lusztig polynomial, since  $L(\nu) \cong \Delta(\nu)$ ; see [CPS93, § 3]. (As noted in [BNP01, Theorem 7.5] and its proof, which also give the above isomorphism for large  $p$ , the right-hand side can be converted to a cohomology group of an irreducible module via a translation taking  $\nu$  to 0. Thus, the relevant Kazhdan–Lusztig polynomial has the form  $P_{w_0, w_0 w}$ , where  $l(w_0 w) = l(w_0) + l(w)$ ,  $w \in W_p$ .) The Lusztig character formula is known to hold when  $p \gg h$  [AJS94], and is conjectured to be true for  $p \geq h$ .

In [FP86, Proposition 3.2], it was observed that  $H^m(G(p), L(\mu)) \cong H_{\text{gen}}^m(G, L(\mu))$  if  $p$  is sufficiently large, depending on the highest weight  $\mu$  and the integer  $m$ . More precisely, in this case, taking  $\mu$  to lie in the closure of the lowest  $p$ -alcove, [FP86, Theorem 3.3, Corollary 3.4] gives a dimension formula, valid when  $p$  is large:

$$\dim H^m(G(p), L(\mu)) = \begin{cases} 0 & m \text{ odd,} \\ \sum_{w \in W} \det(w) \mathfrak{p}_{m/2}(w \cdot \mu) & m \text{ even.} \end{cases} \tag{A.2}$$

Here  $\mathfrak{p}$  is the Kostant partition function. It is interesting to rederive this result in our present context, and compare it with (A.1).

In addition to assuming that  $p > 2b + h$ , also assume that  $p \geq (\mu, \alpha_0^\vee) + h - 1$ . (The last condition just says  $\mu$  lies in the closure of the lowest  $p$ -alcove, so that  $L(\mu) \cong \nabla(\mu)$ .) Theorem 6.5(a) shows that  $H_{\text{gen}}^m(G, L(\mu)) \cong H^m(G, L(\mu)^{[1]}) \cong H^m(G, \nabla(\mu)^{[1]})$ . A formula for the dimension of the latter module is derived in [CPS09, Proposition 4.2], which is precisely that given in the right-hand side of (A.2).

Under the same conditions on  $p$  as in the previous paragraph, but perhaps enlarging  $p$  further, we claim there is an identification of  $H_{\text{gen}}^m(G, L(\mu))$  with  $H^m(G(p^r), L(\mu))$ , for all positive integers  $r$ . For  $r \geq 2$ , this claim follows from Theorem 6.5(c). For the case  $r = 1$ , we also require  $\mu$  to be  $b$ -small, a condition on  $p$  when  $\mu$  is fixed.

Return now to (A.1) in the case of a  $b$ -small  $\mu \in X^+$ . By the argument for Theorem 6.2(b), there is no other  $b$ -small dominant weight  $\widetilde{W}_p$ -conjugate of  $\mu$ . Thus, we can assume  $\nu = \mu$ . Next,  $H_{\text{gen}}^m(G, L(\mu)) \cong H^m(G, L(\mu)^{[1]})$  by Theorem 6.5(a). If this generic cohomology is nonzero, then

$\mu$  belongs to the root lattice. If translation to the principal block is applied to  $L(\mu) \otimes L(\mu^*)^{[1]}$ , using [CPS09, Lemma 3.1], we obtain an irreducible module  $L(\tau) \otimes L(\mu^*)^{[1]}$  with  $\tau$  in lowest  $p$ -alcove, and with highest weight also in  $W_p \cdot 0$ . Therefore,  $\tau = 0$ . Clearly, translation to the principal block takes  $L(\mu)$  to  $L(0)$ . Thus,

$$\begin{aligned} H^m(G(p), L(\mu)) &\cong \text{Ext}_G^m(L(\mu), L(\mu) \otimes L(\mu^*)^{[1]}) \\ &\cong \text{Ext}_G^m(L(0), L(\mu^*)^{[1]}) \\ &\cong H^m(G, L(\mu^*)^{[1]}) \\ &\cong H^m(G, L(\mu)^{[1]}) \\ &\cong H_{\text{gen}}^m(G, L(\mu)) \end{aligned}$$

in this case. If  $H_{\text{gen}}^m(G, L(\mu)) = 0$ , we claim that  $H^m(G(p), L(\mu)) = 0$  also. Otherwise,

$$\text{Ext}_G^m(L(\mu), L(\mu) \otimes L(\mu^*)^{[1]}) \neq 0.$$

Thus,  $\mu$  and  $\mu + p\mu^*$  belong to the same  $W_p$ -orbit, forcing  $\mu^*$  to belong to the root lattice, again giving (by translation arguments)  $H^m(G(p), L(\mu)) = H_{\text{gen}}^m(G, L(\mu)) = 0$ , which is a contradiction. This completes the proof of the claimed identification.

Finally, observe the answer we obtained for  $\dim H^m(G(p), L(\mu))$ , for our rederivation of (A.2), is, when  $\mu$  is  $b$ -small and lies in the root lattice, a Kazhdan–Lusztig polynomial coefficient. (The dimension is independent of  $p$  and has the form  $\dim \text{Ext}_G^m(\Delta(0), L(p\mu))$  with  $p\mu$  in the Jantzen region.) In this case, the Kazhdan–Lusztig polynomial coefficient that gives the right-hand side of (A.1) is associated to  $\text{Ext}_G^m(\Delta(0), L(p\mu^*))$ , which is the same coefficient. (Apply a graph automorphism.) When  $p\mu$  is not in the root lattice,  $\mu + p\mu^*$  is not in  $W_p \cdot \mu$ , and so the right-hand side of (A.1) is zero, as is  $\dim \text{Ext}_G^m(\Delta(0), L(p\mu))$ . Consequently, the combinatorial determination of (A.1) is, for  $b$ -small  $\mu$ , the same as that for (A.2), if Kazhdan–Lusztig polynomial coefficients are used in both cases.

However, the determination of the dimension of (A.1) as a Kazhdan–Lusztig polynomial coefficient or zero applies for all restricted  $\mu$ , if  $p \gg 0$ , not just for those that are  $b$ -small or lie in the closure of the bottom  $p$ -alcove. Thus, in some sense, the discussion above may be viewed as giving a generalization of the determination in [FP86] of  $H^m(G(p), L(\mu))$  for  $p$  sufficiently large, depending on  $m$  and  $\mu$ .

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