ON STABILITY IN THE LARGE FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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1. Autonomous systems. This note concerns the stability of systems of (real) differential equations in the large on Euclidean space E^n and on certain Riemannian manifolds M^n. The results will be refinements of those of Krasovski (**3**), (**4**), (**5**) and of Markus and Yamabe (**8**) and will make clear the role of the various assumptions in the type of theorems under consideration.

In this section, the main theorems are stated for autonomous systems

$$x' = f(x). \tag{1}$$

Their proofs are given in § 2, 3, 4. In § 5, 6, 7, generalizations to non-autonomous systems are made.

The following notation will be used below: Let A^* denote the transpose of the (real) matrix $A = (a_{jk})$, A^H the Hermitian part, $\frac{1}{2}(A + A^*)$, of A. For any two matrices A and B, let $A < B$ mean that $A^H < B^H$, that is, that $B^H - A^H$ is positive definite. Finally, let I be the unit matrix. For points x, y of Euclidean space, $x \cdot y$ denotes the scalar product and $|x| = (x \cdot x)^{\frac{1}{2}} \geqslant 0$. It will generally be assumed that:

(A) $M = M^n$ is a complete Riemannian manifold with a positive, definite, metric tensor $g_{ik}(x)$ of class C^1, and $f(x)$ is a contravariant vector field of class C^1 on M. (The covariant derivative of f is the tensor with components

$$f^k{}_{,m} = \partial f^k/\partial x^m + g^{ik}[jm, i]f^j,$$

where

$$[jm, i] = \tfrac{1}{2}(\partial g_{ji}/\partial x^m + \partial g_{mi}/\partial x^j - \partial g_{jm}/\partial x^i).)$$

The distance between two points x, y of M, considered as a metric space, will be denoted by $d(x, y)$. By $d(x)$ will be meant the distance $d(x, x^0)$ from x to a fixed point x^0 of M.

LEMMA 1. *Assume* (A). *Suppose that the tensor* $e_{ij} = g_{ik}f^k{}_{,j}$ *satisfies*

$$(e_{ij}) < 0. \tag{2}$$

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Then every solution $x = x(t)$ *of* (1) *exists for large* $t > 0$; *furthermore, if* $x = x_1(t)$, $x_2(t)$ *are two distinct solutions of* (1) *for* $t \geqslant 0$, *then*

$$d(x_1(t), x_2(t)) \text{ is decreasing} \tag{3}$$

for $t \geqslant 0$. *In particular, if there exists a stationary point* $x = x_0$,

$$f(x_0) = 0, \tag{4}$$

then every solution $x = x(t) \not\equiv x_0$ *satisfies*

$$d(x_0, x(t)) \downarrow 0, t \to \infty \tag{5}$$

(*where* "$\downarrow$" *signifies* "*decreasing*").

It can be remarked that if the condition (2) is relaxed to

$$(e_{ij}) \leqslant 0, \tag{2'}$$

then the assertion concerning the existence of $x(t)$ for large t remains valid, but (3) must be replaced by

$$d(x_1(t), x_2(t)) \text{ is non-increasing} \tag{3'}$$

and, of course, (4) then does not imply (5) or even $d(x(t), x_0) \to 0$ as $t \to \infty$. Assertion (3′) implies, however, that there is a constant C, depending only on $f(x)$ with the property that if $x = x(t)$ is any solution of (1) for $t \geqslant T$, then

$$d(x(t)) \leqslant d(x(T)) + C(t - T) \text{ for } t \geqslant T. \tag{6}$$

In order to see this, let $x = x_1(t)$ be the solution of (1) satisfying $x_1(0) = x^0$, where x^0 is the reference point of M in the definition of $d(x) = d(x, x^0)$. Let $s > 0$, and consider the solution $x = x_1(t + s)$ of (1). Then, by (3′),

$$d(x_1(t + s), x_1(t)) \leqslant d(x_1(s), x_1(0)) \text{ for } t \geqslant 0.$$

This clearly implies the existence of a constant $C > 0$ such that $d(x_1(t)) \leqslant Ct$ for $t \geqslant 0$. The inequality (6) follows from this fact and (3′), where $x(t) = x_2(t)$.

Lemma 1 is similar to results of Lewis **(7)** and Opial **(9)**. These authors deal with the case where M is replaced by a compact set. One new feature of Lemma 1 is the important remark that (2′) implies that all solutions exist for large t. The end of the proof of Lemma 1 is similar to an argument of LaSalle **(6)**.

In the last part of Lemma 1, (2) need not be required at $x = x_0$.

A consequence of (5) is that $f(x) \neq 0$ for $x \neq x_0$; that is, under the condition (2), there is at most one stationary point. It is of interest to note that a strengthened form of condition (2) implies the existence of a (unique) stationary point. This is the assertion of the following theorem.

(I) *Assume* (A). *Let* $\lambda(r)$ *be a positive, non-increasing function of* r *for* $r \geqslant 0$ *such that*

$$\int^{\infty} \lambda(r) dr = \infty. \tag{7}$$

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Let the tensor $e_{jk} = g_{jm}f^m{}_{,k}$ *satisfy*

(8) $$(e_{jk}(x)) \leqslant -\lambda(d(x))(g_{jk}(x)).$$

Then there exists a unique point $x = x_0$ *of* M *satisfying* $f(x_0) = 0$. (*Hence, by Lemma* 1, *all solutions* $x = x(t) \not\equiv x_0$ *of* (1) *satisfy* (5).)

Markus and Yamabe **(8)**[1] prove a result concerning solutions of (1) in which it is assumed that f satisfies (8), but (7) is replaced by the stronger condition

$$\int^{\infty}\left[\exp(-\epsilon)\int_0^t \lambda(u)du\right]dt < \infty \qquad \text{for all } \epsilon > 0.$$

Although their assumption is stronger than (7), their conclusion is apparently weaker than (5), since they did not notice (3) or that (1) has a *unique* stationary point. For weaker versions of (I) in the case that M^n is Euclidean space E^n (or the vector space R^n with a constant positive definite metric tensor $G = (g_{jk})$), see **(3)**, **(4)**, **(5)**.

If the proof of (I) is combined with that of Lemma 1, there results the estimate

$$d(x(t), x_0) \leqslant d(x(S), x_0)e^{-\lambda(c)(t-S)}$$

for $t \geqslant S$ if $x(t)$ is defined for $t = S$. In this inequality, $c = d(x_0) + d(x(S), x_0)$.

It turns out that most of the assertions of (I) remain valid if (8) is relaxed to

(9) $$e_{jk}f^jf^k \leqslant -\lambda(d(x))g_{jk}f^jf^k.$$

(Ia) *Assume all conditions of* (I) *except that* (8) *is replaced by* (9). *Then:*

(i) *any solution* $x = x(t)$ *of* (1) *defined at* $t = 0$ *exists for* $t \geqslant 0$;

(ii) *the limit* $x(\infty) = \lim x(t)$, $t \to \infty$, *exists and is a stationary point*, $f(x(\infty)) = 0$;

(iii) *if* $x(t) \not\equiv x(\infty)$ *and*

$$v(t) = (g_{jk}f^jf^k)^{\frac{1}{2}} \text{ at } x = x(t),$$

then $v(t) \downarrow 0$, $t \to \infty$;

(iv) *the set of stationary points* $x = x_0$ *of* $f(x)$ *is connected*; *hence*,

(v) *if the stationary points* $x = x_0$ *of* $f(x)$ *are isolated* (*for example, if* $\det(e_{jk}(x_0)) \neq 0$ *whenever* $f(x_0) = 0$), *then* $f(x)$ *has a unique stationary point* $x = x_0$ (*so that* $x(\infty) = x_0$ *is independent of the particular solution* $x(t)$).

The proof of (Ia) gives the following improvements of (i)–(iii): a solution $x = x(t)$, $t \geqslant 0$, of (1) has the *a priori* bound

(10) $$d(x(t)) \leqslant c, \text{ where } c = L_1(L(d(x(0))) + v(0))$$

and $w = L_1(r)$ is the inverse function of

(11) $$L(w) = \int_0^w \lambda(r)dr;$$

also, $d(x(t)) \leqslant c$ implies

[1]Added in proof. See also Osaka Math. J., *12* (1960), 305–317.

(12) $$0 \leqslant v(t) \leqslant v(0)e^{-\lambda(c)t} \text{ for } t \geqslant 0$$

and

(12′) $$d(x(t), x(\infty)) \leqslant (v(0)/\lambda(c))e^{-\lambda(c)t} \text{ for } t \geqslant 0.$$

If condition (7) does not hold, but the initial point $x = x(0)$ of a particular solution $x = x(t)$ is such that the definition of c in (10) is meaningful, then assertions (i)–(iii) are valid for this $x = x(t)$.

The following example shows the need for the additional hypothesis in part (v) of (Ia): Let $n = 2$ and $M = E^2$ be the Euclidean plane with co-ordinates $x = (x^1, x^2)$. The system of differential equations $x' = -(x^1, 0)$ satisfies the analogue of (9) with $\lambda(r) \equiv 1$. The stationary points of this system form the line $x^1 = 0$. The general solution is

$$x = (x_0^1e^{-t}, x_0^2) \to (0, x_0^2)$$

and

$$v(t) = |x_0^1|e^{-t} \downarrow 0, \text{ as } t \to \infty.$$

Lemma 1 implies the following statement for the case that $M = M^n$ is the Euclidean space E^n.

LEMMA 1′. *Let $f(x)$ be of class C^1 on E^n and let $J(x) = (\partial f/\partial x)$ be the Jacobian matrix of f. Let $J(x) < 0$ for all $x \neq x_0$, where x_0 is a stationary point, $f(x_0) = 0$. Then every solution $x = x(t) \not\equiv x_0$ of (1) satisfies $|x(t) - x_0| \downarrow 0$, as $t \to \infty$.*

The following is a corollary of (I) when $M = E^n$ is Euclidean space.

(I′) *Let a map $T: E^n \to E^n$ be given by $y = f(x)$, where $f(x)$ is of class C^1 on E^n, and let $J(x) = (\partial f/\partial x)$. If $J(x) \leqslant -\lambda(|x|)I$, where $\lambda = \lambda(r)$ is as in (I), then T is one-to-one and onto. (Hence all solutions of (1) satisfy $|x(t) - x_0| \downarrow 0$ as $t \to \infty$, where $x = x_0$ is the unique point satisfying $f(x_0) = 0$.)*

It is clear that $J \leqslant -\lambda(|x|)I$ does not imply that T is onto (even in the case $n = 1$) if (7) fails to hold.

2. Proof of Lemma 1. Let $x = x(t; x_1)$ be the unique solution of (1) satisfying the initial condition $x(0; x_1) = x_1$. Let $x_1(t) = x(t; x_1)$ and $x_2(t) = x(t; x_2)$, where x_1, x_2 are distinct arbitrary points of M. Suppose that $x_1(t)$ exists on a closed interval $[0, T]$ where $T > 0$. Let $x = z(u)$, where $0 \leqslant u \leqslant d = d(x_1, x_2)$, be a geodesic of minimal length satisfying $z(0) = x_1$ and $z(d) = x_2$. Finally, let $x = x(t, u) = x(t; z(u))$ be the solution of (1) determined by $x(0, u) = z(u)$.

Let ϵ have the property that if $0 \leqslant u < \epsilon \leqslant d$, then $x(t, u)$ is defined for $0 \leqslant t \leqslant T$. In any case, $x(t, \epsilon)$ exists on some interval $[0, S]$. Let $L(t)$ denote the length of the curve $x = x(t, u)$, where $0 \leqslant u \leqslant \epsilon$, for a fixed t, $0 \leqslant t \leqslant S$. Then

(13) $$L(t) = \int_0^\epsilon (g_{jk}(x)y^jy^k)^{\frac{1}{2}}du,$$

where $x = x(t, u)$ and $y = \partial x(t, u)/\partial u$.

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By (1), y is a solution of

$$y' = J(x)y, \text{ where } J(x) = (\partial f/\partial x), \tag{14}$$

$x = x(t, u)$, and u is fixed. Note that $y(0, u) = \partial x(0, u)/\partial u = \partial z/\partial u \neq 0$; hence $y(t, u) \neq 0$. By (13) and (14), $L' = dL/dt$ is the integral of the product of $\frac{1}{2}(g_{jk}(x)y^j y^k)^{-\frac{1}{2}}$ and of $(g_{jk}y^j y^k)'$. This last factor is

$$(\partial g_{jk}/\partial x^m)f^m y^j y^k + 2g_{jk}y^j(\partial f^k/\partial x^m)y^m.$$

If $[ij, k]$ denotes the Riemann–Christoffel symbol of the first kind, then $\partial g_{jk}/\partial x^m = [jm, k] + [km, j]$ and $\partial f^k/\partial x^m = f^k{}_{,m} - g^{ik}[jm, i]f^j$. Hence, the expression in the last formula line is $2g_{jk}f^k{}_{,m}y^k y^m$; that is,

$$(g_{jk}y^j y^k)' = 2e_{jk}y^j y^k.$$

Thus, $L'(t) < 0$ for $0 \leqslant t \leqslant S$, so that $L(t) < L(0) = d(x_1, z(\epsilon))$.

Since $d(x_1(t), x(t, \epsilon)) \leqslant L(t)$,

$$d(x_1(t), x(t, \epsilon)) < d(x_1, z(\epsilon)) \tag{15}$$

for $0 < t \leqslant S$. Clearly, (15) implies that the solution $x = x(t, \epsilon)$ of (1) exists for $0 \leqslant t \leqslant T$. Hence $x(t, u)$ exists for $0 \leqslant t \leqslant T$ for each fixed u, $0 \leqslant u \leqslant d$. In particular, $x = x_2(t) = x(t, d)$ exists for $0 \leqslant t \leqslant T$.

If the point x_2 in the last argument is chosen to be $x_2 = x_1(T)$, so that $x_2(t) \equiv x(t + T; x_1)$, it follows that $x_1(t)$ exists for $0 \leqslant t \leqslant 2T$. Repetitions of this argument show that $x_1(t)$ is defined for all $t \geqslant 0$. Since x_1 is an arbitrary point of M, the first assertion of Lemma 1 follows. The second assertion (3) follows from the case $\epsilon = d$ of (15).

As to the third assertion, let $x = x(t) \not\equiv x_0$ be defined for $t \geqslant 0$. Then, by (3), $d_0 = \lim d(x_0, x(t))$ exists as $t \to \infty$. Suppose, if possible, that (5) does not hold, so that $d_0 > 0$. Then there are t-values $t_1 < t_2 < \ldots.$ such that $t_m \to \infty$ and $x_1 = \lim x(t_m)$ exists, as $m \to \infty$. Clearly, $d(x_1, x_0) = d_0 > 0$. Let $x = x_m(t) = x(t - t_m; x_1)$ be the solution of (1) determined by the initial condition $x_m(t_m) = x_1$. Then $d(x_m(t), x_0) < d_0$ for $t > t_m$. The continuous dependence of solutions on initial conditions implies, therefore, that $d(x(t_m + 1), x_0) < d_0$ for large m. But this contradicts $d_0 < d(x(t), x_0) \to d_0$, $t \to \infty$. Thus Lemma 1 is proved.

3. Proof of (**I**)–(**Ia**). Let $x = x(t)$ be a solution defined at $t = 0$ and let $y = x'(t)$. Then $y = y(t)$ satisfies the linear equation (14), where $x = x(t)$. Consider the speed

$$v(t) = (g_{ik}y^i y^k)^{\frac{1}{2}}, \text{ where } y = x' = f. \tag{16}$$

It follows that $dv^2/dt = 2e_{km}(x)y^k y^m$; see the calculation of $L'(t)$ in § 2. Thus (8) or (9) implies $dv^2/dt \leqslant -2\lambda(d(x))v^2$ or, since $v \geqslant 0$,

$$v' \leqslant -\lambda(d(t))v, \text{ where } d(t) = d(x(t)). \tag{17}$$

Define a function $w = w(t)$ by

$$w(t) = d(0) + \int_0^t v(s)ds. \tag{18}$$

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By the definition of distance on M and the triangular inequality,

$$(19) \qquad d(t) \leqslant w(t),$$

and so, by the monotony of λ, $\lambda(d(t)) \geqslant \lambda(w(t))$. Since $w' = v \geqslant 0$ and $w'' = v'$, (17) implies that

$$w''(t) \leqslant -\lambda(w(t))w'(t).$$

Hence

$$w'(t) \leqslant w'(0) - \int_{w(0)}^{w(t)} \lambda(w)dw.$$

In view of $w' = v$ and the definition of $L(w)$ in (11), this can be written as

$$v(t) \leqslant v(0) + L(d(0)) - L(w(t)).$$

Since $v(t) \geqslant 0$, (19) implies that

$$L(d(t)) \leqslant L(w(t)) \leqslant L(d(0)) + v(0).$$

This shows that $x = x(t)$ is defined for all t and satisfies (10).

By (10) and (17), $v' \leqslant -\lambda(c)v \leqslant 0$ for $t \geqslant 0$. Hence (12) holds and either $v(t) \equiv 0$ or $v(t) \downarrow 0$ as $t \to \infty$. Thus if $x = x_0$ is any cluster point of $x = x(t)$, $t \to \infty$, then (16) shows that $f(x_0) = 0$. In view of Lemma 1, this completes the proof of (I).

Also assertions (i), (iii) of (Ia) have been proved. The definition (16) of $v(t)$ and the inequality (12) show that the length of the curve $x = x(t)$, $0 \leqslant t < \infty$, is finite,

$$\int_0^\infty (g_{jk}(x(t))x^{j'}(t)x^{k'}(t))^{\frac{1}{2}}dt = \int_0^\infty v(t)dt < \infty.$$

This implies (ii) in (Ia).

Since (v) follows from (iv), it only remains to prove (iv). The verification of (iv) to follow can be modified to show that the set of stationary points of (1) is a retract of M.

In order to prove (iv), let Q be the set of stationary points of (1). Consider a map $P: M \to Q$ defined as follows: if $x = x(t)$ is an arbitrary solution of (1) for $t \geqslant 0$, put $Px(0) = x(\infty)$. It is clear that the range of P is the set Q. Since M is connected, it will follow that Q is connected if it is verified that P is continuous.

To this end, let x_1 be any point of M and M_δ the sphere $d(x_1, x) \leqslant \delta$. The proof of the existence of $x(\infty)$ above shows that if $\epsilon > 0$, then there exists a $T = T(\epsilon) > 0$ independent of δ, $0 < \delta \leqslant 1$, with the property that if $x(0)$ is in M_δ, then $d(x(T), x(\infty)) < \epsilon$; cf. (12′). With $T = T(\epsilon)$ fixed, choose a positive $\delta = \delta(\epsilon) \leqslant 1$ so small that $d(x_1(T), x(T)) < \epsilon$ if $x = x_1(t), x(t)$ are solutions of (1) determined by $x_1(0) = x_1$ and any point $x(0)$ of M_δ, respectively. Thus $x(0)$ in M_δ implies that $d(x_1(\infty), x(\infty)) < 3\epsilon$. This proves the continuity of P and completes the proof of (iv) and of (Ia).

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4. On flat metrics. The proofs of Lemma 1 and (I) are particularly simple if M^n is a real n-dimensional vector space with a metric $G = ||g_{jk}||$, where G is a constant, symmetric, positive definite matrix. If $J[s] = J(x_2 s + x_1 (1 - s))$, then

$$f(x_2) - f(x_1) = \left(\int_0^1 J[s]ds\right)(x_2 - x_1).$$

Hence, for any constant matrix G,

$$(x_2 - x_1)\cdot G(f(x_2) - f(x_1)) = \int_0^1 (x_2 - x_1)\cdot GJ[s](x_2 - x_1)ds.$$

For example, if $GJ < 0$ and $x_1 \neq x_2$, then the integral is negative so that the map $T: M^n \to E^n$ given by $y = f(x)$ is one-to-one.

If $f(x_0) = 0$, then (1) can be written as $(x - x_0)' = f(x) - f(x_0)$. Hence $GJ < 0$ implies

$$(x - x_0)'\cdot G(x - x_0) = \int_0^1 (x - x_0)\cdot GJ[s](x - x_0)ds < 0, \tag{20}$$

for $x \neq x_0$. A simple direct proof of Lemma 1′ follows at once from this.

The equation in (20) does not seem to have been exploited in the study of stability; cf. the comparatively complicated proof in **(1)**, pp. 31-32, of the result of Krasovski which results if $GJ < 0$ is replaced by the stronger assumption $GJ(x) < -\epsilon I < 0$ and (5) by the weaker assertion $x(t) \to x_0$, $t \to 0$.

Another application of (20) will be given for non-autonomous systems in (II′) in the next section.

5. Non-autonomous systems. The results above can be generalized somewhat to systems in which t occurs explicitly,

$$x' = f(t, x). \tag{21}$$

Below it will be assumed that

(B) M, $g_{ik}(x)$, $d(x, y)$, $d(x)$ are as in (A). $f(t, x)$ is a C^1 contravariant vector field on M for every fixed $t \geqslant 0$; also f and its derivatives along M are continuous in (t, x).

The techniques of § 2 above (cf. **(7)**, **(9)**, **(10)**) imply the following analogue of Lemma 1.

LEMMA 2. *Assume* (B). *Let x_0 be a point of M satisfying*

$$f(t, x_0) = 0 \text{ for } t \geqslant 0. \tag{22}$$

Let the tensor $e_{jk}(t, x) = g_{jm}(x) f^m{}_{,k}(t, x)$ satisfy

$$(e_{jk}(t, x)) \leqslant 0 \ [or \ < 0]. \tag{23}$$

Then all solutions $x = x(t)$ of (21) *exist for large t, and $d(x_0, x(t))$ is non-increasing [or decreasing]. If, in addition, for every $c > 0$, there is a non-negative function $\mu(t) = \mu_c(t)$ defined for $t \geqslant 0$ and satisfying*

$$\int^{\infty} \mu(t)dt = \infty \tag{24}$$

and

$$(e_{jk}(t, x)) \leqslant - \mu_c(t)(g_{jk}(x)) \text{ for } t \geqslant 0 \text{ and } d(x) \leqslant c, \tag{25}$$

then every solution $x = x(t)$ *of* (1) *satisfies*

$$d(x_0, x(t)) \to 0 \text{ as } t \to \infty. \tag{26}$$

For the case that $\mu_c(t) > 0$ is independent of c and t, and $g_{jk}(x)$ is independent of x, see **(20)**, and Winter **(10)**; also Krasovski, see **(1**, p. 31**)**.

The obvious way to generalize (I) from (1) to (21) is to require an analogue of (7), (8) for a monotone $\lambda(r)$ and to assume that the length of f, $(g_{jk}(x) f^j(t, x)f^k(t, x))^{\frac{1}{2}}$, is a non-increasing function of t for every fixed x on M. But if, for example, $e_{jk}(t, x)$ satisfies

$$(e_{jk}(t, x)) \leqslant - \lambda(d(x))(g_{jk}(x)) \tag{27}$$

at $t = 0$, where $\lambda = \lambda(r)$ is as in (I), then it follows that there is a unique $x = x_0$ satisfying $f(0, x_0) = 0$. Thus, when the length of f is a non-increasing function of t, one has trivially that $f(t, x_0) = 0$ for $t \geqslant 0$.

A different generalization of (I) is given by

(II) *Assume* (B) *and that* $f_t = \partial f/\partial t$ *exists and is continuous in* (t, x). *Let* $\lambda(r)$ *be as in* (I) *and put*

$$L(w) = \int_0^w \lambda(r)dr. \tag{28}$$

Let $a(t)$ *be a non-negative, continuous function integrable over* $0 \leqslant t < \infty$,

$$A = \int_0^{\infty} a(t)dt < \infty. \tag{29}$$

Let $N(w)$ *be a non-decreasing function of* w *for* $w \geqslant 0$ *satisfying*

$$L(w) - AN(w) \to \infty \text{ as } w \to \infty. \tag{30}$$

Assume that $e_{jk}(t, x)$ *satisfies* (9) *and that the length of* $f_t(t, x)$ *satisfies*

$$0 \leqslant [g_{jk}(x)f^j{}_t(t, x)f^k{}_t(t, x)]^{\frac{1}{2}} \leqslant a(t)N(d(x)) \tag{31}$$

for $t \geqslant 0$ *and* x *in* M. *Then:*

(i) *the limit* $f(x) = \lim f(t, x)$, $t \to \infty$, *exists uniforml yon compact subsets of* M;

(ii) *every solution* $x = x(t)$ *of* (21) *exists for large* t *and tends to a limit point* $x(\infty)$ *which satisfies* $f(x(\infty)) = 0$;

(iii) *the function*

$$v(t) = (g_{jk}x^{j\prime}x^{k\prime})^{\frac{1}{2}}$$

tends to 0 *as* $t \to \infty$;

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(iv) *if, in addition, there is a positive function* $\nu = \nu(c)$ *for* $c > 0$ *such that*

$$(e_{jk}(t, x)) \leqslant -\nu(c)(g_{jk}(x)) \text{ for } t \geqslant 0,\ d(x) \leqslant c,$$

then the limit function $f(x)$ *has a unique zero* $x = x_0$ *(so that* $x(\infty) = x_0$ *does not depend on the solution* $x = x(t)$*).*

The proof will furnish *a priori* bounds for $d(x(t))$ and *a priori* estimates for the $o(1)$-functions $d(x(t), x(\infty))$ and $v(t)$ depending only on the initial conditions for $x = x(t)$.

One of the main difficulties in the proof of (iv) in (II) is the fact that the limit function $f(x)$ need not be of class C^1 or even Lipschitz continuous, so that, *a priori*, it is not clear that the solutions of $x' = f(x)$ are locally unique. Local uniqueness will be proved by the use of a theorem of van Kampen **(2)**. In any case, the assertion (iv) in (II) cannot be obtained from (Ia).

If all assumptions of (II) hold except (30) and if $N(w) \leqslant \text{const.}\, L(w)$, then (II) becomes applicable when $0 \leqslant t < \infty$ is replaced by $T \leqslant t < \infty$ for a sufficiently large T (since A is then replaced by an arbitrarily small constant).

Under the assumptions of (II), it follows that $f(t, x)$ is a bounded function of t for fixed x. This suggests the following:

(II′) *Assume* (B) *and that* $M = E^n$. *Let* G *be a positive definite, constant matrix and* $\lambda = \lambda(r)$ *a positive, non-increasing function of* $r(\geqslant 0)$. *Suppose that* α, β *are positive constants satisfying* $\alpha^2 I \leqslant G \leqslant \beta^2 I$, *that*

$$GJ(t, x) \leqslant -\lambda(|x|)I, \tag{32}$$

that $f(t, 0)$ *is a bounded function of* $t \geqslant 0$, *and that*

$$\infty \geqslant (\alpha/\beta^3) \limsup_{r\to\infty} \lambda(r) r > \operatorname*{l.u.b.}_{0\leqslant t<\infty} |f(t, 0)|. \tag{33}$$

Then every solution $x = x(t)$ *of* (21) *exists for large* t *and is bounded as* $t \to \infty$.

It will also be clear from the proof that if, in addition, either $f(t, 0) \to 0$ as $t \to \infty$ or

$$\int_0^\infty |f(t, 0)| dt < \infty, \tag{34}$$

then $|x(t)| \to 0$ as $t \to \infty$. Furthermore, if conditions (32) and (33) are replaced by the assumptions $GJ(t, x) \leqslant 0$ and (34), then the conclusions of (II′) remain valid and $\lim x(t) \cdot Gx(t)$ exists as $t \to \infty$; cf. (47) below.

6. Proof of (II). The first part of the proof of (II) is similar to that of (I). Let $x = x(t)$ be a solution of (21) on some interval $(0 \leqslant)\ S \leqslant t \leqslant T$. Define $v = v(t)$ by (16). Then

$$(v^2)' = 2e_{km}(t, x) y^k y^m + 2g_{jk} f^j f^k{}_t.$$

By Schwarz's inequality,

$$|g_{jk}f^j f^k{}_t| \leqslant |g_{jk}f^j f^k|^{\frac{1}{2}}\, |g_{jk}f^j{}_t f^k{}_t|^{\frac{1}{2}}.$$

Thus, by (16), (9), and (31),

(35) $\qquad v' \leqslant -\lambda(d(t))v + a(t)N(d(t))$, where $d(t) = d(x(t))$.

Define $w = w(t)$ by (18), so that (19) holds and

$$w'' \leqslant -\lambda(w)w' + a(t)N(w).$$

A quadrature over $[S, t]$ gives

$$w'(t) \leqslant C - L(w(t)) + AN(w(t)),$$

where $C = w'(S) + L(w(S))$ and the justification for the last term is the fact that $N(w)$, $w(t)$ are non-decreasing in w, t, respectively.

Since $w' = v \geqslant 0$, it is clear from (30) that there does not exist any $T_0 (< \infty)$ such that $w(t) \to \infty$ as $t \to T_0 - 0$. Hence $x(t)$ exists for all $t \geqslant S$ and is bounded; in fact, $d(x(t)) \leqslant c$ for $t \geqslant S$ if $L(c) - AN(c) > C$.

Let $d(t) \leqslant c$ for $t \geqslant S$, then (35) gives

$$v' \leqslant -\lambda(c)v + N(c)a(t).$$

Hence, for $t \geqslant S$,

(36) $$0 \leqslant v(t) \leqslant v(S)e^{-\lambda(c)(t-S)} + N(c)\int_S^t e^{\lambda(c)(s-t)}a(s)ds,$$

so that (29) implies $v(t) \to 0$ as $t \to \infty$. Thus, the definition of v shows that

(37) $$f(t, x(t)) \to 0 \text{ as } t \to \infty.$$

Integrating (36) for $S \leqslant t \leqslant T$ gives

$$\int_S^T vdt \leqslant (v(S)/\lambda(c))(1 - e^{-\lambda(c)(T-S)}) + N(c)\int_S^T e^{-\lambda(c)t}\int_S^t e^{\lambda(c)s}a(s)dsdt.$$

An integration by parts shows that the last (iterated) integral is $1/\lambda(c)$ times

$$-e^{-\lambda(c)T}\int_S^T e^{-\lambda(c)t}a(t)dt + \int_S^T a(t)dt;$$

hence

$$\lambda(c)\int_S^\infty vdt \leqslant v(S) + N(c)\int_S^\infty a(t)dt < \infty.$$

Consequently, $x(\infty) = \lim x(t)$, $t \to \infty$, exists and satisfies

$$\lambda(c)d(x(t), x(\infty)) \leqslant v(t) + N(c)\int_t^\infty a(s)ds.$$

The assertion (i) of (II) concerning the existence and uniformity of the limit $f(x) = \lim f(t, x)$, $t \to \infty$, is clear from (29) and (31). Furthermore, (37) implies that $f(x(\infty)) = 0$. Thus (i)–(iii) are proved.

In order to prove (iv), it is sufficient to verify the following:

(*) *Assume the conditions of* (II), *including those of* (iv) *concerning* $\nu = \nu(c)$. *Let* p *denote a point of* M. *Then solutions of*

$$p' = f(p) \tag{38}$$

are uniquely determined by initial conditions; all solutions exist for large $t > 0$; *and* $d(p_1(t), p_2(t))$ *is a decreasing function of* t *if* $p = p_1(t), p_2(t)$ *are distinct solutions on a common* t*-interval.*

To this end, let $x = x_1(t)$ and $x = x_2(t)$ be two distinct solutions of (21) for $t \geqslant S$. Let $z = z(u)$, where $0 \leqslant u \leqslant d$, be a geodesic of minimal length joining $x = x_1(S), x_2(S)$ and let $x = x(t, u)$ be the solution of (21) determined by $x(S, u) = z(u)$. As in § 2, define $L(t) = L_\epsilon(t)$ by (13) for $t \geqslant S$, $0 \leqslant \epsilon \leqslant d$. Then $(e_{jk}(t, x)) \leqslant 0$ implies that $L_\epsilon(t)$ is a non-increasing function of t. Since $x = x_1(t)$ is bounded as $t \to \infty$, it follows that there exists a constant $c > 0$ such that $d(x(t, u)) \leqslant c$ for $t \geqslant S$, $0 \leqslant u \leqslant d$. Hence, by (27),

$$dL_\epsilon(t)/dt \leqslant -\nu(c)L_\epsilon(t) \quad \text{for } t \geqslant S;$$

cf. the derivation of $L'(t) < 0$ in § 2. Since $d(x_1(t), x_2(t)) \leqslant L_d(t)$,

$$d(x_1(t), x_2(t)) \leqslant d(x_1(S), x_2(S))e^{-\nu(c)(t-S)} \tag{39}$$

for $t \geqslant S$.

Let M_1 be a bounded (open) set of M. Consider the family of solutions $x = x(t; t_0, x_1)$ of (1) determined by the initial condition $x(t_0; t_0, x_1) = x_1$, where $t_0 \geqslant 0$ and x_1 is a point of M_1. Then the derivation of (39) shows that there is a constant $c = c(M_1)$ such that $d(x(t; t_0, t_1)) \leqslant c$ for $t \geqslant t_0 \geqslant 0$. Hence (39) holds for $t_0 \leqslant S \leqslant t < \infty$ if $x_1(t) = x(t; t_0, x_1)$, $x_2(t) = x(t; t_0, x_2)$, and x_1, x_2 are points of M_1.

Let $y = \partial x(t; t_0, x_1)/\partial u$, where $u \neq t_0$ is one of the parameters determining the solution $x = x(t; t_0, x_1)$. Then the length of y, $(g_{jk}y^j y^k)^{\frac{1}{2}}$, is a decreasing function of t $(\geqslant t_0)$; cf. the derivation of (39). In particular, $y(t; t_0, x_1)$ is uniformly bounded for $t \geqslant t_0$ and x_1 in M. Consequently, $x(t; t_0, x_1)$ is uniformly bounded and uniformly Lipschitz continuous with respect to t and x_1 for $t \geqslant t_0 \geqslant 0$ and x_1 in M_1.

It follows that there is a sequence of t-values $t_1 < t_2 < \ldots$ such that $t_n \to \infty$ and

$$p(t; x_1) = \lim_{n\to\infty} x(t + t_n; t_n; x_1) \tag{40}$$

exists uniformly for x_1 in M_1 and bounded $t \geqslant 0$. Furthermore, (40) is uniformly Lipschitz continuous with respect to x_1 in M_1 for $t \geqslant 0$. Note that $p_n = x(t + t_n; t_n, x_1)$ is a solution of the initial value problem

$$p_n' = f(t + t_n, p_n), \quad p_n(0) = x_1.$$

Hence (40) is a solution of

$$p' = f(p) \text{ and } p(0) = x_1. \tag{41}$$

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Also, an obvious limit process in (39) shows that

(42) $$d(p(t; x_1), p(t; x_2)) \text{ is decreasing in } t \text{ if } x_1 \neq x_2,$$

$t \geqslant 0$ and x_1, x_2 in M_1.

Through any point of M, there passes at most one path of the family $x = p(t; x_1)$; that is,

(43) $$p(t + s; x_1) = p(t; p(s; x_1)).$$

In order to prove this, let $y = y(t; t_0, x_1)$ be defined by

(44) $$y = dx(t + t_0; t_0, x_1)/dt_0.$$

Then $y = x' + \partial x/\partial t_0$, and so

$$y' = x'' + J(\partial x/\partial t_0) = Jx' + f_t + J(\partial x/\partial t_0).$$

Consequently,

(45) $$y' = Jy + f_t, \qquad y(0) = 0,$$

where the argument of J and f_t is $(t + t_0, x(t + t_0; t_0, x_1))$. The condition $y(0) = 0$ in (45) is clear from $x(t_0; t_0, x_1) = x_1$. If $Y = (g_{jk}y^jy^k)^{\frac{1}{2}}$ is the length of y, then

$$(Y^2)' = 2e_{jk}y^jy^k + 2g_{jk}y^jf^k{}_t.$$

The derivation of (35) shows that

$$Y' \leqslant -\lambda(c)Y + N(c)a(t + t_0) \leqslant N(c)a(t + t_0).$$

Since $Y(0) = 0$,

$$Y(t) \leqslant N(c)\int_{t_0}^{t+t_0} a(s)ds \to 0 \text{ as } t_0 \to \infty.$$

As Y is the length of y, (44) implies that

(46) $$x(t + s + t_0; s + t_0, x_1) - x(t + t_0; t_0, x_1) \to 0 \text{ as } t_0 \to \infty$$

uniformly for x_1 in M_1 and bounded $s, t \geqslant 0$. The relation

$$x(t + s + t_0; t_0, x_1) = x(t + s + t_0; s + t_0, x(s + t_0; t_0, x_1)),$$

the uniform Lipschitz continuity of $x(t; t_0, x_1)$ with respect to x_1, and (40) give

$$x(t + s + t_n; t_n, x_1) = x(t + t_n; t_n, p(s; x_1)) + o(1),$$

where $o(1) \to 0$ as $n \to \infty$ uniformly for x_1 in M_1 and bounded $s, t \geqslant 0$. The equation (43) for $s, t \geqslant 0$ follows from this. Clearly, (43) is valid for those s, t for which the quantities in (43) are meaningful.

A theorem of van Kampen **(2)** implies that (41) has a unique solution locally. The conditions of van Kampen's theorem are that $f(p)$ is continuous; that (38) possesses a family of solutions $p = p(t; x_1)$, where $p(0; x_1) = x_1$ and $p(t; x_1)$ is defined on an open interval which can depend on x_1; that $p(t; x_1)$ is locally, uniformly Lipschitz continuous with respect to x_1; finally, that

(43) holds whenever the quantities in (43) are meaningful. The conclusion is that (41) has a unique solution locally.

This uniqueness assertion, together with (42), gives assertion (*). Hence the proof of (II) is complete.

Remark. If $\nu(c) = 0$ is permitted, (*) remains valid if "$d(p_1(t), p_2(t))$ is a decreasing" is replaced by "$d(p_1(t), p_2(t))$ is a non-increasing."

7. Proof of (II′). Let $x = x(t)$ be a solution of (21) defined at $t = S$. Write (21) as

$$x' = [f(t, x) - f(t, 0)] + [f(t, 0)].$$

Then an analogue of (20) is

$$x \cdot Gx' = \int_0^1 x \cdot GJ[s]x ds + x \cdot Gf(t, 0),$$

where $J[s] = J(t, xs)$. Thus, if $r^2 = x \cdot Gx$, it follows from $\alpha|x| \leqslant r \leqslant \beta|x|$, (32) and the monotony of x, that

$$r' \leqslant [-(r/\alpha)(\alpha/\beta^3)\lambda(r/\alpha) + |f(t, 0)|]\beta. \tag{47}$$

Assumption (33) implies that there exists a constant R such that

$$R > r(S) \text{ and } (R/\alpha)(\alpha/\beta^3)\lambda(R/\alpha) > |f(t, 0)| \text{ for } t \geqslant S.$$

Since $r(S) < R$, it is clear from (47) that $x = x(t)$ exists and that $r(t) < R$ for $t \geqslant S$.

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