COMPOSITIO MATHEMATICA

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Compositio Math. **159** (2023), 1153–1187.

doi:10.1112/S0010437X2300711X

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY
EST. 1865

Compositio Math. 159 (2023) 1153–1187
doi:10.1112/S0010437X2300711X

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Abstract

We formalize the concepts of holomorphic affine and projective structures along the leaves of holomorphic foliations by curves on complex manifolds. We show that many foliations admit such structures, we provide local normal forms for them at singular points of the foliation, and we prove some index formulae in the case where the ambient manifold is compact. As a consequence of these, we establish that a regular foliation of general type on a compact algebraic manifold of even dimension does not admit a foliated projective structure. Finally, we classify foliated affine and projective structures along regular foliations on compact complex surfaces.

1. Introduction

For a one-dimensional holomorphic foliation on a complex manifold, a *foliated projective structure* is a family of complex projective structures along the leaves of the foliation that vary holomorphically in the transverse direction. Particular cases of such structures are foliated translation structures, corresponding to global holomorphic vector fields tangent to the foliation and vanishing only at its singular points, and foliated affine structures, which are a key tool in the study of holomorphic vector fields without multivalued solutions carried out in [GR12]. Some interesting families of foliated projective structures are the isomonodromic foliations on moduli spaces of branched projective structures [Vee93, CDF14, McM14, GP17]; foliated projective structures also appear prominently in Zhao's classification of birational Kleinian groups [Zha21]. As we will see, there are plenty more of examples, and it seems that a theory deserves to be developed. The aim of this article is to begin a systematic study both of these structures and of the closely related affine ones. It concerns chiefly the problems of the existence of such structures on compact foliated manifolds, of their local description at the singular points of the foliation, and of the relations of their local invariants with the global topology of the foliation and the manifold.

On a manifold of dimension n, a foliation $\mathcal{F}$, in a neighborhood of a singular point p, may be defined by a vector field Z with singular set of codimension at least two, unique up to multiplication by a nonvanishing holomorphic function; the projectivization of the linear part of Z at p is a local invariant of $\mathcal{F}$, but the linear part in itself is not. In the presence of a generic foliated projective structure, a distinguished linear part of a vector field tangent to $\mathcal{F}$ at p may be defined up to sign: the eigenvalues of this linear part become the *ramification indices* $\nu_1, \ldots, \nu_n$ of the structure at p (§ 3.2.2); they are well-defined up to ordering and up to a simultaneous

Received 27 April 2021, accepted in final form 29 August 2022, published online 15 May 2023.
2020 Mathematics Subject Classification 32M25, 57M50, 53C12 (primary), 32S65 (secondary).
Keywords: holomorphic foliation, projective structure, affine structure, Kodaira fibration.

change of sign (the ambiguity of the sign can be lifted for foliated affine structures). When the vector field is nondegenerate and linearizable with semisimple linear part, these indices encode the 'cone angles' induced by the foliated projective structure on each one of the n separatrices. In Theorem 3.7 we prove that, generically, a foliated projective structure in the neighborhood of a singular point is determined by these indices.

Our main result is an index theorem which concerns the global properties of foliated projective structures, assuming that the ambient manifold M is compact and that the foliated projective structure satisfies a nondegeneracy condition at the singular points. Theorem 5.1 affirms that, given a symmetric homogeneous polynomial $\varphi(x_1, \ldots, x_{n+1})$ of degree $n+1$, if φ_{odd} denotes the odd part of φ in the variable x_{n+1}, the quantity

$$\sum_{p \in \mathrm{Sing}(\mathcal{F})} \frac{\varphi_{\mathrm{odd}}(\nu_1, \ldots, \nu_n, 1)}{\nu_1 \ldots \nu_n}$$

can be expressed as an explicit polynomial in the Chern classes of TM and of the tangent bundle of the foliation, $T_{\mathcal{F}}$. Some instances of our result are (Examples 5.2 and 5.3): if n, the dimension of M, is odd, $n = 2k+1$,

$$\sum_{p \in \mathrm{Sing}(\mathcal{F})} \frac{\nu_1 + \cdots + \nu_n}{\nu_1 \cdots \nu_n} = c_1^{2k}(T_{\mathcal{F}}) c_1(TM - T_{\mathcal{F}}); \tag{1.1}$$

and if it is even,

$$\sum_{p \in \mathrm{Sing}(\mathcal{F})} \frac{1}{\nu_1 \cdots \nu_n} = c_1^{n}(T_{\mathcal{F}}). \tag{1.2}$$

This last formula only makes sense if n is even, but if the projective structure reduces to an affine one, it is also valid for n odd (Theorem 4.1). As usual, in these formulae, the left-hand side vanishes if the singular set is empty.

For all this to be of interest, we need to have a good knowledge of the foliations which admit foliated affine and projective structures.

Foliated affine structures are quite common. Some foliations can be shown to admit them almost by construction (like for the 'evident' foliations on Inoue or Hopf surfaces; see Examples 2.5 and 2.6), or because they admit a description that makes this patent, like elliptic fibrations (Example 2.8) or foliations on complex projective spaces (Example 2.4). In general, on the manifold M, for a foliation $\mathcal{F}$ with canonical bundle $K_{\mathcal{F}}$, there corresponds a class $\alpha_{\mathcal{F}}$ in $H^1(M, K_{\mathcal{F}})$ that measures the obstruction for $\mathcal{F}$ to admit a foliated affine structure (§ 2.1.3). There are situations where this cohomology group is altogether trivial; this allows, for instance, to prove that all foliations whose 'canonical bundle of the space of leaves' is ample carry a foliated affine structure (Lemma 2.9; by adjunction, given a sufficiently positive line bundle on the manifold M, every foliation on M having it for its cotangent bundle supports a foliated affine structure). Further instances of foliated affine structures may be given by constructing foliated connections on some line bundles, and propagating them to the tangent bundle of the foliation (§ 2.1.4). Following this strategy, we prove that any foliation on a Calabi–Yau manifold (Corollary 2.11) or on a generic hypersurface of $\mathbf{P}^3$ (Example 2.13) admits a foliated affine structure.

Foliated projective structures can be directly shown to exist in some cases, like on Hilbert foliations, suspensions, and turbulent foliations (Examples 2.16 and 2.17). In a way similar to that of the affine version, for a given foliation $\mathcal{F}$ there is a class $\beta_{\mathcal{F}}$ in $H^1(M, K_{\mathcal{F}}^2)$ which vanishes if and only if it admits a foliated projective structure. This obstruction may be calculated in some cases, and, by doing so, we show that every foliation in the product of a curve with $\mathbf{P}^1$ admits a foliated projective structure, while not always a foliated affine one (Proposition 2.20).

Despite these positive results, there exist foliations that do not support any foliated projective structure. For instance, Zhao proved that no Kodaira fibration (considered as a foliation) carries such a structure [Zha19]. A consequence of our index formulae is that if a compact complex surface has a regular foliation admitting a foliated projective structure, its signature vanishes (Corollary 6.1), giving an alternative proof of Zhao's result (yet another has been recently given in [EWF21]). More generally, our index theorem implies that, in even dimensions, regular foliations of general type do not support foliated projective structures (Proposition 6.3).

These results allow us to fully classify foliated affine and projective structures along regular foliations on surfaces (Corollary 6.2, § 6.2). For example, from Brunella's classification of regular foliations on surfaces [Bru97] and the previously mentioned Corollary 6.1, if a regular foliation on a surface of general type which is not a fibration admits a foliated projective structure, the surface is a quotient of the bidisk, with the foliation being either the vertical or horizontal one. (In turn, this last result of ours constitutes a key ingredient in Zhao's classification of birational Kleinian groups in dimension two [Zha21].)

There are some situations that are closely related to those discussed here, but which do not fall within the scope of this article. The structures we consider are defined on the actual manifold, and not on an infinite cover of it, as in Griffith's work on the uniformization of Zariski-open subsets of algebraic varieties [Gri71] (which uses foliated projective structures along a covering of a pencil) or in the 'covering tubes' of a foliation, as in Ilyashenko's notion of *simultaneous uniformization* (see [Il'06] and references therein). Holomorphic foliations by curves which are hyperbolic as Riemann surfaces carry naturally a leafwise hyperbolic (hence projective) structure; the hyperbolic metric varies continuously in the transverse direction [Ver87] (even in the presence of singular points [LN94, CGM95]) and, moreover, plurisubharmonically [Bru03], but the leafwise hyperbolic geometry will very seldomly give a foliated projective structure in the sense we consider here. In the real setting, objects related to those studied here have been considered, for instance, in [IM93] and [Mal02].

We assume that the reader is familiar with both the local and global theory of foliations by curves on complex manifolds, like the material covered in the first chapters of [Bru04].

2. Definitions and the problem of existence

We recall the notions of affine and projective structures on curves, and define similar notions for singular holomorphic foliations by curves. We also give various existence criteria showing that many foliations carry such structures, and examples of foliations that do not.

2.1 Foliated affine structures

An *affine structure* on a curve is an atlas for its complex structure taking values in $\mathbf{C}$ whose changes of coordinates lie within the affine group $\{z \mapsto az + b\}$.

The *affine distortion* of a local biholomorphism between open subsets of $\mathbf{C}$ is the operator

$$\mathcal{L}(f) := \frac{f''}{f'}\, dz,$$

which plays a fundamental role in the study of affine structures. It vanishes precisely when f is an affine map. A simple computation shows that, for the composition of two germs of

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biholomorphisms between open sets of $\mathbf{C}$,

$$\mathcal{L}(f \circ g) = \mathcal{L}(g) + g^*\mathcal{L}(f). \tag{2.1}$$

Hence, the affine distortion of a biholomorphism between open subsets of curves equipped with affine structures does not depend on the chosen affine charts. Given two affine structures on a curve C, the affine distortion of the identity map measured in the corresponding affine charts, namely the one-form $\mathcal{L}(\psi \circ \phi^{-1})$ for ϕ a chart of the first affine structure and ψ a chart of the second, gives a globally well-defined one-form on C which vanishes if and only if the affine structures agree. Reciprocally, given an affine structure and a one-form α on C, if α reads $a(z)\,dz$ in some affine chart of the affine structure, the maps given in this chart by the solutions ψ of $\psi'' = a\psi'$ give a second globally-defined affine structure on C. An easy consequence of (2.1) is that this provides the moduli space of affine structures on C with the structure of an affine space directed by the vector space of holomorphic one-forms on C (see also [Gun66, § 9]).

Given an affine structure on a curve, the family of vector fields which are constant in the coordinates of the affine structure is well-defined. Such a family is the one of flat sections of a holomorphic connection on the tangent bundle of C. Reciprocally, given a holomorphic connection on the tangent bundle of the curve, one can define the atlas of charts where the flat sections of the connection are constant vector fields. A change of coordinates of this atlas maps a constant vector field to another constant vector field, and hence it belongs to the affine group, thus retrieving an affine complex structure on the curve. We deduce that there is a canonical correspondence between affine structures on a curve and connections on its tangent bundle. In particular, the only compact curves admitting affine structures are elliptic ones (see [Ben60], or Theorem 4.1 in § 4). On such a curve, there is a canonical affine structure coming from its uniformization by $\mathbf{C}$, or, equivalently, by the integration of a non-identically-zero holomorphic one-form.

We will adopt both points of view in order to extend the definition of affine structures on curves to the context of unidimensional singular holomorphic foliations on complex manifolds.

2.1.1 *The foliated setting.* Let us begin by recalling that a *singular holomorphic foliation $\mathcal{F}$ of dimension one* on a complex manifold M is defined by the data of a cover by open sets $\{U_i\}_{i\in I}$ of M and a family $\{Z_i\}_{i\in I}$ of holomorphic vector fields Z_i on U_i, such that the vanishing locus of Z_i in U_i has codimension at least two, and that on the intersection $U_i \cap U_j$ of two open sets of the cover, the vector fields Z_i and Z_j are proportional, namely, $Z_i = g_{ij}Z_j$ for a function $g_{ij} : U_i \cap U_j \to \mathbf{C}^*$. The set where the vector fields vanish is the *singular set* of the foliation, denoted by $\mathrm{Sing}(\mathcal{F})$. When defined in this way, two foliations are regarded as equivalent if the subsheafs of the sheaf of sections of the tangent bundle TM of M generated by the vector fields Z_i are the same. This subsheaf is called the *tangent sheaf* of the foliation; it is locally free, and corresponds to the sheaf of sections of a holomorphic line bundle, the *tangent bundle* of the foliation, that we denote by $T_{\mathcal{F}}$. We then have a morphism $T_{\mathcal{F}} \to TM$, which vanishes only over the singular set of $\mathcal{F}$. This map completely characterizes $\mathcal{F}$, and can be used as an alternative definition of a foliation. The *canonical* bundle of the foliation is the bundle $K_{\mathcal{F}} := T_{\mathcal{F}}^*$.

A first definition of a *foliated affine structure* is the following.

Definition 2.1. Let M be a complex manifold and $\mathcal{F}$ a singular holomorphic foliation by curves on M. A *holomorphic foliated affine structure* on $\mathcal{F}$ is an open cover $\{U_i\}$ of $M \setminus \mathrm{Sing}(\mathcal{F})$ and submersions $\phi_i : U_i \to \mathbf{C}$ transverse to $\mathcal{F}$ such that, in restriction to a leaf L of $\mathcal{F}$, $(\phi_i|_L) \circ (\phi_j|_L)^{-1}$ is an affine map of $\mathbf{C}$.

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No condition is explicitly imposed on the singular set of the foliation, but Hartogs's phenomena will implicitly do so. The affine geometry of the leaves as they approach the singular set is studied in § 3.

Remark 2.2. If nonempty, *the set of foliated affine structures on the foliation $\mathcal{F}$ on the manifold M is an affine space directed by the vector space $H^0(K_{\mathcal{F}})$.* By considering a foliated version of the construction described at the beginning of § 2.1, on the restriction of $\mathcal{F}$ to $M \setminus \operatorname{Sing}(\mathcal{F})$, the difference between two foliated affine structures is a family of holomorphic one-forms along the leaves of $\mathcal{F}$ varying holomorphically in the transverse direction, this is, a section of $K_{\mathcal{F}}$ over $M \setminus \operatorname{Sing}(\mathcal{F})$, vanishing identically if and only if the affine structures coincide. By Hartogs's theorem, this section of $K_{\mathcal{F}}$ over $M \setminus \operatorname{Sing}(\mathcal{F})$ extends holomorphically to a section of $K_{\mathcal{F}}$ over M.

There are foliations without any foliated affine structure, e.g. those having a compact leaf of genus different from one. Notwithstanding, and in contrast with the scarcity of curves having affine structures, there are many foliations that support them.

Example 2.3. A holomorphic vector field with isolated singularities on a manifold of dimension at least two, e.g. a holomorphic vector field on a compact Kähler manifold [Kob72], induces a foliated affine structure whose changes of coordinates are not only affine but are actually translations (we call these *foliated translation structures*).

Example 2.4. The orbits of a homogeneous polynomial vector field on $\mathbf{C}^{n+1}$ are preserved by homotheties, and the vector field defines a foliation on $\mathbf{P}^n$; moreover, the vector field also endows this foliation with a foliated affine structure: it induces a translation structure along its phase curves, and the homotheties of $\mathbf{C}^{n+1}$ act affinely in the translation charts. As we shall see in Proposition 4.3, for a foliation $\mathcal{F}$ on $\mathbf{P}^n$ of degree strictly greater than one, every foliated affine structure on $\mathcal{F}$ may be obtained in this way: if X is a homogeneous vector field on $\mathbf{C}^{n+1}$ of degree $d > 1$ generating $\mathcal{F}$, the foliated affine structures on $\mathcal{F}$ are those induced by the vector fields of the form $X + PR$, where $R = \sum_i z_i \partial/\partial z_i$ is the radial vector field and P is a homogeneous polynomial of degree $d - 1$.

Example 2.5 (Inoue surfaces). The Inoue surfaces S_M, $S_N^{(+)}$, and $S_N^{(-)}$ are compact complex non-Kähler surfaces which are quotients of $\mathbf{H} \times \mathbf{C}$ by groups of affine transformations of $\mathbf{C}^2$ (see [Ino74]). For the surfaces S_M, the associated action preserves the two foliations of $\mathbf{H} \times \mathbf{C}$ and is affine on the leaves of both of them (with respect to the tautological affine structure on $\mathbf{C}$, to the one inherited from the inclusion $\mathbf{H} \subset \mathbf{C}$ on $\mathbf{H}$). The two foliations induced in S_M thus admit foliated affine structures. For the surfaces $S_N^{(+)}$ and $S_N^{(-)}$, the action on $\mathbf{H} \times \mathbf{C}$ preserves the foliation given by the fibers of the first factor, and acts affinely upon its leaves. The induced foliations are also endowed with foliated affine structures.

Example 2.6 (Hopf surfaces). Hopf surfaces are compact complex surfaces whose universal covering is biholomorphic to $\mathbf{C}^2 \setminus \{0\}$ (see [BHPV04, Ch. V, Section 18]). The primary ones are quotients of $\mathbf{C}^2 \setminus \{0\}$ by contractions of the form

$$(x, y) \mapsto (\alpha x + \lambda y^n, \beta y), \tag{2.2}$$

with $n \geq 1$ and $\lambda = 0$ if $\alpha \neq \beta^n$. In the *elliptic* case (when $\alpha = \beta^n$ and $\lambda = 0$), when x and y are given weights n and 1, respectively, every quasihomogeneous polynomial vector field on $\mathbf{C}^2$ is preserved by the contraction up to a constant factor. The associated foliations on the Hopf surface have thus a foliated affine structure (which may be a translation one in particular cases). In the general (nonelliptic) case, the linear diagonal vector fields $Ax\partial/\partial x + By\partial/\partial y$ ($AB \neq 0$)

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if $\lambda = 0$, or the 'Poincaré–Dulac' ones $(nx + \mu y^n)\partial/\partial x + y\partial/\partial y$ ($\mu \in \mathbf{C}$) if $\lambda \neq 0$ are preserved by the contraction, and induce nowhere-vanishing vector fields on the Hopf surface. Further, the coordinate vector field $\partial/\partial x$ is preserved up to a constant factor, and the foliation it induces has a natural foliated affine structure. By Brunella's classification [Bru97, Section 5], there are no further foliations on primary Hopf surfaces: *every foliation on a primary Hopf surface supports a foliated affine structure*.

2.1.2 *Foliated connections.* Now let us turn to a more intrinsic equivalent definition of a foliated affine structure in terms of the notion of *foliated connection*, which enables the construction of more examples.

Given a foliation $\mathcal{F}$ and a sheaf $\mathcal{S}$ of $\mathcal{O}_S$-modules, a *foliated connection on $\mathcal{S}$ relative to $\mathcal{F}$* is a differential operator $\nabla : \mathcal{S} \to \mathcal{O}(K_{\mathcal{F}}) \otimes \mathcal{S}$ which, away from the singular points of $\mathcal{F}$, satisfies the Leibniz rule

$$\nabla(fs) = d_{\mathcal{F}} f \otimes s + f\nabla s,$$

for every $f \in \mathcal{O}$ and every $s \in \mathcal{S}$, where $d_{\mathcal{F}}$ denotes the differential along the leaves of $\mathcal{F}$. (In general, we will consider $\mathcal{F}$ as fixed and omit it from the discussion.) A foliated connection on a holomorphic vector bundle is a foliated connection on its sheaf of sections. Foliated connections appear in the work of Baum and Bott under the name of *partial connections* (see [BB70, Section 3]).

In particular, after extending to the singularities of $\mathcal{F}$ via Hartogs's theorem, a foliated connection on $T_{\mathcal{F}}$ is a map $\nabla : T_{\mathcal{F}} \to T_{\mathcal{F}} \otimes K_{\mathcal{F}} = \mathcal{O}_M$ which to a vector field Z tangent to $\mathcal{F}$ assigns a holomorphic function $\nabla(Z)$, its *Christoffel symbol*, satisfying the Leibniz rule

$$\nabla(fZ) = Zf + f\nabla(Z). \tag{2.3}$$

Let us see that such a connection is equivalent to a foliated affine structure.

Given a foliated connection ∇ on $T_{\mathcal{F}}$ and $p \notin \mathrm{Sing}(\mathcal{F})$, if Z is a vector field tangent to $\mathcal{F}$ that does not vanish at p and such that $\nabla(Z) \equiv 0$ (if Z is *parallel*), if ϕ is a function such that $d\phi(Z) \equiv 1$, ϕ is part of an atlas of a foliated affine structure that depends only on ∇ (it is not difficult to see that such a Z and such a ϕ always exist).

For the other direction, let $\mathcal{F}$ be a foliation endowed with a foliated affine structure σ_0. Let Z be a vector field defined on the open set $U \subset M$, tangent to $\mathcal{F}$ (with a singular set of codimension two), and denote by σ_Z the foliated affine structure induced by Z on U. As in Remark 2.2, the difference $\sigma_Z - \sigma_0$ is a section α of $K_{\mathcal{F}}$ over U. Consider the holomorphic function $\alpha(Z)$ on U, and define a foliated connection ∇ on $T_{\mathcal{F}}$ by setting $\nabla(Z) = \alpha(Z)$. Let us verify that Leibniz's rule (2.3) takes place. We will do so locally on a curve, in a coordinate z where $Z = \partial/\partial z$. A chart for the affine structure induced by $f(z)\partial/\partial z$ is $\int^z d\xi/f(\xi)$ and, thus, $\sigma_Z - \sigma_{fZ} = -f'/f\, dz$. Hence, the contraction of $\sigma_{fZ} - \sigma_0$ with fZ yields $Zf + f\alpha(Z)$, in agreement with formula (2.3).

Observe that the definition of foliated affine structures via foliated connections has the advantage of not needing to distinguish between regular and singular points of the foliation.

Lemma 2.7 (Extension Lemma [GR12, Proposition 8]). *Let M be a manifold, $\mathcal{F}$ a foliation on M, $p \in M$. Let X be a meromorphic vector field defined on a neighborhood of p whose divisor of zeros and poles D is invariant by $\mathcal{F}$ and which is tangent to $\mathcal{F}$ away from it. Then, on a neighborhood of p, the foliated affine structure induced by X away from D extends to D in a unique way.*

Proof. Let Z be a nonvanishing holomorphic vector field defining $\mathcal{F}$ on a neighborhood of p. Let $X = f_1^{n_1} \cdots f_k^{n_k} Z$, for $n_i \in \mathbf{Z}$ and reduced holomorphic functions f_i such that Zf_i divides f_i, say

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$Zf_i = h_i f_i$ for some holomorphic function h_i. In the complement of the divisor of zeros and poles of X, X induces a foliated connection ∇ such that $\nabla(X) \equiv 0$, for which

$$\nabla(Z) = \nabla\Big(\Big(\prod f_i^{-n_i}\Big)X\Big) = \Big(\prod f_i^{n_i}\Big)Z\Big(\prod f_i^{-n_i}\Big) = -\sum_{i=1}^{k}\frac{n_i}{f_i}Zf_i = -\sum_{i=1}^{k}n_i h_i,$$

and ∇ extends holomorphically to a full neighborhood of p. □

Let us see how one can concretely apply this lemma to produce foliated affine structures.

Example 2.8 (Elliptic fibrations). Let $\mathcal{F}$ be an elliptic fibration. There is a natural foliated affine structure in the complement of the singular fibers: a smooth elliptic curve carries a canonical affine structure (given by the integration of any nowhere-vanishing holomorphic form) which varies holomorphically with the curve. (For the universal elliptic curve, the existence of a foliated affine structure is just a corollary of the fact that the Hodge bundle of abelian differentials on the fibers exists [Zvo12]; the fact that the Chern class of this bundle does not vanish shows that one cannot reduce this affine structure to a translation one.) This structure can be extended to the singular fibers as follows. First, build a non-identically-zero meromorphic section of $T_{\mathcal{F}}$ whose divisor of zeros and poles is supported on a union of fibers. (To do so, one can apply Corollary 12.3 in [BHPV04, Ch. V] to get a meromorphic volume form ω on the total space whose divisor of zeros and poles is supported on a finite union of fibers, the desired section of $T_{\mathcal{F}}$ being the symplectic gradient of a meromorphic function defined on the base with respect to ω.) On the fibers on which this vector field is regular, it induces the canonical affine structure. Lemma 2.7 then shows that this foliated affine structure extends to the whole surface.

2.1.3 *A cohomological obstruction.* For general fibered spaces, there is a classical cohomological obstruction for the existence of a connection. In our setting, there is a natural class $\alpha_{\mathcal{F}}$ in $H^1(M, K_{\mathcal{F}})$, whose vanishing is equivalent to the existence of a foliated connection on $T_{\mathcal{F}}$, or, equivalently, of a foliated affine structure on $\mathcal{F}$. Let us recall this construction in our case. Observe that, locally, foliated affine structures exist, e.g. the translation structures associated to vector fields generating $\mathcal{F}$. Let $\{U_i\}_{i\in I}$ be a cover by open sets of M so that a foliated connection on $T_{\mathcal{F}}$, ∇_i, is defined on each U_i. In the intersection $U_i \cap U_j$, the difference $\nabla_i - \nabla_j$ is a section α_{ij} of $K_{\mathcal{F}}$ on $U_i \cap U_j$. Moreover, $(\alpha_{ij})_{ij}$ is a cocycle. It is easy to see that the cohomology class $\alpha_{\mathcal{F}}$ in $H^1(M, K_{\mathcal{F}})$ induced by $(\alpha_{ij})_{ij}$ does not depend on the choices made. To construct a globally defined connection, we need to modify each affine connection ∇_i on U_i by the addition of a section α_i of $K_{\mathcal{F}}$, $\nabla_i' = \nabla_i + \alpha_i$, so that the ∇_i' coincide on the intersection of their domains. This means that on $U_i \cap U_j$, $\alpha_i - \alpha_j = \alpha_{ij}$, which amounts to saying that the class $\alpha_{\mathcal{F}}$ in $H^1(M, K_{\mathcal{F}})$ is trivial. Hence, *a foliated affine structure exists if and only if the class $\alpha_{\mathcal{F}}$ vanishes.*

We next derive a criterion for the existence of foliated affine structures. For a manifold M of dimension n and a foliation by curves $\mathcal{F}$ on M, let $K_{M/\mathcal{F}}$ denote the line bundle on M whose sections are the $(n-1)$-forms that vanish along the foliation. We will call it the *canonical bundle of the space of leaves* by a slight abuse of both notation and terminology inspired by [McQ05, Section IV].

Lemma 2.9. *Assume that $\mathcal{F}$ is a singular holomorphic foliation by curves on a compact manifold M of dimension $n > 1$. If $K_{M/\mathcal{F}}$ is ample, $\mathcal{F}$ carries a holomorphic foliated affine structure.*

Proof. Since M has an ample line bundle, it is projective, and thus Kähler. Recall the Kodaira vanishing theorem: given an ample divisor D on the Kähler manifold M, $H^q(K_M + D) = 0$ for any $q > 0$. By the adjunction formula, $K_M = K_{M/\mathcal{F}} + K_{\mathcal{F}}$, where, as usual, K_M is the canonical

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bundle of M, so if $K_{M/\mathcal{F}}$ is ample, $H^1(K_{\mathcal{F}}) = 0$, which implies that $\mathcal{F}$ admits a foliated affine structure. □

2.1.4 *More on foliated connections.* For the problem of establishing the existence of foliated connections on the tangent bundle of a foliation $\mathcal{F}$, investigating the existence of foliated connections on other line bundles might prove rewarding, since the set of isomorphism classes of line bundles admitting foliated connections forms a group and is closed under the operations of taking powers and extracting roots: foliated connections on other line bundles might propagate up to $T_{\mathcal{F}}$. An interesting problem is thus that of determining, for a singular holomorphic foliation on a complex manifold, which are the holomorphic line bundles having foliated connections.

A fundamental example of a foliated connection is the *Bott connection* [Bot72] on $K_{M/\mathcal{F}}$: it is defined through the exterior derivative operator

$$d : K_{M/\mathcal{F}} \to \mathcal{O}(K_M) \simeq \mathcal{O}(K_{\mathcal{F}}) \otimes K_{M/\mathcal{F}},$$

where the last isomorphism is given by adjunction.

On a closed Kähler manifold, every holomorphic line bundle with trivial first Chern class carries a flat unitary connection, which, by restriction, induces a foliated connection. Hence, in this setting, the problem consists in determining which are the Chern classes of line bundles which carry foliated connections. This set is a subgroup of the Néron–Severi group which contains all the torsion points. As we have shown, it contains the first Chern class of $K_{M/\mathcal{F}}$, but, in general, it seems difficult to say more. There are, however, situations where this point of view allows the existence of foliated affine structures to be established. Let us give some examples.

Lemma 2.10. *On a compact Kähler manifold with vanishing first Chern class, any singular holomorphic foliation carries a foliated affine structure.*

Proof. Since the manifold has vanishing first Chern class, its canonical bundle has a unitary flat connection. By the adjunction formula, the tensor product of this connection with the Bott connection produces a flat connection on the cotangent bundle of the foliation and, hence, by duality, a foliated affine structure. □

Corollary 2.11. *Any foliation on a Calabi–Yau manifold has a foliated affine structure.*

Lemma 2.12. *If the Picard number of a compact Kähler manifold M is one, then:*

- *if the first Chern class of $K_{M/\mathcal{F}}$ is not a torsion element in the Néron–Severi group, there is a foliated affine structure;*
- *otherwise, $\mathcal{F}$ has a transverse invariant pluriharmonic form.*

Proof. If the first Chern class of $K_{M/\mathcal{F}}$ is not a torsion element in the Néron–Severi group, then any line bundle over S has a foliated connection; this is due to the fact that having a foliated connection is stable under taking power or roots. In particular, the tangent bundle carries a foliated affine structure. If not, $K_{M/\mathcal{F}}$ carries a unitary flat connection over S. Given a flat section ω, naturally considered as a holomorphic form of degree $n-1$, the product $\omega \wedge \overline{\omega}$ is a well-defined pluriharmonic form on S which vanishes on the foliation $\mathcal{F}$. Such a form is closed because S is Kähler and, hence, defines a family of transverse pluriharmonic forms. □

Example 2.13 (Hypersurfaces of $\mathbf{P}^3$). Well-known examples of surfaces having Picard number one are generic hypersurfaces of $\mathbf{P}^3$ of degree at least four, by a theorem of Noether, see [Del73]. These are simply connected by the hyperplane section theorem of Lefschetz and, in particular, it is impossible in this case for the normal bundle to a foliation to have a torsion first Chern class.

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Indeed, if it were the case, the normal bundle would be holomorphically trivial, and so would be its dual, and consequently we would have a holomorphic form on the surface vanishing on the foliation. However, such a form does not exist because the surface has a vanishing first Betti number. In other words, we have proved that *on a generic surface in* $\mathbf{P}^3$*, every singular holomorphic foliation carries a foliated affine structure.* Note that this property holds on the explicit examples produced in [Shi81], namely the surfaces defined in homogeneous coordinates by $w^m + xy^{m-1} + yz^{m-1} + zx^{m-1} = 0$ for $m \geq 5$ a prime number.

2.2 Foliated projective structures

A projective structure on a curve is an atlas for its complex structure taking values in $\mathbf{P}^1$ whose changes of coordinates lie within the group of projective transformations $\{z \mapsto (az+b)/(cz+d)\}$. In this case, the *Schwarzian derivative*

$$\{f(x), x\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2, \tag{2.4}$$

plays a role analogous to the one played by the affine distortion in the context of affine structures.

Given two projective structures on a curve C with charts $\{(U_i, \phi_i)\}$ and $\{(V_j, \psi_j)\}$, the quadratic form on $U_i \cap V_j$ given by

$$\{f(z), z\}\, dz^2, \tag{2.5}$$

for $f = \psi_j \circ \phi_i^{-1}$, gives a globally well-defined quadratic form on C, which vanishes if and only if the projective structures coincide. This is due to the fact that the operator (2.5) satisfies

$$\{f \circ g, z\}\, dz^2 = \{g, z\}\, dz^2 + g^*(\{f, w\}\, dw^2).$$

Reciprocally, given a projective structure with charts $\{(U_i, \phi_i)\}$ and a quadratic form β on C, if β reads $\beta_i(z)\, dz^2$ in U_i, the charts locally given by the solutions of the Schwarzian differential equation $\{f, z\} = \beta_i$ give a globally well-defined projective structure on C. In this way, on a curve, *the projective structures form an affine space directed by the vector space of holomorphic quadratic differentials.*

Projective structures are much more flexible than affine ones: they exist on *any* curve, and, for instance, for curves of genus $g \geq 2$, their moduli is an affine space of dimension $3g - 3$. Projective structures associated to particular geometries (spherical for genus zero, Euclidean in the case of genus one, and hyperbolic for genus at least two) are given by the Uniformization Theorem [dSG10]. Nevertheless, the existence of unrestricted projective structures can be very easily established independently from it, as Poincaré was well aware of; see [Gun66, § 9] for a modern presentation.

Definition 2.14. Let M be a complex manifold, $\mathcal{F}$ a singular holomorphic foliation by curves on M. A *holomorphic foliated projective structure* on M over $\mathcal{F}$ is an open cover $\{U_i\}$ of $M \setminus \mathrm{Sing}(\mathcal{F})$ and submersions $\phi_i : U_i \to \mathbf{P}^1$ transverse to $\mathcal{F}$ such that, in restriction to a leaf L of $\mathcal{F}$, $(\phi_i|_L) \circ (\phi_j|_L)^{-1}$ belongs to $\mathrm{PSL}(2, \mathbf{C})$.

Remark 2.15. In a way analogous to Remark 2.2, if not empty, *the moduli space of foliated projective structures on a given singular holomorphic foliation* $\mathcal{F}$ *is an affine space directed by the vector space* $H^0(K_{\mathcal{F}}^2)$. The difference between two foliated projective structures gives, on $M \setminus \mathrm{Sing}(\mathcal{F})$, a family of quadratic differentials along the leaves of $\mathcal{F}$ varying holomorphically in the transverse direction, a section of $K_{\mathcal{F}}^2$ over $M \setminus \mathrm{Sing}(\mathcal{F})$ that extends, by Hartogs's theorem, to all of M, and which vanishes identically if and only if the foliated projective structures coincide.

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Foliated projective structures may also be defined in terms of foliated projective connections: a *foliated projective connection* (on $T_{\mathcal{F}}$) is a map $\Xi : T_{\mathcal{F}} \to \mathcal{O}(M)$ that to a vector field Z tangent to $\mathcal{F}$ associates a holomorphic function $\Xi(Z)$, its *Christoffel symbol*, satisfying the modified Leibniz rule:

$$\Xi(fZ) = f^2\Xi(Z) + fZ^2(f) - \frac{1}{2}(Zf)^2. \tag{2.6}$$

(When restricted to curves, this definition is equivalent to those found in [Tyu78, Def. 1.3.1] and [Gun67, Section 4].)

For instance, if $\nabla : T_{\mathcal{F}} \to \mathcal{O}(M)$ is a foliated connection on $T_{\mathcal{F}}$, the associated foliated projective connection Ξ is

$$\Xi(Z) = -\frac{1}{2}(\nabla(Z))^2 + Z(\nabla(Z)). \tag{2.7}$$

Let us see that a foliated projective structure is equivalent to a foliated projective connection. Let $\mathcal{F}$ be a foliation endowed with a foliated projective structure ρ_0. Let Z be a vector field tangent to $\mathcal{F}$ with singular set of codimension at least two, and consider the foliated projective structure ρ_Z that it defines. From Remark 2.15, the difference $\rho_Z - \rho_0$ is a section α of $K_{\mathcal{F}}^2$. Define $\Xi(Z)$ as $\alpha(Z^{\otimes 2})$. Let us prove that it satisfies condition (2.6). As before, it is sufficient to do so locally in a curve. Consider a curve endowed with a projective structure ρ_0, Z a holomorphic vector field and z a local coordinate in which $Z = \partial/\partial z$. Let $\alpha(z)\, dz^2$ be the quadratic form $\rho_Z - \rho_0$. The projective structure defined by fZ has $\int^z d\xi/f(\xi)$ as a chart and, thus,

$$\rho_Z - \rho_{fZ} = \left(\frac{1}{2}\left(\frac{f'}{f}\right)^2 - \frac{f''}{f}\right) dz^2.$$

Hence, the contraction of $\rho_{fZ} - \rho_0$ with $(fZ)^{\otimes 2}$ gives $f^2\alpha(Z^{\otimes 2}) + Z^2 f - \frac{1}{2}(Zf)^2$, establishing (2.6). Reciprocally, if Ξ is a foliated projective connection, $p \notin \operatorname{Sing}(\mathcal{F})$ and Z is a holomorphic vector field tangent to $\mathcal{F}$ that does not vanish at p, and such that $\Xi(Z) \equiv 0$, if ϕ is a function defined in a neighborhood of p such that $d\phi(Z) \equiv 1$, ϕ defines a foliated projective structure in the sense of Definition 2.14 that depends only on Ξ.

Example 2.16 (Suspensions). A nonsingular foliation by curves $\mathcal{F}$ on a compact surface S is a *suspension* if there exists a fibration onto a curve $\pi : S \to C$ which is everywhere transverse to $\mathcal{F}$. *On a suspension, every foliated projective structure is the pull-back of a projective structure on the base.* In fact, if $\Xi_{\mathcal{F}}$ is a foliated projective connection and Z is a nonvanishing vector field defined in an open subset U of C, if $\pi_{\mathcal{F}}^* Z$ denotes the pull-back of Z tangent to $\mathcal{F}$, then since $\Xi_{\mathcal{F}}(\pi_{\mathcal{F}}^* Z)$ is a holomorphic function, it is constant along the fibers of π. In this way, the projective structure on C given by $\Xi_C(Z) := \Xi_{\mathcal{F}}(\pi_{\mathcal{F}}^* Z)$ is well-defined; the foliated one is its pull-back.

Example 2.17 (Turbulent foliations). Let S be a compact surface, $\pi : S \to C$ an elliptic fibration, $\mathcal{F}$ a turbulent foliation on S adapted to π, i.e. almost every fiber of π is everywhere transverse to $\mathcal{F}$. We refer the reader to [BHPV04, Ch. 5, § 7] and [Bru04, Ch. 4, Section 3] for facts around elliptic fibrations and turbulent foliations that we will use leisurely. We will restrict to the cases where π is relatively minimal, and show that $\mathcal{F}$ admits a foliated projective structure.

Let us begin with the case where $\mathcal{F}$ is regular and where the fibers of π are simple. This implies that, as a foliation, the fibration is also regular, and, in particular, that it does not have singular fibers. Let $C_0 \subset C$ be the subset above which π and $\mathcal{F}$ are transverse. By the arguments in Example 2.16, the projective structures on C_0 and the foliated ones on $\pi^{-1}(C_0)$ are in correspondence. We will establish a condition on the projective structure on the base that guarantees that the foliated one extends to the invariant fibers. Let F be a fiber such that,

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for $p = \pi(F)$, $p \notin C_0$. The fibration around F is given by the projection $\pi : \mathbf{D} \times \mathbf{C}/\Lambda \to \mathbf{D}$ for some elliptic curve $\mathbf{C}/\Lambda$. For some local coordinates z and w in $\mathbf{D}$ and $\mathbf{C}/\Lambda$ centered at p, $\mathcal{F}$ is given by the nonvanishing holomorphic vector field $Z = z^n\partial/\partial z + B(z)\partial/\partial w$, with B a holomorphic nonvanishing function, and where n is the multiplicity with which F appears in the tangency divisor between $\mathcal{F}$ and the fibration. Let Ξ_0 be a projective connection on $\mathbf{D} \setminus \{0\}$, and let Ξ be the corresponding foliated projective connection on $\pi^{-1}(\mathbf{D} \setminus \{0\})$. In the spirit of Lemma 2.7, by formula (2.6), since $\pi_* Z = z^n\partial/\partial z$,

$$\Xi(Z) = \Xi_0\left(z^n\frac{\partial}{\partial z}\right) = z^{2n}\Xi_0\left(\frac{\partial}{\partial z}\right) + \frac{1}{2}n(n-2)z^{2(n-1)}. \tag{2.8}$$

If this expression is holomorphic (in particular, if the projective structure on C_0 extends as a regular one to p), the foliated projective structure extends to F; this extension is unique.

If Γ is a finite group acting on S with isolated fixed points, preserving the fibration, the foliation and the foliated projective structure, it induces a foliated projective structure on the nonsingular part of S/Γ. Let us prove that the foliated projective structure extends to the minimal resolution of the singular points. Let $q \in S$ be a point with nontrivial stabilizer. In a neighborhood of q, this stabilizer is a cyclic group $\langle g \rangle$ of order m of biholomorphisms fixing q such that the derivative of g at q has as eigenvalues two primitive mth roots of unity. Suppose that, in a neighborhood of q, the nonvanishing vector field Z gives both the foliation and its projective structure, and choose local coordinates where $Z = \partial/\partial x$. The action of g on the leaf space of Z may be linearized while preserving the expression of Z, and we may suppose that it is given by $y \mapsto \omega y$ for a primitive mth root of unity ω. Since the foliation induced by Z is preserved, the action of g on Z must consist in multiplying it by a function f such that $f^m \equiv 1$ and, thus, $f \equiv \omega^n$ for some $n < m$, $(m, n) = 1$. In particular, the vector field $y^{m-n}Z$, which by the Leibniz formula (2.3) induces the same affine structure than Z away from $y = 0$, is preserved by g. It induces a vector field on the quotient as well as on its minimal resolution. Since in this resolution the divisor contracting to the singular point is invariant by the foliation coming from Z, the affine structure extends to this divisor by Lemma 2.7.

All turbulent foliations adapted to relatively minimal elliptic fibrations are constructed through such a process of taking quotients and resolving singularities. Thus, by suitably choosing a possibly singular projective structure on the base of the associated minimal elliptic fibration, we obtain a foliated projective structure: *every turbulent foliation adapted to a relatively minimal elliptic fibration admits a foliated projective structure.*

Not all foliations support foliated projective structures. As we mentioned in the introduction, by the work of Zhao [Zha19], no Kodaira fibration admits one (we will give another proof of this fact through Corollary 6.1; one more has recently appeared in [EWF21]). Despite the generality of this result, we thought it worthwhile to include a concrete, hands-on, self-contained instance of it.

Example 2.18 (An explicit Kodaira fibration without a foliated projective structure). Recall that a *Kodaira fibration* is a smooth holomorphic fibration $S \to B$ from a complex surface over a curve which is not a holomorphic fiber bundle (Kodaira gave the first examples of such fibrations [Kod67], see also [BHPV04, p. 220]). Through a construction close to Atiyah's [Ati69], we here construct an explicit Kodaira fibration with fibers of genus six which does not support a foliated projective structure, i.e. such that there is no family of projective structures on the fibers varying holomorphically.

Start with a curve C of genus two, and let $\pi : C' \to C$ be a connected nonramified double covering (C' has genus three). For every $x \in C$, we construct 64 curves of genus six: the ramified double coverings $C'' \to C'$ ramified over $\pi^{-1}(x)$. Such a covering is determined by a morphism from $H_1(C' \setminus \pi^{-1}(x), \mathbf{Z})$ to $\mathbf{Z}/2\mathbf{Z}$ that maps the peripheral cycles around each one of the two punctures to 1; the number of such coverings is 64. Construct the Kodaira fibration $F : S \to B$ by putting all these surfaces over the point $x \in C$, and taking a connected component (we do not know, in general, if the resulting surface is connected, a case that would lead to a genus 65 base B). Hence, a point y in B is the data of a point $x \in C$ and of a double covering $\delta_y : F^{-1}(y) \to C'$ ramified over $\pi^{-1}(x)$. Let d be the degree of the covering $B \to C$.

We claim that this fibration does not carry a foliated projective structure. Assume by contradiction that there exists a family of projective structures $\{\sigma_y\}_{y\in B}$ on the fibers of F that vary holomorphically with y. Introduce a family of branched projective structures $\{\beta_y\}_{y\in B}$ on the fibers $F^{-1}(y)$, β_y being the pull-back of a (nonbranched) projective structure ν on C' by δ_y. The Schwarzian derivative of β_y in the charts given by σ_y gives a family of meromorphic quadratic differentials on $F^{-1}(y)$ that vary holomorphically with the y parameter, and which have poles of order two located at the points $\delta_y^{-1}(\pi^{-1}(x))$, with residue $-3/2$ (as quadratic differentials). Indeed, if u, v are charts of σ and β at such a point, we have $v = c_2u^2 + \cdots$ where $c_2 \neq 0$, hence $\{v, u\} = -\frac{3}{2}u^{-2} + \cdots$. We denote by Q_y the quadratic differential on $F^{-1}(y)$.

For each y in B, we denote by i_y the involution on $F^{-1}(y)$ that exchanges the points in the fiber of δ_y, and we define $R_y = Q_y + i_y^*Q_y$. This is an i_y-invariant meromorphic quadratic differential on $F^{-1}(y)$ having poles at $\delta_y^{-1}(\pi^{-1}(x))$ of order two and residues -3. Hence, there is a meromorphic quadratic differential S_y on C' such that $R_y = \delta_y^*S_y$. This differential has poles on the set $\pi^{-1}(x)$, and is holomorphic elsewhere. We claim that the poles on $\pi^{-1}(x)$ are of order two, and that the residues are $-3/4$. To see this, take coordinates v, w in $F^{-1}(y)$ and in C', respectively, such that δ_y is the map $v \mapsto w = v^2$. The quadratic differential R_y is expressed in the v-coordinates by $R_y = (-3v^{-2} + c_0 + c_2v^2 + \cdots)\, dv^2$, because it is invariant by the involution $v \mapsto -v$. Hence, $S_y = \frac{1}{4}(-3w^{-2} + c_0w^{-1} + c_2w + \cdots)\, dw^2$, proving the claim.

We now define, for $x \in C$, the meromorphic quadratic differential T_x on C' by $T_x = \sum S_y$ for all the coverings $y \in B$ corresponding to the point x. We see that T_x has poles only at $\pi^{-1}(x)$, that these are of order two and that the residues are $-3d/4$. Let j the involution on C' which exchanges the fibers of π, and let U_x be the meromorphic quadratic differential on C which satisfies $\pi^*U_x = T_x + j^*T_x$. The family $\{U_x\}_{x\in C}$ is a holomorphic family of meromorphic quadratic differentials on C having a unique pole on C at x of order two and of residue $-3d/2$.

We claim that such a family of meromorphic quadratic differentials cannot exist. Indeed, choose a point x_0 in C which is not fixed by any nontrivial involution and such that there exists a holomorphic quadratic differential μ on C that does not vanish at x_0. Consider the holomorphic function $f : C \setminus \{x_0\} \to \mathbf{C}$ given at x by the evaluation of U_x/μ at x_0. It extends meromorphically to x_0, having there a pole of order two, because for a local coordinate z centered at x_0, $U_x/\mu = -\frac{3}{2}(z - x)^{-2} + \cdots$. Hence, f extends to a ramified double covering from C to $\mathbf{P}^1$, and the involution exchanging its fibers fixes x_0. This is a contradiction.

The existence of a foliated projective structure is equivalent to the vanishing of a class $\beta_{\mathcal{F}}$ in $H^1(M, K_{\mathcal{F}}^2)$ whose definition mimics the definition of the class $\alpha_{\mathcal{F}}$ introduced in the context of foliated affine structures. Namely, take a covering of M by open sets U_i on which we have foliated projective connections Ξ_i, and consider the cocycle $\beta = (\beta_{ij})_{ij}$, where $\beta_{ij} = \Xi_i - \Xi_j$ is a section of $K_{\mathcal{F}}^2$ over $U_i \cap U_j$. Its cohomology class $\beta_{\mathcal{F}} \in H^1(M, K_{\mathcal{F}}^2)$ is well-defined. To construct a globally defined foliated projective connection, one needs to modify each Ξ_i in U_i by adding

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some section β_i of $K^2_{\mathcal{F}}$, in such a way that the resulting connections on the U_i coincide in the intersection of their domains. This is equivalent to solving the equation $\beta_i - \beta_j = \beta_{ij}$, so *there exists a foliated projective structure if and only if* $\beta_{\mathcal{F}} = 0$.

If M is a curve and $\mathcal{F}$ is the foliation whose only leaf is M, then by Serre duality $h^1(M, K^2_M) = h^0(M, TM)$, and we recover the fact that every compact curve of higher genus has a projective structure. Notice, however, that this argument does not allow to conclude that rational and elliptic curves have such structures.

Despite Example 2.18, it is quite common for a singular holomorphic foliation to carry a foliated projective structure. The following criteria is a consequence of Kodaira's vanishing theorem.

LEMMA 2.19. *Let M be a compact manifold of dimension $n > 1$ and $\mathcal{F}$ a foliation by curves on M such that $K^2_{\mathcal{F}} \otimes K^*_M$ is ample. Then, there exists a foliated projective structure on $\mathcal{F}$.*

Proof. Under the assumption, M is projective, and by Kodaira's vanishing theorem, $H^1(K^2_{\mathcal{F}}) = 0$, so $\beta_{\mathcal{F}}$ vanishes and the claim follows. □

Let us illustrate the use of this lemma.

PROPOSITION 2.20. *Any singular holomorphic foliation on the product of a curve with the projective line carries a foliated projective structure.*

Proof. Let C be a curve of genus g, $S = C \times \mathbf{P}^1$, and let $\mathcal{F}$ be a foliation on S. Curves of the form $\{*\} \times \mathbf{P}^1$ will be called *vertical*; those of the form $C \times \{*\}$, *horizontal*. Let $V \in H^2(S, \mathbf{Z})$ be the Poincaré dual of a vertical curve, $H \in H^2(S, \mathbf{Z})$ that of a horizontal one. These generate $H^2(S, \mathbf{Z})$. If $\mathcal{F}$ is either the vertical or the horizontal foliation, the proposition follows from the existence of projective structures on curves, so we suppose that we are in neither case. Let us denote by n_h (respectively, n_v) the number of tangencies of $\mathcal{F}$ with a generic horizontal (respectively, vertical) curve. We call n_h the horizontal degree and n_v the vertical one. They are both nonnegative. We claim that

$$n_h \geq 2g - 2. \tag{2.9}$$

The foliation $\mathcal{F}$ is defined by a morphism $i : T_{\mathcal{F}} \to TS$ that vanishes on the singular set of $\mathcal{F}$ (a finite number of points). Since $TS = \mathrm{pr}_1^*(TC) \oplus \mathrm{pr}_2^*(T\mathbf{P}^1)$, the morphism i is given by sections of $K_{\mathcal{F}} \otimes \mathrm{pr}_1^*(T_C)$ and of $K_{\mathcal{F}} \otimes \mathrm{pr}_2^*(T\mathbf{P}^1)$ that vanish simultaneously on a finite set. Since the foliation is not the vertical one, the first section does not vanish identically. Since $K_{\mathcal{F}} = K_S \otimes N_{\mathcal{F}}$, $c_1(K_{\mathcal{F}} \otimes \mathrm{pr}_1^*(TC)) = n_v H + (n_h - 2g + 2)V$, and such a section can only exist if both $n_v \geq 0$ and $n_h - 2g + 2 \geq 0$, proving the claim.

Since $K_S = \mathrm{pr}_1^*(K_C) \otimes \mathrm{pr}_2^*(K_{\mathbf{P}^1})$, $c_1(K_S) = (2g-2)V - 2H$. Let $c_1(N_{\mathcal{F}}) = aH + bV$ for some $a, b \in \mathbf{Z}$. On a horizontal curve that is not invariant by $\mathcal{F}$, a meromorphic section of $N^*_{\mathcal{F}}$ induces a meromorphic one-form having $n_h - (aH + bV) \cdot H$ zeros, so $n_h - b = 2g - 2$. The same reasoning shows that $n_v - a = -2$. To sum up,

$$c_1(N_{\mathcal{F}}) = (n_v + 2)H + (n_h - 2g + 2)V.$$

From $K_{\mathcal{F}} = K_S \otimes N_{\mathcal{F}}$, $c_1(K_{\mathcal{F}}) = n_v H + n_h V$, so

$$c_1(K^2_{\mathcal{F}} \otimes K^*_S) = (2n_v + 2)H + (2n_h - (2g-2))V. \tag{2.10}$$

If $g \neq 1$ or $n_h > 0$, we infer from (2.9) and (2.10) that $K^2_{\mathcal{F}} \otimes K^*_S$ intersects positively H and V and, hence, every algebraic curve in S. This implies, by Nakai's criterion [BHPV04, Ch. IV,

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Corollary 6.4], that it is ample, and, by the previous lemma, $\mathcal{F}$ admits a foliated projective structure.

Finally, if $g = 1$ and $n_h = 0$, the foliation is turbulent, adapted to a nonsingular elliptic fibration. We have already shown in Example 2.17 that these foliations admit foliated projective structures as well. □

Remark 2.21. It would be interesting to investigate the existence of foliated projective structures on general foliated ruled surfaces (the work of Gómez-Mont [GM89] seems a natural starting point). Most foliations on these seem to have foliated affine structures. For instance, we leave to the reader the following: a more detailed inspection of the proof of Proposition 2.20, together with the use of Lemma 2.9, shows that, apart from suspensions (vanishing vertical degree) and eventually foliations of horizontal degree $2g - 2$ (the lower bound for the horizontal degree of a nonvertical foliation), foliations on a product with a rational curve carry foliated affine structures. We have not been able to decide whether the foliations of horizontal degree $2g - 2$ carry or not such structures.

3. Local normal forms

At a nonsingular point of a foliation, there are no local invariants neither for affine nor for projective foliated structures. There are indeed local invariants at the singular points, beginning with those of the foliation itself. The main results of this section, Theorems 3.2 and 3.7, give local normal forms for generic foliated affine and projective structures on generic foliations. We prove that, in all dimensions, in the neighborhood of a generic singular point of a foliation, a generic foliated projective structure is induced by an affine one, and that a generic foliated affine structure is given by a linear vector field having a constant Christoffel symbol. In particular, we prove that the spaces of generic foliated affine and projective structures over a generic germ of singular foliation have both dimension one. We also introduce the affine and projective *ramification indices*, the main local invariants of foliated affine and projective structures at singular points of foliations, in terms of which the results of the following sections will be stated.

3.1 The affine case

3.1.1 *Affine structures with singularities on curves.* Let $U \subset \mathbf{C}$ be a neighborhood of 0, $U^* = U \setminus \{0\}$ and consider an affine structure on U^*. Let α be the one-form in U^* measuring the difference from an auxiliary affine structure on U to the original one. We say that 0 is a *singularity* for the affine structure if α does not extend holomorphically to 0 (if α extends holomorphically to 0, so does the affine structure), and that it is a *Fuchsian* one if α has a simple pole at 0. In this case, the residue of α at 0 does not depend on the choice of the auxiliary affine structure on U. The *(normalized) affine angle* at 0 of the affine structure with singularities, $\measuredangle(0) \in \mathbf{C}$, is $\measuredangle(0) = \operatorname{Res}(\alpha, p) + 1$. The normalized affine angle of a nonsingular point is 1. The normalized affine angle of the affine structure with developing map $z \mapsto z^\theta$ is θ; that of the one with developing map $z \mapsto \log(z)$ vanishes. Following [GR12, Def. 4], we define the (affine) *ramification index* of a singular affine structure as the reciprocal of the normalized affine angle.

We have a classification of germs of singular affine structures with Fuchsian singularities on curves, which may be attributed to Fuchs. It implies that, generically, the affine angle determines the singular affine structure.

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PROPOSITION 3.1. *Consider an affine structure on a neighborhood of* 0 *in* $\mathbf{C}$ *having a Fuchsian singularity at* 0 *with normalized affine angle* $\theta \in \mathbf{C}$. *There exists a coordinate* z *around* 0 *where the affine structure has as a developing map:*

- $\log(z)$ *if* $\theta = 0$;
- $z \mapsto z^\theta$ *if* $\theta \notin \mathbf{Z}^-$;
- *either* $z \mapsto z^\theta$ *or* $z \mapsto z^\theta + \log(z)$ *if* $\theta \in \mathbf{Z}^-$.

Proof. From the affine structure induced by a local coordinate z, the difference with the singular affine structure has the form $((\theta-1)/z + A(z))\,dz$ for some holomorphic function A. The developing map of the affine structure is, thus, a nonconstant solution of $zf'' - ((\theta-1)+zA)f' = 0$. The homogenized equation $zf'' - (\theta-1)f'$ has the solutions z^0 and z^θ (the original equation has *indices* 0 and θ). According to Fuchs's theorem [Inc44, § 15.3], if θ is neither zero nor a negative integer, there is a solution of the form $z^\theta h(z)$ with $h(z)$ holomorphic and nonzero at 0. In this case, in the coordinate $w = zh^{1/\theta}(z)$, the developing map is w^θ. If θ is zero or a strictly negative integer, Fuchs's theorem affirms that there is a solution (in our setting, a developing map) of the form $c\log(z) + z^\theta h(z)$, for some holomorphic function h taking the value 1 at 0 and some constant c (nonzero if $\theta = 0$). If $c = 0$ we are in a case identical to the previous one. Otherwise, if $q(z)$ is such that $q(0) = 0$ and $e^{\theta q(z)} + cz^{-\theta}q(z) - h(z) = 0$, then in the coordinate $w = ze^{q(z)}$ the developing map reads $c\log(w) + w^\theta$ (the existence of such a q follows from the implicit function theorem). By conveniently scaling w and normalizing the developing map by post-composition by an affine map, we get the desired result. □

3.1.2 *The foliated case.* Let $\mathcal{F}$ be a foliation defined on a neighborhood of 0 in $\mathbf{C}^n$ and endowed with a foliated affine structure induced by the foliated connection ∇. Let Z be a vector field tangent to $\mathcal{F}$ and $\gamma = \nabla(Z)$ its Christoffel symbol, as defined in § 2.1, and recall that γ is a holomorphic function at 0. It follows from (2.3) that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of Z at 0, the ratio $[\lambda_1 : \cdots : \lambda_n : \gamma(0)]$ is an invariant of the foliated affine structure.

In dimension one, this invariant may be expressed in terms of the previously defined affine ramification index. Consider a singular affine structure on a neighborhood of 0 in $\mathbf{C}$ given by the connection ∇. Let $\gamma = \nabla(\lambda z\partial/\partial z)$. The difference between the affine structure induced by the coordinate z and the first one is $(\gamma(z)/\lambda - 1)\,dz/z$ and, thus, for the ramification index ν of the original affine structure,

$$\nu = \frac{\lambda}{\gamma(0)}. \tag{3.1}$$

In the foliated case in $(\mathbf{C}^n, 0)$, if the eigenvalues at 0 of the vector field are $\lambda_1, \ldots, \lambda_n$ and its Christoffel symbol γ does not vanish at 0 we say that $\nu_i = \lambda_i/\gamma(0)$ is a *principal ramification index*. From (3.1),

$$[\lambda_1 : \cdots : \lambda_n : \gamma(0)] = [\nu_1 : \cdots : \nu_n : 1]. \tag{3.2}$$

In the generic nondegenerate case there will be n curves $C_1, \ldots, C_n$ through 0, invariant by $\mathcal{F}$, pairwise transverse, and tangent to the eigenspaces of the linear part of the vector field, and ν_i will be the ramification index of the affine structure on C_i at 0.

Generically, the ratio (3.2) determines the foliated affine structure.

THEOREM 3.2. *Let* $\mathcal{F}$ *be a foliation on a neighborhood of* 0 *in* $\mathbf{C}^n$, *with a singularity at* 0, *tangent to a nondegenerate vector field* Z *satisfying Brjuno's condition* (ω). *For a generic foliated connection* ∇ *on* $T_{\mathcal{F}}$, *there exist coordinates around* 0 *where* $\mathcal{F}$ *is tangent to a linear vector field* Z' *whose Christoffel symbol* $\nabla(Z')$ *is constant.*

We refer the reader to [Arn80, Ch. 5] for details on Brjuno's condition (ω), and only mention that it is satisfied by generic (in a measure-theoretic sense) linear parts. The genericity of the affine structure will be made precise further on. Note that this theorem introduces a notion of equivalence that is different from that of having the same foliated atlas (Remark 2.2), in which no change of coordinates is involved. The proof of our theorem is an application of the following general result.

Theorem 3.3 (Brjuno and Pöschel). *Let $Z=\sum_i \lambda_i z_i \partial/\partial z_i$ be a linear vector field on $(\mathbf{C}^n,0)$. Let F be a holomorphic function defined in the neighborhood of $(0,0)$ in $\mathbf{C}\times\mathbf{C}^n$ such that $F(0,0)=0$, and consider the differential equation $Zf=F(f,z)$ subject to the condition $f(0)=0$. Let $\mu=\partial F/\partial f|_{(0,0)}$ and suppose that $\mu\neq\langle K,\lambda\rangle$ for every $K\in(\mathbf{Z}_{\geq 0})^n$ with $|K|\geq 2$. Let*

$$\omega'(m)=\min_{2\leq|K|\leq m}|\langle K,\lambda\rangle-\mu|.$$

Then, if

$$-\sum_{\nu\geq 0}2^{-\nu}\log\omega'(2^{\nu+1})<\infty, \tag{3.3}$$

the equation has a holomorphic solution (which is, moreover, unique).

In this theorem, the function f will be a solution of the differential equation if and only if the vector field $Z\oplus F(\zeta,z)\partial/\partial\zeta$, defined in a neighborhood of the origin of $\mathbf{C}^n\times\mathbf{C}$, has $\zeta=f(z)$ as an invariant manifold. The condition $\mu\neq\langle K,\lambda\rangle$ guarantees the existence of a formal solution, and (3.3) guarantees its convergence. For $n=1$, the hypothesis on (ω') is superfluous, and the result reduces to Briot and Bouquet's theorem [Inc44, §12.6].

Theorem 3.3 does not exactly appear in the literature in the above formulation. Brjuno's announcement [Brj74] gives a similar statement, and we can find in [Pös86] an analogous result by Pöschel in the context of invariant manifolds for germs of diffeomorphisms; the proof of the latter may be adapted in a straightforward way to give a complete proof of the above theorem. (For the case where the λ_i belong to the Poincaré domain, see also [Kap79, CS14]; see [Cha88, § IX] for an analogous result under Siegel-type Diophantine conditions.)

Proof of Theorem 3.2. Since Z satisfies Brjuno's condition (ω), it is linearizable, so we may suppose that it is already linear. Suppose that f is a function such that

$$Zf=\gamma(0)-f\gamma,\quad f(0)=1. \tag{3.4}$$

The existence of such a function follows, generically, from Theorem 3.3, which we may apply to (3.4). In terms of the statement of Theorem 3.3, $\mu=-\gamma(0)$; generically, $\mu\neq\sum_i m_i\lambda_i$, and condition (3.3) is satisfied. The Christoffel symbol of the vector field $Z'=fZ$, is, by construction, the constant $\gamma(0)$. It remains constant in the coordinates where Z' is linear. □

Note that the condition $-\gamma(0)\neq\sum_i m_i\lambda_i$ may be expressed solely in terms of the principal affine ramification indices.

3.2 The projective case

3.2.1 *Projective structures with singularities on curves.* Let $U\subset\mathbf{C}$ be a neighborhood of 0, $U^*=U\setminus\{0\}$, and consider a projective structure on U^*. Let β be the quadratic form in U^* measuring the difference from an auxiliary projective structure on U to this one. We say that 0 is a *singularity* for the projective structure if β does not extend holomorphically to 0. A singularity of a projective structure is said to be *Fuchsian* if β has at most a double pole at 0. The quadratic residue $\mathrm{Q}(\beta,0)$ of the quadratic form β at 0, $\mathrm{Q}((r/z^2+\cdots)\,dz^2,0)=r$, does not depend on the choice of the auxiliary projective structure. In this case, we define the *(normalized)*

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projective angle at 0 of the projective structure with singularities as $\measuredangle(0) = \sqrt{1 - 2\mathrm{Q}(\beta, 0)}$. It is only well-defined up to sign. The normalized projective angle of the projective structure with developing map $z \mapsto z^\theta$ is $\pm\theta$. We define the *projective ramification index* at 0 as the reciprocal of the normalized projective angle. Again, it is only well-defined up to sign.

We also have a local classification of projective structures with Fuchsian singularities in dimension one.

PROPOSITION 3.4. *Consider a projective structure on a neighborhood of* 0 *in* $\mathbf{C}$ *having a Fuchsian singularity at* 0 *with normalized projective angle* $\theta \in \mathbf{C}$. *Then, there exists a singular affine Fuchsian structure in its class. In particular, there exists a coordinate* z *around* 0 *where the developing map is given as in Proposition 3.1.*

Proof. The difference from the projective structure induced by a local coordinate z to the singular one has the form $S(z)\,dz^2$, $S(z) = \frac{1}{2}(1-\theta^2)z^{-2} + \cdots$. From (2.4), the affine structure with invariant $g(z)\,dz$ is in the projective class of the original projective structure if g is a solution to the Riccati equation $g' = S + \frac{1}{2}g^2$ (if there is some f for which $f''/f' = g$ and $\{f, z\} = S$). For $u = zg$, this equation reads

$$zu' = z^2 S(z) + u + \tfrac{1}{2}u^2. \tag{3.5}$$

Let θ be a root of θ^2 that is not a strictly positive integer. By the theorem of Briot and Bouquet [Inc44, §12.6], (3.5) has a holomorphic solution $u(z)$ with $u(0) = \theta - 1$. The affine structure induced by $g\,dz = u\,dz/z$ is, thus, Fuchsian and induces the original projective structure. □

3.2.2 *The foliated case.* Let $\mathcal{F}$ be a foliation tangent to a nondegenerate vector field defined in a neighborhood of 0 in $\mathbf{C}^n$ and endowed with a foliated projective structure induced by the projective connection Ξ. Let Z be a vector field tangent to $\mathcal{F}$ and $\rho = \Xi(Z)$ its Christoffel symbol, as defined in § 2.2. From (2.6), if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of Z at 0, in the weighted projective space $\mathbf{P}(1, \ldots, 1, 2)$, the ratio $[\lambda_1 : \cdots : \lambda_n : \rho(0)]$ is an invariant of the foliated projective structure. Let us relate this invariant, in dimension one, to the previously defined projective ramification index. Consider a singular projective structure on $(\mathbf{C}, 0)$ and let $\rho = \Xi(\lambda z \partial/\partial z)$. The difference of the projective structure with coordinate z and the singular one is $\frac{1}{2}(1 + 2\rho/\lambda^2)\,dz^2/z^2$ and, thus, for the projective ramification index ν, $\nu^2 = -\frac{1}{2}\lambda^2/\rho(0)$. In particular,

$$[\lambda_1 : \cdots : \lambda_n : -2\rho(0)] = [\nu_1 : \cdots : \nu_n : 1] \quad \text{in } \mathbf{P}(1, \ldots, 1, 2). \tag{3.6}$$

The numbers ν_i are said to be the *principal projective ramification indices* of the foliated projective structure at 0.

Remark 3.5. The individual principal projective ramification indices are only well-defined up to sign. More generally, only their even functions are well-defined.

Remark 3.6. If a foliated affine structure is considered as a projective one, its affine and projective ramification indices coincide (within the limitations given by the previous remark).

The analogue of Theorem 3.2 for foliated projective structures is the following one.

THEOREM 3.7. *Let* $\mathcal{F}$ *be a foliation on a neighborhood of* 0 *in* $\mathbf{C}^n$, *with a singularity at* 0, *generated by a nondegenerate vector field* Z *satisfying Brjuno's condition* (ω). *For a generic foliated projective structure on* $\mathcal{F}$:

- *there exists a foliated affine structure in its class; and*
- *there exist coordinates where* $\mathcal{F}$ *is tangent to a linear vector field having a constant Christoffel symbol.*

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Proof. Let $\mathcal{F}$ be a foliation endowed with a foliated projective structure with foliated projective connection Ξ. Let Z be a vector field tangent to $\mathcal{F}$, and suppose that it is linear. Let $\rho = \Xi(Z)$ be its Christoffel symbol. From formula (2.7), if γ is a function such that $Z\gamma = \frac{1}{2}\gamma^2 + \rho$, there exists, like in Proposition 3.4, a foliated affine structure inducing the given projective one, with connection ∇, such that $\nabla(Z) = \gamma$. We may resort to Theorem 3.3 to establish the existence of a solution to this equation with one of the initial conditions $\gamma(0)$ such that $\gamma^2(0) + 2\rho(0) = 0$. For the hypothesis of the theorem, $\mu = \gamma(0)$, and according to it, we have solutions to the equation whenever $\gamma(0) \neq \sum_i m_i\lambda_i$ and condition (3.3) is satisfied. Theorem 3.2 and formula (2.7) establish the second part of our claim. □

4. An index theorem for foliated affine structures

The existence of an affine structure on a curve imposes topological restrictions on it, and the only compact curves admitting them are elliptic. Similarly, the existence of a foliated affine structure imposes topological restrictions on both the foliation and the ambient manifold, and conditions the local behavior of the foliated affine structure at its singular points.

We will make this precise through an index theorem relating the affine ramification indices defined in the previous section with some topological data depending only on the foliation. Some index theorems of the like follow directly from Baum and Bott's theorem [BB70] because, generically, from (3.2), the ratios of the eigenvalues of a vector field tangent to a foliation at a singular point (in terms of which the Baum–Bott index theorem is expressed in many situations) are the ratios of the principal affine ramification indices. We are nevertheless interested in results that truly depend on the foliated affine structure and not just on the foliation that supports it.

Theorem 4.1. *Let M be a compact complex manifold of dimension n, and $\mathcal{F}$ a holomorphic foliation by curves on M having only isolated nondegenerate singularities $p_1, \ldots, p_k$. Consider a foliated affine structure subordinate to $\mathcal{F}$ having at each one of the singularities a nonvanishing Christoffel symbol, and let $\nu_1^{(i)}, \ldots, \nu_n^{(i)}$ be the principal affine ramification indices at p_i. Then,*

$$(-1)^n \sum_{i=1}^{k} \frac{1}{\nu_1^{(i)} \cdots \nu_n^{(i)}} = c_1^n(T_{\mathcal{F}}).$$

Here, $c_1(T_{\mathcal{F}}) \in H^2(M, \mathbf{Z})$ is the first Chern class of $T_{\mathcal{F}}$, $c_1^n(T_{\mathcal{F}}) \in H^{2n}(M, \mathbf{Z})$, and, as usual, we have identified cohomology classes of top degree with their evaluations on the fundamental class. For instance, if $\mathcal{F}$ is a foliation of degree d on $\mathbf{P}^n$, $c_1^n(T_{\mathcal{F}}) = (1-d)^n$. In the case where M is a curve ($n = 1$), there are no singularities ($k = 0$) and $T_{\mathcal{F}} = TM$: the result reduces to $c_1(TM) = 0$, and implies that M is an elliptic curve.

A foliated affine structure along the foliation $\mathcal{F}$ defines naturally a *geodesic vector field* on $T_{\mathcal{F}}$. Theorem 4.1 will follow from applying Lehmann's index theorem [Leh91] to the foliation induced by this vector field relative to the zero section.

4.1 The geodesic vector field

Consider a foliated affine structure along the foliation $\mathcal{F}$. For every $v \in T_{\mathcal{F}}$ such that $\pi(v)$ is not a singular point of $\mathcal{F}$, for some $U \subset \mathbf{C}$, with $0 \in U$, there is a geodesic of the associated connection, $c : (U, 0) \to (M, \pi(v))$, tangent to $\mathcal{F}$, such that $c'(0) = v$ (the germ of c at 0 is unique). The derivative gives a lift $\widetilde{c} : U \to T_{\mathcal{F}}$ with $\pi(\widetilde{c}(t)) = c(t)$ and $\widetilde{c}(t) = c'(t)$. The vector field on $T_{\mathcal{F}} \setminus \pi^{-1}(\mathrm{Sing}(\mathcal{F}))$ that has this curve as its integral one through v extends, by Hartogs's theorem, to all of $T_{\mathcal{F}}$. This is the *geodesic vector field* of the foliated affine structure. Since, for $\alpha \in \mathbf{C}^*$, $c(\alpha t)$ is a geodesic whenever $c(t)$ is, the geodesic vector field enjoys the following equivariance property: if X denotes the geodesic vector field and $h_\alpha : T_{\mathcal{F}} \to T_{\mathcal{F}}$ the homothety by α on the fibers,

$$X(h_\alpha v) = \alpha D h_\alpha(X(v)). \tag{4.1}$$

Remark 4.2. This gives yet another definition of a foliated affine structure: *a vector field X on $T_{\mathcal{F}}$ projecting onto $\mathcal{F}$ satisfying relation (4.1)*. For such a vector field, the associated foliated affine structure on $\mathcal{F}$ is given by the projections of its solutions.

This characterization allows some foliated affine structures to be classified.

Proposition 4.3. *Let $\mathcal{F}$ be a foliation on $\mathbf{P}^n$ of degree strictly greater than one. Every foliated affine structure on $\mathcal{F}$ is induced by a homogeneous vector field on $\mathbf{C}^{n+1}$ whose projectivization gives $\mathcal{F}$ together with its foliated affine structure (as explained in Example 2.4).*

Proof. If $\mathcal{F}$ is a foliation of degree d on $\mathbf{P}^n$, $T_{\mathcal{F}} = O(1-d)$, which equals $O^{\otimes(d-1)}(-1)$. Since $d > 1$, a vector field on $T_{\mathcal{F}}$ satisfying (4.1) may be lifted to a vector field Z on the total space of $O(-1)$ such that, for $\alpha \in \mathbf{C}^*$, $Z(h_\alpha v) = \alpha^{d-1} D h_\alpha(Z(v))$. The total space of the line bundle $O(-1)$ on $\mathbf{P}^n$ identifies to the blowup of $\mathbf{C}^{n+1}$ at 0, so Z induces a vector field on $\mathbf{C}^{n+1}$ which is homogeneous of degree d and whose projectivization gives $\mathcal{F}$ together with its foliated affine structure. □

Remark 4.4. The situation is different for linear foliations of $\mathbf{P}^n$ (case $d = 1$ in the previous proposition). The vector fields inducing such a foliation $\mathcal{F}$ are those of the form $X + cR$ with X linear homogeneous, $R = \sum_i z_i \partial/\partial z_i$ and $c \in \mathbf{C}$: they all induce the same foliated affine structure (actually, a translation one). However, the space of foliated affine structures on $\mathcal{F}$ has dimension one ($T_{\mathcal{F}}$ is trivial, and so is $K_{\mathcal{F}}$), and some foliated affine structures on $\mathcal{F}$ are not induced by homogeneous vector fields on $\mathbf{C}^{n+1}$.

Local expressions for geodesic vector fields may be given as follows. Let $\{U_i\}_{i\in I}$ be a cover of M by open subsets such that, in U_j, $\mathcal{F}$ is given by the vector field Z_j. If $U_i \cap U_j \neq \emptyset$, let $g_{ij} : U_i \cap U_j \to \mathbf{C}^*$ be the function such that $Z_i = g_{ij} Z_j$. The line bundle $T_{\mathcal{F}}$ is obtained by gluing the sets in $\{U_i \times \mathbf{C}\}_{i\in I}$ by means of the identification

$$(u, \zeta_j) = (u, g_{ij}\zeta_i) \tag{4.2}$$

if $U_i \cap U_j \neq \emptyset$. Now let the foliated affine structure come into play. Let $\gamma_j : U_j \to \mathbf{C}$ be the Christoffel symbol $\nabla(Z_j)$. Consider, in $U_j \times \mathbf{C}$, the vector field

$$X_j = \zeta_j Z_j - \gamma_j \zeta_j^2 \frac{\partial}{\partial \zeta_j}. \tag{4.3}$$

In $(U_i \times \mathbf{C}) \cap (U_j \times \mathbf{C})$, under (4.2), this vector field reads $g_{ij}\zeta_i Z_j - (g_{ij}\gamma_j + Z_j g_{ij})\zeta_i^2 \partial/\partial\zeta_i$, which, by Leibniz's rule (2.3), equals X_i. This shows that (4.3) defines a global holomorphic vector field X on the total space of $T_{\mathcal{F}}$. We now establish that this is the geodesic vector field of ∇.

The vector field H on $T_{\mathcal{F}}$ given by $\zeta_j \partial/\partial\zeta_j$ in $U_j \times \mathbf{C}$ is globally well-defined. We have the relation $[H, X] = X$. In its integral form, it implies that if $(z(t), \zeta_j(t))$ is a solution to X_j and $\alpha \in \mathbf{C}$, $(z(\alpha t), \alpha\zeta_j(\alpha t))$ is also a solution, which is exactly condition (4.1). Since all the solutions above a given point may be constructed in this way, the vector field X gives a class of parametrizations of the leaves of $\mathcal{F}$ that is invariant under precompositions by affine maps. The inverses of these parametrizations form the atlas of charts of a foliated affine structure.

Let us prove that the foliated affine structure associated to X is exactly the one we started with, that the parametrized solutions of X project onto the geodesics of our original foliated

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affine structure. If the vector field Z_j is such that $\nabla(Z_j)$ vanishes identically, then, on the one hand, the geodesics of ∇ are the integral curves of Z_j (with their natural parametrization) and its constant multiples; on the other hand, X_j reduces to $\zeta_j Z_j$ (ζ_j is a first integral), and the integral curves of the latter project also onto the integral curves of the constant multiples of Z_j. We conclude that X is the geodesic vector field of ∇.

Remark 4.5. The alternative definition of Remark 4.2 may be infinitesimally expressed as follows: *a foliated affine structure on $\mathcal{F}$ is a vector field X on $T_{\mathcal{F}}$ projecting onto $\mathcal{F}$ such that $[H,X]=X$.* Such a vector field has local expressions of the form (4.3), and the foliated connection $\nabla : T_{\mathcal{F}} \to \mathcal{O}(M)$ locally defined by $\nabla(Z_j)=\gamma_j$ gives a globally well-defined connection.

Remark 4.6. The geodesic vector field X is quasihomogeneous, for $[H,X]=X$. The singularities of $\mathcal{F}$ are in correspondence with the fibers of $T_{\mathcal{F}}$ along which $H\wedge X=0$; those with nonvanishing Christoffel symbols correspond to the fibers where X does not vanish identically. Let $p\in\mathrm{Sing}(\mathcal{F})$ be one of these, and let $\nu_1,\dots,\nu_n$ be its principal ramification indices. For the solution of X contained in the fiber above p, one can define its *Kowalevsky exponents*, complex numbers that localize some integrability properties of X. In our case, with the normalizations found in [Gor00], these are $-1,\nu_1,\dots,\nu_n$.

The vector field X induces a foliation by curves $\mathcal{G}$ on $T_{\mathcal{F}}$ that leaves the zero section invariant. This is exactly the setting of Lehmann's theorem.

4.2 Lehmann's theorem

For the proof of Theorem 4.1, we will use an index theorem due to Lehmann which generalizes the Camacho–Sad index theorem to higher dimensions [Leh91]. Let us recall it in the generality that will suit our needs. We follow the normalizations and sign conventions found in [Suw98].

Let V be a manifold of dimension $n+1$, $M\subset V$ a codimension-one smooth compact submanifold with normal bundle N_M, and $\mathcal{G}$ a foliation by curves on V leaving M invariant. Suppose that the singularities of the foliation induced by $\mathcal{G}$ on M are isolated. For such a singularity p, in coordinates $(z_1,\dots,z_n,w)$ centered at p, where M is given by $w=0$ and $\mathcal{G}$ is induced by $X=\sum_{i=1}^n a_i(z,w)\partial/\partial z_i + wb(z,w)\partial/\partial w$, define

$$\mathrm{Res}_{\mathcal{G}}(c_1^n,M,p)=(-1)^n\left(\frac{i}{2\pi}\right)^n\int_T \frac{b^n(z,0)}{\prod_{i=1}^n a_i(z,0)}\,dz_1\wedge\cdots\wedge dz_n,$$

with $T=\{w=0\}\cap(\bigcap_{i=1}^n\{\|a_i(z,0)\|=\epsilon\})$ for some sufficiently small ϵ. (This number is well-defined.) Lehmann's theorem affirms that

$$\sum_{p\in\mathrm{Sing}(\mathcal{G}|_M)}\mathrm{Res}_{\mathcal{G}}(c_1^n,M,p)=c_1^n(N_M),$$

where $c_1(N_M)\in H^2(M,\mathbf{Z})$ denotes the first Chern class of N_M.

If the restriction of X to $w=0$ is nondegenerate at p and the eigenvalues of the linear part of this restriction are $\lambda_1,\dots,\lambda_n$, we have that $\mathrm{Res}_{\mathcal{G}}(c_1^n,M,p)=b^n(0)(\lambda_1\cdots\lambda_n)^{-1}$.

Proof of Theorem 4.1. Let $\mathcal{G}$ be the foliation on $T_{\mathcal{F}}$ induced by the geodesic vector field X. It leaves the zero section M invariant. If $\mathcal{F}$ is generated by $Z=\sum_i a_i(z)\partial/\partial z_i$ in a neighborhood of p, $\mathcal{G}$ is, in a neighborhood of p in $T_{\mathcal{F}}$, tangent to the vector field $\sum_{i=1}^n a_i(z)\partial/\partial z_i-\zeta\gamma(z)\partial/\partial\zeta$. If $\gamma(0)\neq 0$ and $\nu_1,\dots,\nu_n$ are the principal affine ramification indices of the foliated affine structure at p, then, by (3.1),

$$\mathrm{Res}_{\mathcal{G}}(c_1^n,M,p)=\frac{(-\gamma(0))^n}{\lambda_1\cdots\lambda_n}=\frac{(-1)^n}{\nu_1\cdots\nu_n}.$$

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On the other hand, by construction, N_M is exactly $T_{\mathcal{F}}$. A straightforward application of Lehmann's theorem yields Theorem 4.1. □

5. An index theorem for foliated projective structures

Every compact curve admits a projective structure but, as we have seen, not every foliation in a surface admits a foliated one. When foliated projective structures do exist, the foliation and the ambient manifold impose conditions on the behavior of the structure at the singular points of the foliation. The results in this section cast these in a precise form.

The Baum–Bott index theorem [BB70] is behind the formulation and the proof of our result, and we begin by recalling some of the notions and terms that appear in its statement, which will be briefly recalled in § 5.3.

Let $\varphi(x_1, \dots, x_k)$ be a symmetric homogeneous polynomial of degree k with complex coefficients. Define the polynomial $\widetilde{\varphi}$ through the equality $\widetilde{\varphi}(\sigma_1, \dots, \sigma_k) = \varphi(x_1, \dots, x_k)$, where $\sigma_i = \sum_{j_1 < \dots < j_i} x_{j_1} \dots x_{j_i}$ is the ith elementary symmetric polynomial in $x_1, \dots, x_k$. For a vector bundle V on the manifold N, let $c_i(V)$ denote the ith Chern class of V, $c(V)$ its total Chern class, and let $\varphi(c(V)) \in H^{2k}(N, \mathbf{Z})$ be given by $\widetilde{\varphi}(c_1(V), \dots, c_k(V))$. This definition extends to the context of virtual vector bundles, elements of the K-theory of N. We refer the reader to [Bot69] for facts around virtual vector bundles and their Chern classes.

Let $\varphi(x_1, \dots, x_{n+1})$ be a symmetric homogeneous polynomial of degree $n+1$. We distinguish the variable x_{n+1}. For $i = 0, \dots, n+1$, define the symmetric homogeneous polynomial of degree i in n variables $\widehat{\varphi}_i$ through the equality

$$\varphi(x_1, \dots, x_n, x_{n+1}) = \sum_{i=0}^{n+1} x_{n+1}^{n+1-i} \widehat{\varphi}_i(x_1, \dots, x_n). \tag{5.1}$$

In particular, for the odd part (with respect to x_{n+1}) φ_{odd} of φ,

$$\varphi_{\text{odd}}(x_1, \dots, x_n, x_{n+1}) = \sum_{j=0}^{\lfloor n/2 \rfloor} x_{n+1}^{2j+1} \widehat{\varphi}_{n-2j}(x_1, \dots, x_n).$$

THEOREM 5.1. *Let M be a compact complex manifold of dimension n, and $\mathcal{F}$ a holomorphic foliation by curves on M having only isolated nondegenerate singularities $p_1, \dots, p_k$. Consider a holomorphic foliated projective structure subordinate to $\mathcal{F}$ for which the Christoffel symbols do not vanish at the singularities, and let $\nu_1^{(i)}, \dots, \nu_n^{(i)}$ be the principal projective ramification indices at p_i. Let $\varphi(x_1, \dots, x_{n+1})$ be a symmetric homogeneous polynomial of degree $n+1$. Then, with the previous notation,*

$$\sum_{i=1}^{k} \frac{\varphi_{\text{odd}}(\nu_1^{(i)}, \dots, \nu_n^{(i)}, 1)}{\nu_1^{(i)} \cdots \nu_n^{(i)}} = \sum_{j=0}^{\lfloor n/2 \rfloor} c_1^{2j}(T_{\mathcal{F}}) \widehat{\varphi}_{n-2j}(c(TM - T_{\mathcal{F}})). \tag{5.2}$$

The summands in the left-hand side of (5.2) are, in agreement with Remark 3.5, well-defined.

Example 5.2. If n is even, for $\varphi = \sum_{i=1}^{n+1} x_i^{n+1}$, $\varphi_{\text{odd}}(x_1, \dots, x_{n+1}) = x_{n+1}^{n+1}$, and (5.2) becomes simply formula (1.2).

Example 5.3. If n is odd, $n = 2m+1$, for $\varphi = \sum_{i \neq j} x_i^n x_j$,

$$\varphi_{\text{odd}} = x_{n+1}^n \sum_{i=1}^{n} x_i + x_{n+1} \sum_{i=1}^{n} x_i^n,$$

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and (5.2) becomes

$$\sum_{i=1}^{k}\frac{\nu_1^{(i)}+\cdots+\nu_n^{(i)}}{\nu_1^{(i)}\cdots\nu_n^{(i)}}+\sum_{i=1}^{k}\frac{\big(\nu_1^{(i)}\big)^n+\cdots+\big(\nu_n^{(i)}\big)^n}{\nu_1^{(i)}\cdots\nu_n^{(i)}}=c_1^{2m}(T_{\mathcal{F}})c_1(TM-T_{\mathcal{F}})+\psi(TM-T_{\mathcal{F}}).$$

The Baum–Bott index theorem (see § 5.3) implies that the second summands in each side are equal, and the equality reduces to (1.1).

In particular, in even dimensions, with Remark 3.6 taken into account, Theorem 5.1 extends Theorem 4.1 to the projective setting. (In odd dimensions, Theorem 4.1 is exclusively affine because, to begin with, from Remark 3.5, its statement does not make sense in the projective case.)

Example 5.4. Let $\mathcal{F}$ be the foliation on $\mathbf{P}^2$ given by the pencil of conics through four points in general position. There are three singular conics, pairs of lines associated to the three ways in which the four points in the base can be taken in pairs; these degenerations correspond to the 'pinching' of a loop. Beyond the four base points of the pencil, $\mathcal{F}$ has three other singular points, those where the lines in each pair intersect (all of them are nondegenerate). Fix a foliated projective structure on $\mathcal{F}$. For a nonsingular conic in the pencil, the base points give four Fuchsian singularities of the projective structure (as defined in § 3.2), with ramification indices $\nu_1, \ldots, \nu_4$, which are independent of the conic. In a degenerate conic in the pencil the pinched loop produces, in each one of its lines, a Fuchsian singular point of the projective structure having a ramification index μ_i, independent of the line. Theorem 5.1 in the instance of Example 5.2 gives that

$$\frac{1}{\nu_1^2}+\frac{1}{\nu_2^2}+\frac{1}{\nu_3^2}+\frac{1}{\nu_4^2}-\frac{1}{\mu_1^2}-\frac{1}{\mu_2^2}-\frac{1}{\mu_3^2}=1,$$

and, in particular, that the three degenerations are not independent.

Remark 5.5. The canonical bundle of a foliation of degree two on $\mathbf{P}^2$ is $O(1)$, its square $O(2)$, and the dimension of the affine space of foliated projective structures is six. For such a foliation, the rational map that to each foliated projective structure subordinate to it associates the seven expressions $\nu_1^{(i)}\nu_2^{(i)}$ takes values in the six-dimensional variety given by (1.2). A calculation we have made with a computer algebra system shows that, at a generic foliated projective structure subordinate to the foliation of the previous example, the differential of this map has full rank. In particular, together with Corollary 3.3 in [Gui06], this implies that *in the space of foliations of degree two of* $\mathbf{P}^2$ *endowed with a foliated projective structure, the projective ramification indices determine a generic element (both the foliation and the projective structure) up to a finite indeterminacy.*

Remark 5.6. In Theorem 5.1, singularities of $\mathcal{F}$ with vanishing Christoffel symbols may be considered (still under the hypothesis of nondegeneracy). The contribution to the left-hand side of (5.2) of such a singularity reduces to its Baum–Bott index associated to $\widehat{\varphi}_n$. Details are left to the reader.

For the proof of Theorem 5.1, we construct a geodesic vector field for the foliated projective structure and apply the Baum–Bott index theorem to a foliation associated to it. This vector field is defined in the total space of a particular vector bundle which we now describe.

5.1 Some foliated jet bundles

From a vector bundle over a manifold $V \to M$, one may construct the vector bundle $J^kV \to M$ of k-jets of sections of V (see, for instance, [KS72]). In the presence of a singular foliation by curves $\mathcal{F}$ on M, we may consider the bundle $J^k_{\mathcal{F}}V$ of k-jets of sections of V 'along the leaves of $\mathcal{F}$', which exists, at least, away from the singularities of $\mathcal{F}$. The bundle need not extend to the singular points of the foliation, but if it does, this extension is unique. (See Theorem 1 and Proposition 7 in [Ser66].)

For $V = T_{\mathcal{F}}$, the bundle $J^k_{\mathcal{F}}T_{\mathcal{F}}$ can be defined as a vector bundle (of rank $k+1$) on the whole of M. Let us show this concretely by exhibiting explicit local trivializations. Let Z be a vector field tangent to $\mathcal{F}$ defined on an open set $U \subset M$, which vanishes only on the singular set of $\mathcal{F}$, and let $A = \sum(1/l!)a_l z^l \in \mathbf{C}[z]$ be a polynomial of degree at most k. Build a holomorphic section $\sigma(Z,A)$ of $J^k_{\mathcal{F}}T_{\mathcal{F}}$ over U in the following way: at a regular point p of $\mathcal{F}$ in U, for the vector field fZ such that $Z^l(f)|_p = a_l$ for $l = 0,\dots,k$, $\sigma(Z,A)(p)$ is the k-jet along $\mathcal{F}$ at p of the vector field fZ. The section $\sigma(Z,1)$ is the one obtained by taking the k-jet of the vector field Z itself. By construction, the sections $\sigma(Z,1), \sigma(Z,z), \dots, \sigma(Z,z^k)$ are linearly independent at every regular point of $\mathcal{F}$ in U, so they induce a trivialization of $J^k_{\mathcal{F}}T_{\mathcal{F}}$ on $U \setminus \mathrm{Sing}(\mathcal{F})$. The existence of these local trivializations suffices to show that the bundle $J^k_{\mathcal{F}}T_{\mathcal{F}}$ can be extended over the singular set in a unique way, because the cocycle that expresses one trivialization in terms of the other consists of holomorphic functions $U \setminus \mathrm{Sing}(\mathcal{F}) \to \mathrm{GL}(k+1,\mathbf{C})$ that, by Hartogs's theorem, extend to the singular set as functions taking values in the set of invertible matrices (the determinant cannot vanish only on a set of codimension two or more). We thus construct $J^k_{\mathcal{F}}T_{\mathcal{F}} \to M$, the *vector bundle of k-jets of $T_{\mathcal{F}}$ along $\mathcal{F}$*. We have the linear projections $j_{k-i} : J^k_{\mathcal{F}}T_{\mathcal{F}} \to J^{k-i}_{\mathcal{F}}T_{\mathcal{F}}$, and, naturally, $J^0_{\mathcal{F}}T_{\mathcal{F}} = J^0T_{\mathcal{F}} = T_{\mathcal{F}}$.

We are interested in the case $k = 1$, for which it is useful to have an explicit expression for the cocycle defined above. Let $\{U_i\}_{i\in I}$ be a cover of M by open subsets such that $\mathcal{F}$ is generated by the holomorphic vector field Z_i in U_i. Let $g_{ij} : U_i \cap U_j \to \mathbf{C}^*$ be such that $Z_i = g_{ij}Z_j$; it is the cocycle associated to $T_{\mathcal{F}}$. Let $p \in U \setminus \mathrm{Sing}(\mathcal{F})$, and let z be a coordinate along the leaf L of $\mathcal{F}$ through p, centered at p, where the restriction of Z_j to L reads $\partial/\partial z$. In restriction to L, $Z_i = g_{ij}(z)\partial/\partial z$ and, thus, along L,

$$Z_i = \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{d^l}{d\zeta^l} g_{ij}(\zeta)\bigg|_{\zeta=0}\right) z^l \frac{\partial}{\partial z} = \sum_{l=0}^{\infty} \frac{1}{l!} Z_j^l(g_{ij})|_p z^l \frac{\partial}{\partial z}.$$

On the other hand, $\sigma(Z,z)$ is independent of Z: if a vector X field tangent to $\mathcal{F}$ vanishes at a nonsingular point p in the leaf L of $\mathcal{F}$, its first jet along $\mathcal{F}$ at p is the eigenvalue of the linear part at p of its restriction to L. From these facts, on $U_i \cap U_j$,

$$\begin{pmatrix} \sigma(Z_i,1) \\ \sigma(Z_i,z) \end{pmatrix} = \begin{pmatrix} g_{ij} & Z_j(g_{ij}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma(Z_j,1) \\ \sigma(Z_j,z) \end{pmatrix}.$$

Summing up, $J^1_{\mathcal{F}}T_{\mathcal{F}}$ can be obtained by gluing the sets in $\{U_i \times \mathbf{C}^2\}_{i\in I}$ through the identifications

$$\begin{pmatrix} \zeta_j \\ \xi_j \end{pmatrix} = \begin{pmatrix} g_{ij} & 0 \\ Z_j(g_{ij}) & 1 \end{pmatrix} \begin{pmatrix} \zeta_i \\ \xi_i \end{pmatrix}, \tag{5.3}$$

with (ζ_i,ξ_i) linear coordinates on $\mathbf{C}^2$. In these, the section of $J^1_{\mathcal{F}}T_{\mathcal{F}}$ over U_i generated by fZ_i is given by

$$fZ_i \mapsto (f, Z_i(f)). \tag{5.4}$$

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The projections $(\zeta_i, \xi_i) \mapsto \zeta_i$ glue into the natural linear projection $j_0 : J^1_{\mathcal{F}} T_{\mathcal{F}} \to T_{\mathcal{F}}$. The line subbundle $\ker(j_0)$ is trivial.

5.2 The geodesic vector field and its projectivization

If $\mathcal{F}$ is endowed with a foliated projective structure and $p \in M \setminus \mathrm{Sing}(\mathcal{F})$, a *geodesic* through p is a parametrized curve $f : (U, 0) \to (M, p)$, $U \subset \mathbf{C}$, $0 \in U$, which is tangent to $\mathcal{F}$ and which induces the given projective structure on the leaf of $\mathcal{F}$ through p (which is the inverse of a projective chart). For the tautological projective structure on $\mathbf{P}^1$, in the affine chart $[z : 1]$, the geodesics through 0 are those of the form $t \mapsto at/(1 - bt)$ with $a \in \mathbf{C}^*$, $b \in \mathbf{C}$. Their corresponding velocity vector fields are $a^{-1}(a + bz)^2 \partial/\partial z$. Each one of them is characterized by its 1-jet at 0. Furthermore, every 1-jet of vector field with a nonvanishing 0-jet is realized by the velocity vector field of a geodesic. In this way, above the regular part of $\mathcal{F}$, the geodesics of a foliated projective structure lift into $J^1_{\mathcal{F}} T_{\mathcal{F}} \setminus \ker(j_0)$ through their velocity vector fields, and there is a lift of a unique geodesic through every point in $J^1_{\mathcal{F}} T_{\mathcal{F}} \setminus \ker(j_0)$. There is, thus, a natural vector field on $J^1_{\mathcal{F}} T_{\mathcal{F}} \setminus \ker(j_0)$ associated to a foliated projective structure. This is its *geodesic vector field*.

If $f : U \to M$ is a geodesic defined in a neighborhood of $t = 0$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbf{C})$ is sufficiently close to the identity, $f((at + b)/(ct + d))$ is also a geodesic: we have a local action of $\mathrm{SL}(2, \mathbf{C})$ on $J^1_{\mathcal{F}} T_{\mathcal{F}} \setminus \ker(j_0)$ induced by the foliated projective structure. Let us restrict this action to the one-parameter subgroups of the standard basis of $\mathfrak{sl}(2, \mathbf{C})$. The vector fields on $J^1_{\mathcal{F}} T_{\mathcal{F}}$ associated to these will satisfy the same Lie-algebraic relations. Let $f : U \to M$ be a geodesic. That the action via reparametrization of the one-parameter subgroup $\left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$ preserves geodesics is equivalent to the fact that if $f(t)$ is a parameterized geodesic, so is $f(t + s)$ for every (sufficiently small) fixed s. This reparametrization comes thus from the flow of the geodesic vector field. The one induced by the one-parameter subgroup $\left(\begin{smallmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{smallmatrix}\right)$ yields the geodesic $t \mapsto f(e^s t)$. It multiplies the velocity vector field of f by the factor e^s. In coordinates, it multiplies each one of the two components of (5.4) by e^s, and is thus induced by the vector field H on $J^1_{\mathcal{F}} T_{\mathcal{F}}$ that, in $U_j \times \mathbf{C}^2$, reads $\zeta_j \partial/\partial \zeta_j \oplus \xi_j \partial/\partial \xi_j$ (it retains its expression in the other charts and is globally well-defined). Now let us consider the more interesting case of the one-parameter subgroup $\left(\begin{smallmatrix} 1 & 0 \\ -s & 1 \end{smallmatrix}\right)$. Let us suppose that we are on a curve where we have a local coordinate z and that $f(0) = 0$. The reparametrization gives the geodesic $f(t/(1 - st))$, whose velocity vector field is $(1 + sf^{-1}(z)) f'(f^{-1}(z)) \partial/\partial z$. With respect to the vector field $Z = v(z)\partial/\partial z$, the 1-jet of its velocity vector field, as in (5.4), is

$$\left(\frac{f(0)}{v(0)}, -\frac{v'(0)}{v(0)} + \frac{f''(0)}{f'(0)} + 2s \right).$$

The action of this one-parameter subgroup induces the vector field Y on $J^1_{\mathcal{F}} T_{\mathcal{F}}$ that reads $2\partial/\partial \xi_j$ on $U_j \times \mathbf{C}^2$. As expected, $[H, Y] = -Y$.

If X is a vector field on $J^1_{\mathcal{F}} T_{\mathcal{F}}$ giving the geodesic vector field of a foliated projective structure, its integral curves project to curves of $\mathcal{F}$ and, together with the vector fields H and Y, satisfies the $\mathfrak{sl}(2, \mathbf{C})$ relations

$$[Y, X] = 2H, \quad [H, X] = X, \quad [H, Y] = -Y. \tag{5.5}$$

The integral form of these relations gives the reparametrization of the geodesics on $J^1_{\mathcal{F}} T_{\mathcal{F}}$: if $(z(t), \zeta_j(t), \xi_j(t))$ is a solution of X_j and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbf{C})$ is sufficiently close to the identity,

$$\left(z\left(\frac{at + b}{ct + d}\right), \frac{1}{(ct + d)^2} \zeta_j\left(\frac{at + b}{ct + d}\right), \frac{1}{(ct + d)^2} \xi_j\left(\frac{at + b}{ct + d}\right) - \frac{2c}{ct + d} \right)$$

is also a solution of X.

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Remark 5.7. The above formula is similar to the 'invariance condition' enjoyed by Halphen's system [Hal81]. This is not surprising, because the latter is essentially an extension to the punctures of the geodesic flow of the projective structure on the thrice-puncture sphere given by uniformization; see [Gui07, Section 3].

Let us explicitly construct the vector field X associated to a foliated projective connection $\Xi : T_{\mathcal{F}} \to \mathcal{O}_M$. Let $\rho_i : U_i \to \mathbf{C}$ be the Christoffel symbol $\Xi(Z_i)$ of Z_i. Consider, on $U_j \times \mathbf{C}^2$, the vector field

$$X_j = \zeta_j Z_j \oplus \zeta_j \xi_j \frac{\partial}{\partial \zeta_j} \oplus \left(\frac{1}{2}\xi_j^2 - \rho_j \zeta_j^2\right)\frac{\partial}{\partial \xi_j}.$$

In $(U_i \times \mathbf{C}^2) \cap (U_j \times \mathbf{C}^2)$, under (5.3), X_j reads

$$g_{ij}\zeta_i Z_j \oplus \zeta_i \xi_i \frac{\partial}{\partial \zeta_i} \oplus \left(\frac{1}{2}\xi_i^2 - \left[g_{ij}^2 \rho_j + g_{ij} Z_j^2 g_{ij} - \frac{1}{2}(Z_j g_{ij})^2\right]\zeta_i^2\right)\frac{\partial}{\partial \xi_i},$$

which, by the Leibniz rule (2.6), is exactly X_i. Thus, these vector fields glue into a globally-defined holomorphic vector field X on $J^1_{\mathcal{F}} T_{\mathcal{F}}$. It satisfies the relations (5.5); since the projections of its integral curves onto M differ by precompositions with fractional linear transformations, it induces a foliated projective structure.

Let us identify the foliated projective structure induced by the vector field X just defined. Let us do so in dimension one, in a coordinate z where Z is $\partial/\partial z$, this is, for the vector field $X = \zeta\partial/\partial z + \zeta\xi\partial/\partial\zeta + (\frac{1}{2}\xi^2 - \rho\zeta^2)\partial/\partial\xi$ and $\widehat{\pi}(z,\zeta,\xi) = z$. Let $(z(t),\zeta(t),\xi(t))$ be a solution to X. Comparing the projective structures induced by z and t on the base, we have

$$\{t, z(t)\} = -\frac{1}{(z'(t))^2}\{z(t), t\} = \rho(z).$$

It follows from this formula that for the projective structure induced by X, the Christoffel symbol of $\partial/\partial z$ is ρ, and coincides, as we sought to establish, with the one induced by the projective connection Ξ.

Note that X is defined as a holomorphic vector field on all of $J^1_{\mathcal{F}} T_{\mathcal{F}}$ and that it is transverse to π over $M \setminus \operatorname{Sing}(\mathcal{F})$ away from $\ker(j_0)$.

Remark 5.8. The conditions (5.5) that the geodesic vector field of a foliated projective structure must satisfy are also sufficient ones. From this, we have an equivalent formulation for our definition: *a foliated projective structure subordinate to $\mathcal{F}$ is a vector field X on $J^1_{\mathcal{F}} T_{\mathcal{F}}$ that projects onto $\mathcal{F}$ and that satisfies the relations (5.5) with the vector fields H and Y.*

Let $\pi : \mathbf{P}(J^1_{\mathcal{F}} T_{\mathcal{F}}) \to M$ be the projectivization of $J^1_{\mathcal{F}} T_{\mathcal{F}}$. The foliation by curves induced by X on $J^1_{\mathcal{F}} T_{\mathcal{F}}$ is invariant by the flow of H and, thus, the total space of $\mathbf{P}(J^1_{\mathcal{F}} T_{\mathcal{F}})$ inherits a foliation by curves $\mathcal{G}$ that projects onto $\mathcal{F}$. This will be the main object in the proof of Theorem 5.1.

For the expression of $\mathcal{G}$ in local coordinates, cover $U_j \times \mathbf{P}^1$ by charts $U_j^+ = U_j \times \mathbf{C}$ and $U_j^- = U_j \times \mathbf{C}$, where, in U_j^+ (respectively, U_j^-), an affine coordinate u_j (respectively, v_j) for the second factor is given by $[u_j : 1] = [\xi_j : \zeta_j]$ (respectively, $[1 : v_j] = [\xi_j : \zeta_j]$). In U_j^+, $\mathcal{G}$ is tangent to the vector field

$$Z_j - \left(\frac{1}{2}u_j^2 + \rho_j\right)\frac{\partial}{\partial u_j} \tag{5.6}$$

and, in U_j^-, to $Z_j + (\frac{1}{2} + \rho_j v_j^2)\partial/\partial v_j$. These last two glue together into a vector field with isolated singularities on $U_i^+ \cup U_i^-$ and, from (5.3), in $U_j^+ \cap U_i^+$, $g_{ij} u_j = u_i + Z_j(g_{ij})$.

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These calculations show that $T_{\mathcal{G}} = \pi^* T_{\mathcal{F}}$. They also exhibit the fact that, even if X is not transverse to π along $\ker(j_0)$, $\mathcal{G}$ is transverse to π above $M \setminus \mathrm{Sing}(\mathcal{F})$.

For an integral curve C of $\mathcal{F}$, the restriction of $\mathcal{G}$ to $\pi^{-1}(C)$ is a Riccati foliation with respect to the rational fibration $\pi|_C$. There is a section $\sigma : M \to \mathbf{P}(J^1_{\mathcal{F}} T_{\mathcal{F}})$ of π given by the projectivization of the subbundle $\ker(j_0)$, which is everywhere transverse to $\mathcal{G}$ and which inherits, in consequence, a foliated projective structure subordinate to $\mathcal{F}$. We claim that this projective structure is the one we started with. Let us prove this in dimension one, in a coordinate where Z is $\partial/\partial z$, where $\mathcal{G}$ is generated by $\partial/\partial z - (\frac{1}{2}\rho(z) + \widetilde{u}^2)\partial/\partial\widetilde{u}$ for $\widetilde{u} = \frac{1}{2}u$. The charts of the projective structure induced on the line $\widetilde{u} = \infty$ by the orbits of this vector field are the solutions h of the Schwarzian equation $\{h, z\} = \rho(z)$ (see [LMP09, Proposition 2.1]). This proves our claim.

Remark 5.9. For a projective structure on a curve, the triple $(\mathbf{P}(J^1_{\mathcal{F}} T_{\mathcal{F}}), \mathcal{G}, \sigma)$ gives the *graph* of the projective structure (see [LMP09, Section 1.5]). Our construction actually gives more: since $\mathcal{G}$ comes from the quotient of X, it is naturally endowed with a foliated affine structure in the complement of the image of $\ker(j_0)$, and π identifies the projective classes of the foliated affine structures with the foliated projective structure of $\mathcal{F}$.

5.3 Proof of Theorem 5.1

Let us briefly recall the Baum–Bott index theorem in the generality that we need. Let N be a compact complex manifold of dimension m, $\mathcal{H}$ a holomorphic foliation by curves on N having only finitely many singularities, all of them nondegenerate. We use the terminology around symmetric polynomials introduced at the beginning of § 5. For $p \in \mathrm{Sing}(\mathcal{H})$, let A_p be the linear part at p of a vector field generating $\mathcal{H}$ in a neighborhood of p. Define $\sigma_i(A_p)$ by $\det(\mathbf{I} + tA_p) = \sum_{i=0}^m \sigma_i(A)t^i$ and, for a homogeneous symmetric polynomial $\varphi(x_1, \dots, x_m)$ of degree m, let $\varphi(A_p) = \widetilde{\varphi}(\sigma_1(A_p), \dots, \sigma_m(A_p))$.

The Baum–Bott index theorem [BB70] affirms that

$$\sum_{p \in \mathrm{Sing}(\mathcal{H})} \frac{\varphi(A_p)}{\det(A_p)} = \varphi(TN - T_{\mathcal{H}}). \tag{5.7}$$

Theorem 5.1 follows from applying it to the foliation $\mathcal{G}$ on $\mathbf{P}(J^1_{\mathcal{F}} T_{\mathcal{F}})$, for the same φ appearing in its statement.

We begin by calculating the left-hand side of (5.7) for the foliation $\mathcal{G}$ on $\mathbf{P}(J^1_{\mathcal{F}} T_{\mathcal{F}})$. Let $p \in U_j$ be a singular point of $\mathcal{F}$, Z a vector field generating $\mathcal{F}$ in a neighborhood of p, $\rho = \Xi(Z)$ the Christoffel symbol of Z, which, by hypothesis, does not vanish at p. In $\mathbf{P}(J^1_{\mathcal{F}} T_{\mathcal{F}})$, above p, there are two singular points of $\mathcal{G}$. At these, from (5.6), the ratios of the eigenvalues of a vector field tangent to $\mathcal{G}$ are $[\lambda_1 : \cdots : \lambda_n : \sqrt{-2\rho(p)}]$ and $[\lambda_1 : \cdots : \lambda_n : -\sqrt{-2\rho(p)}]$. The sum of the contributions of these two points to the left-hand side of (5.7) is

$$\begin{aligned} &\frac{\varphi(\lambda_1, \dots, \lambda_n, \sqrt{-2\rho(p)})}{\lambda_1 \cdots \lambda_n \sqrt{-2\rho(p)}} - \frac{\varphi(\lambda_1, \dots, \lambda_n, -\sqrt{-2\rho(p)})}{\lambda_1 \cdots \lambda_n \sqrt{-2\rho(p)}} \\ &\quad = \frac{\varphi(\nu_1, \dots, \nu_n, 1) - \varphi(\nu_1, \dots, \nu_n, -1)}{\nu_1 \cdots \nu_n} = 2\frac{\varphi_{\mathrm{odd}}(\nu_1, \dots, \nu_n, 1)}{\nu_1 \cdots \nu_n}, \end{aligned}$$

where $\nu_1, \dots, \nu_n$ are the principal projective ramification indices of the foliated projective structure at p, and where the first equality follows from (3.6). This last expression is well-defined (Remark 3.5). Since there are no further singular points of $\mathcal{G}$, the sum of these terms over the singular points of $\mathcal{F}$ gives twice the total sum in the left-hand side of (5.2).

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Now let us come to the right-hand side of (5.7) for the foliation $\mathcal{G}$ on $\mathbf{P}(J^1_{\mathcal{F}}T_{\mathcal{F}})$. In order to express this right-hand side in terms of data in M, we need a better understanding of the Chern classes of $\mathbf{P}(J^1_{\mathcal{F}}T_{\mathcal{F}})$. Grothendieck's approach [Gro58] is particularly well adapted to the study of Chern classes of projective bundles.

Let $\widetilde{\pi} : V \to M$ be a rank-two vector bundle, $\pi : \mathbf{P}(V) \to M$ the associated projective bundle. We denote by $L \to \mathbf{P}(V)$ the dual of the tautological bundle, and by $\zeta \in H^2(\mathbf{P}(V), \mathbf{Z})$ the Chern class of L. By Grothendieck's definition of Chern classes,

$$\zeta^2 + \pi^* c_1(V)\zeta + \pi^* c_2(V) = 0, \tag{5.8}$$

where $c_k(V) \in H^{2k}(M, \mathbf{Z})$ is the kth Chern class of V. From the short exact sequence

$$0 \to \ker(D\pi) \longrightarrow T\mathbf{P}(V) \xrightarrow{D\pi} \pi^* TM \to 0,$$

for the total Chern classes we have

$$c(T\mathbf{P}(V)) = c(\ker(D\pi))c(\pi^* TM).$$

PROPOSITION 5.10. *We have an isomorphism* $\ker(D\pi) \simeq L^{\otimes 2} \otimes \det(V)$. *In particular,* $c(\ker D\pi) = 1 + 2\zeta + c_1(\det(V))$.

Proof. We have the following canonical isomorphism: given $x \in M$ and $l \in \mathbf{P}(V_x)$ (a line in V_x) we have

$$T_l \mathbf{P}(V_x) \simeq \mathrm{Hom}(l, V_x/l). \tag{5.9}$$

Indeed, the derivative of the projectivization $p : V_x \setminus \{0\} \to \mathbf{P}(V_x)$ induces for each $u \in l$ an isomorphism $Dp_u : V_x/l \to T_x\mathbf{P}(V_x)$ that satisfies

$$\lambda Dp_{\lambda u} = Dp_u \quad \text{for any } u \in l \text{ and } \lambda \in \mathbf{C}, \tag{5.10}$$

because p is invariant by multiplication by $\lambda \in \mathbf{C}$. The isomorphism (5.9) is then defined by the formula

$$v \in T_l\mathbf{P}(V_x) \text{ corresponds to } u \in l \xrightarrow{\varphi_v} (Dp_u)^{-1}(v) \in V_x/l,$$

and (5.10) shows that $v \mapsto \varphi_v$ is linear.

Given $\omega \in \wedge^2 V_x^*$ (a dual of the determinant bundle) and $\psi \in \mathrm{Hom}(l, V_x/l)$ we can form the quadratic polynomial φ on l (an element of $L^{\otimes 2}$) by the formula $\varphi(u) = \omega(u, \psi(u))$. This operation produces the desired isomorphism of line bundles over $\mathbf{P}(V)$. □

Associated to $\pi : \mathbf{P}(V) \to M$ we have the *integration along the fibers* (or *transfer*) map $\pi^! : H^l(\mathbf{P}(V), \mathbf{Z}) \to H^{l-2}(M, \mathbf{Z})$ (see [Dol80, Ch. VIII]). It satisfies the product formula $\pi^!(\alpha \cdot \pi^*\beta) = \pi^!\alpha \cdot \beta$ as well as the Fubini relation $\langle \alpha, [\mathbf{P}(V)]\rangle = \langle \pi^!\alpha, [M]\rangle$ (here, $\langle , \rangle$ denotes the cohomology–homology pairing and $[\cdot]$ the fundamental class). We have $\pi^!\zeta = 1$. In particular,

$$\langle \zeta \cdot \pi^*\beta, [\mathbf{P}(V)]\rangle = \langle \pi^!(\zeta \cdot \pi^*\beta), [M]\rangle = \langle \pi^!\zeta \cdot \beta, [M]\rangle = \langle \beta, [M]\rangle. \tag{5.11}$$

In our setting, $J^1_{\mathcal{F}}T_{\mathcal{F}}$ is an extension of $T_{\mathcal{F}}$ by the trivial bundle, so $\det(J^1_{\mathcal{F}}T_{\mathcal{F}}) \simeq T_{\mathcal{F}}$ and, for the total Chern class, $c(J^1_{\mathcal{F}}T_{\mathcal{F}}) = c(T_{\mathcal{F}})$, this is, $c_1(J^1_{\mathcal{F}}T_{\mathcal{F}}) = c_1(T_{\mathcal{F}})$ and $c_2(J^1_{\mathcal{F}}T_{\mathcal{F}}) = 0$. From (5.8), for $V = J^1_{\mathcal{F}}T_{\mathcal{F}}$, $\zeta^2 = -\pi^* c_1(T_{\mathcal{F}})\zeta$, and, for all $k \geq 1$,

$$\zeta^k = (-\pi^* c_1(T_{\mathcal{F}}))^{k-1}\zeta. \tag{5.12}$$

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From the previously established identification $T_{\mathcal{G}} = \pi^* T_{\mathcal{F}}$,

$$c(T\mathbf{P}(J^1_{\mathcal{F}}T_{\mathcal{F}}) - T_{\mathcal{G}}) = \frac{c(T\mathbf{P}(J^1_{\mathcal{F}}T_{\mathcal{F}}))}{c(T_{\mathcal{G}})} = \frac{c(\ker(D\pi))c(\pi^*TM)}{c(\pi^*T_{\mathcal{F}})}$$
$$= (1 + c_1(\ker(D\pi)))\pi^* c(TM - T_{\mathcal{F}}). \tag{5.13}$$

If, as in (5.1), $\varphi(x_1, \ldots, x_{n+1}) = \sum_{i=0}^{n+1} x_{n+1}^i \widehat{\varphi}_{n+1-i}(x_1, \ldots, x_n)$, from (5.13) we have, denoting $c_1(\ker(D\pi))$ by κ,

$$\varphi(c(T\mathbf{P}(J^1_{\mathcal{F}}T_{\mathcal{F}}) - T_{\mathcal{G}})) = \sum_{i=0}^{n+1} \kappa^i \cdot \widehat{\varphi}_{n+1-i}(\pi^* c(TM - T_{\mathcal{F}})). \tag{5.14}$$

From the equality $\kappa = 2\zeta + \pi^* c_1(T_{\mathcal{F}})$ established in Proposition 5.10 and from (5.12), $\kappa^2 = \pi^* c_1^2(T_{\mathcal{F}})$ and, thus,

$$\kappa^i = \begin{cases} \pi^* c_1^i(T_{\mathcal{F}}) + 2\zeta \cdot \pi^* c_1^{i-1}(T_{\mathcal{F}}) & \text{if } i \text{ is odd,} \\ \pi^* c_1^i(T_{\mathcal{F}}) & \text{if } i \text{ is even.} \end{cases}$$

Hence, (5.14) equals

$$\pi^* \sum_{i=0}^{n+1} c_1^i(T_{\mathcal{F}}) \widehat{\varphi}_{n+1-i}(c(TM - T_{\mathcal{F}})) + 2\zeta \cdot \pi^* \sum_{j=0}^{\lfloor n/2 \rfloor} c_1^{2j}(T_{\mathcal{F}}) \widehat{\varphi}_{n-2j}(c(TM - T_{\mathcal{F}})),$$

but the first summand is trivial, because it is the pull-back of classes in M whose degree exceeds the dimension of M. We conclude that, on $\mathbf{P}(J^1_{\mathcal{F}}T_{\mathcal{F}})$,

$$\varphi(c(T\mathbf{P}(J^1_{\mathcal{F}}T_{\mathcal{F}}) - T_{\mathcal{G}})) = 2\zeta \cdot \pi^* \sum_{j=0}^{\lfloor n/2 \rfloor} c_1^{2j}(T_{\mathcal{F}}) \widehat{\varphi}_{n-2j}(c(TM - T_{\mathcal{F}})).$$

By (5.11), this expression equals twice the right-hand side of (5.2). This finishes the proof of Theorem 5.1.

Remark 5.11. Another approach, likely leading to relations like those in Theorem 5.1, would consist in constructing an Atiyah algebroid associated to a foliated projective structure (such as the one described for manifolds in [BD19] and extended to geometries transverse to a foliation in [BD18]), and then adapting the methods of Bruzzo and Rubstov [BR12] to its study. We ignore if this could lead to new relations, and/or if all of our relations can be obtained in this way.

6. Regular foliations

The index theorems of the previous sections impose severe restrictions on foliated affine and projective structures along regular foliations. On surfaces, they make possible a full classification of these structures.

6.1 Some consequences of the index theorems

For a compact surface, the existence of a regular foliation supporting a foliated projective structure greatly limits its topology.

Corollary 6.1. *A compact complex surface admitting a regular foliation that supports a foliated projective structure has vanishing signature.*

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Proof. If M is a compact complex surface that admits a regular foliation $\mathcal{F}$, a consequence of the Baum–Bott index theorem is that $c_1^2(T_{\mathcal{F}}) = c_1^2(M) - 2c_2(M)$; see [Bru97, Section 2]. (From Hirzebruch's formula, the signature $\tau(M)$ equals $\frac{1}{3}(c_1^2(M) - 2c_2(M))$; see [BHPV04, Ch. I, § 3].) If such an $\mathcal{F}$ supports a foliated projective structure, from the instance of Theorem 5.1 in Example 5.2, $c_1^2(T_{\mathcal{F}}) = 0$, and the signature of M vanishes. □

Since Kodaira fibrations have nonvanishing signature [BHPV04, Ch. V, § 14], this gives another proof of the fact that they do not support foliated projective structures.

This obstruction, together with the classification of regular foliations on surfaces provided by Brunella [Bru97], permits to list all the regular foliations on complex surfaces that admit foliated projective structures. (In § 6.2 we give a classification of the foliated affine and projective structures on them.)

COROLLARY 6.2. *The regular foliations on compact complex surfaces that admit foliated projective structures are: isotrivial fibrations, suspensions, linear foliations on tori, turbulent foliations, evident foliations on Hopf or Inoue surfaces, and evident foliations on quotients of the bidisk.*

Proof. Regular foliations on surfaces were classified by Brunella [Bru97, Thm. 2]. Other than those in the previous list, there are nonisotrivial fibrations and some transversely hyperbolic foliations with dense leaves. But nonisotrivial fibrations are necessarily Kodaira ones, because regular elliptic fibrations are necessarily isotrivial, and among the transversely hyperbolic foliations with dense leaves, those supported on surfaces of vanishing signature are quotients of the bidisk (see the closing remarks in [Bru97]).

There remains to exhibit foliated projective structures for all the foliations in the above list. Linear foliations on tori are tangent to holomorphic vector fields, and have a foliated translation structure. Rational fibrations have foliated projective structures (say, by Savel'ev's theorem [Sav82], the fibration is locally holomorphically trivial), elliptic fibrations carry foliated affine structures by the results in Example 2.8, and fibrations of higher genus supporting foliated projective structures are isotrivial, and hence have foliated projective structures (e.g. the complete hyperbolic ones along the fibers). The existence of foliated affine or projective structures for suspensions, elliptic fibrations, turbulent foliations, and foliations on Inoue and primary Hopf surfaces has already been addressed in Examples 2.16, 2.17, 2.5, and 2.6. In § 6.2.7 we show that all foliations on secondary Hopf surfaces admit foliated affine structures. □

Corollary 6.2 shows that a regular foliation on an algebraic compact complex surface carries a foliated projective structure if and only if it is not a foliation of general type. One direction can be directly proved, more generally, for all manifolds of even dimension.

PROPOSITION 6.3. *On a compact algebraic manifold of even dimension, a regular foliation of general type cannot support a foliated projective structure.*

Proof. Let M be the manifold, n its dimension, $\mathcal{F}$ the foliation. The general type assumption on $\mathcal{F}$ says that $K_{\mathcal{F}}$ is big, namely

$$\limsup_{m\to\infty} \frac{\log h^0(K_{\mathcal{F}}^{\otimes m})}{\log m} = n.$$

By [McQ01, Theorem 2, p. 51], $K_{\mathcal{F}}$ is also nef. A nef and big line bundle L on an algebraic variety of dimension n satisfies $c_1^n(L) > 0$; this can be deduced from the asymptotic Riemann–Roch formula, stating that for a nef line bundle L,

$$h^0(L^m) = \frac{1}{n!}c_1^n(L)m^n + O(m^{n-1});$$

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see, for example, [Laz04, Corollary 1.4.41]. Applying this to $L = K_{\mathcal{F}}$ gives $(-1)^n c_1^n(T_{\mathcal{F}}) = c_1^n(K_{\mathcal{F}}) > 0$. On the other hand, if the manifold has even dimension and admits a foliated projective structure, by the particular case of Theorem 5.1 described in Example 5.2, $c_1^n(T_{\mathcal{F}}) = 0$. □

In higher dimensions, there are regular foliations which are not of general type, but which do not support foliated projective structures. Take, for instance, the product of a Kodaira fibration on a surface by a curve, producing a fibration which is not of general type (as a foliation), but which does not have any foliated projective structure.

We do not know if Proposition 6.3 holds true in odd dimensions. We have nevertheless the following weak version of it.

Proposition 6.4. *On a compact algebraic manifold, a regular foliation of general type cannot support a foliated affine structure.*

Proof. In the presence of a foliated affine structure, it follows from Theorem 4.1 that $c_1^n(T_{\mathcal{F}}) = 0$, the left-hand side vanishing by the absence of singular points. The arguments in the proof of Proposition 6.3 allow us to conclude. □

Many families of regular foliations are given by *characteristic foliations* on hypersurfaces of general type in compact symplectic manifolds. They are those generated by the distribution given by the kernel of the restriction of the symplectic form to the hypersurface, see [HV10]. By the adjunction formula and the fact that the canonical of the leaf space of the foliation is trivial, the canonical bundle of such a foliation is isomorphic to the canonical bundle of the hypersurface, so the characteristic foliation is of general type, and does not carry a foliated affine structure by Proposition 6.4. We do not know if these foliations admit foliated projective structures; our index formulae do not give any obstructions whatsoever in this case (we leave to the reader to check that, for all these foliations and in all instances of Theorem 5.1, the right-hand side gives always zero).

These last foliations occur in odd dimensions. In even ones, beyond the case of surfaces, we do not know the extent to which our index formulae give relevant obstructions for the existence of foliated projective structures. We do not seem to have enough examples of regular foliations on manifolds of even dimension.

6.2 A classification for regular foliations on surfaces

We now classify the foliated affine and projective structures for the foliations appearing in Corollary 6.2. We begin with the following lemma.

Lemma 6.5. *Let M be a compact manifold, X a nowhere-vanishing vector field on M, $\mathcal{F}$ the regular foliation induced by X.*

- *The spaces of foliated affine and projective structures of $\mathcal{F}$ are both one-dimensional.*
- *Let $\sigma : M \to M$ be a fixed-point-free involution such that $D\sigma(X) = -X$, let $N = M/\sigma$ and let $\mathcal{G}$ be the foliation on N induced by X. Then the space of foliated projective structures on $\mathcal{G}$ is one-dimensional, and the only foliated affine structure on $\mathcal{G}$ is the one induced by X.*

Proof. A foliated translation structure along $\mathcal{F}$ is induced by X: both the space of foliated affine and projective structures are nonempty. Since any section of $K_{\mathcal{F}}$ (respectively, $K_{\mathcal{F}}^2$) is determined by the constant value it takes on X (respectively, $X^{\otimes 2}$), $H^0(K_{\mathcal{F}})$ (respectively, $H^0(K_{\mathcal{F}}^2)$) is one-dimensional. For the second part, note that the foliated affine structure induced by X on $\mathcal{F}$ is invariant by σ, and descends to a foliated affine structure on $\mathcal{G}$. The sections of $K_{\mathcal{G}}$

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(respectively, $K_{\mathcal{G}}^2$) are in correspondence with the invariant sections of $K_{\mathcal{F}}$ (respectively, $K_{\mathcal{F}}^2$). All sections of $K_{\mathcal{F}}^2$ are invariant, but only the zero section of $K_{\mathcal{F}}$ is. □

Now let us come to the classification.

6.2.1 *Linear foliations on tori.* Foliated affine and projective structures are induced by nowhere-vanishing vector fields and are covered by the first part of Lemma 6.5.

6.2.2 *Suspensions.* As we have shown in Example 2.16, foliated affine (respectively, projective) structures on suspensions are in a one-to-one correspondence with affine (respectively, projective) structures on the base.

6.2.3 *Inoue surfaces.* The existence of foliated affine structures on these surfaces was discussed in Example 2.5.

Consider an Inoue surface S_M (see [Ino74, § 2]). It is the quotient of $\mathbf{H} \times \mathbf{C}$ under the action of a semidirect product $\mathbf{Z} \ltimes \Gamma$; let $\pi : \mathbf{H} \times \mathbf{C} \to S_M$ denote the quotient map. Let $\mathcal{F}$ be either the vertical or the horizontal foliation on S_M. Let $i \in \{1, 2\}$ and let ω be a section of $K_{\mathcal{F}}^i$. The preimage of $\mathcal{F}$ in $\mathbf{H} \times \mathbf{C}$ is generated by a coordinate vector field X. The contraction of $X^{\otimes i}$ with $\pi^*\omega$ gives a holomorphic function on $\mathbf{H} \times \mathbf{C}$ which is invariant under the action of Γ and which, by [Ino74, Lemma 3], is constant. In particular, the function $\pi^*\omega(X^{\otimes i})$ descends to S_M. If it were not the zero constant, $X^{\otimes i}$ would descend to S_M as well, but S_M has no holomorphic vector fields [Ino74, Proposition 2] and neither do its double covers, which are Inoue surfaces of the same kind. We conclude that ω vanishes identically, that $K_{\mathcal{F}}^i$ has no nonzero sections, and that the natural foliated affine structures are rigid both as affine and as projective ones.

Now consider an Inoue surface $S^{(+)}$ (see [Ino74, § 3]). It is the quotient of $\mathbf{H} \times \mathbf{C}$ under the action of a group that preserves the coordinate vector field on the second factor, and which induces a nowhere-vanishing vector field X on $S^{(+)}$. The foliated affine and projective structures on the induced foliation are described by the first part of Lemma 6.5.

Lastly, consider an Inoue surface $S^{(-)}$ (see [Ino74, § 4]). It has an Inoue surface of type $S^{(+)}$ as an unramified double cover $\rho : S^{(+)} \to S^{(-)}$ induced by a fixed-point free involution of $S^{(+)}$ acting upon X by changing its sign. The second part of Lemma 6.5 classifies the foliated structures on the associated foliation.

By the results in [Bru97], these are the only foliations on Inoue surfaces, and the above arguments give a complete classification of foliated affine and projective structures on them.

6.2.4 *Quotients of the bidisk.* In the quotient of $\mathbf{D} \times \mathbf{D}$ under the action of a lattice in $\mathrm{Aut}(\mathbf{D} \times \mathbf{D})$ which is not virtually a product, the vertical (or horizontal) foliation $\mathcal{F}$ carries, by construction, a foliated projective structure which is not affine. For this foliation, $\mathrm{kod}(\mathcal{F}) = -\infty$ (see [Bru04, Ch. 9, Section 5]), and, in particular, $h^0(K_{\mathcal{F}}^2) = 0$: the foliated projective structure is rigid.

6.2.5 *Turbulent foliations.* In this case, the results of Example 2.17 already give a classification of foliated projective structures on them. For instance, in a regular turbulent foliation adapted to a fibration with simple fibers where in the tangency divisor of the foliation and the fibration the nontransverse fibers appear simply, the foliated projective structures are in correspondence with the projective structures on the base which are either regular or have Fuchsian singularities at the points corresponding to the nontransverse fibers.

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These arguments and results easily adapt to foliated affine structures. For instance, if in Example 2.17 we had chosen an affine partial connection ∇_0 instead of the projective one Ξ_0, formula (2.8) would read $\nabla(Z) = z^n \nabla_0(\partial/\partial z) + nz^{n-1}$. This implies, for the existence problem, that elliptic fibrations without singular fibers that are not suspensions support foliated affine structures, and allows us to go from foliated affine structures to some singular affine structures on the base and back.

6.2.6 *Isotrivial fibrations.* Let $\pi : S \to C$ be a regular isotrivial fibration of with typical (nonrational) fiber F. There exists a ramified Galois cover $C' \to C$ with Galois group Γ and an action of Γ on F such that S is given by the quotient of $F \times C'$ under the diagonal action of Γ, π by the projection to C'/Γ (see [Ser96]). By pull-back, we obtain a foliated projective structure on $F \times C'$. Since, on a product, foliated projective structures are constant (such a structure is given by a map from the base of the fibration to the moduli of projective structures on a fixed curve, which is affine), the foliated projective structures on the fibration are in correspondence with the projective structures on F invariant by Γ.

6.2.7 *(Nonelliptic) Hopf surfaces.* Following Brunella's classification, every regular foliation on an elliptic surface is either the elliptic fibration or a turbulent foliation adapted to it. Having already discussed the foliated structures on these, we restrict to Hopf surfaces which are not elliptic, that is, whose group contains a contraction of the form (2.2) with either $\lambda \neq 0$ or $\alpha \neq \beta^n$ (otherwise, the map $(x, y) \mapsto [x : y^n]$ induces an elliptic fibration). The existence of foliated affine structures for foliations on primary Hopf surfaces has already been discussed in Example 2.6.

Secondary Hopf surfaces are unramified quotients of primary ones, quotients of $\mathbf{C}^2 \setminus \{0\}$ under the action of the semidirect product of the infinite cyclic group G generating the associated primary surface and a finite group H that normalizes it. Foliations on secondary Hopf surfaces are induced by foliations on the associated primary ones, and we establish that *every foliation on a secondary Hopf surface has a foliated affine structure*. We owe a classification of Hopf surfaces and their groups to Kato [Kat75, Kat89], and rely on it for the discussion that follows.

For foliations on Hopf surfaces induced by a linear or 'Poincaré–Dulac' vector field on $\mathbf{C}^2$, the contraction (2.2) preserves the vector field. Kato's classification shows that the action of H must also preserve it, inducing a nowhere-zero vector field tangent to foliation on the Hopf surface, and thus endowing it with a translation structure. For these foliations, the first part of Lemma 6.5 describes the foliated structures along them.

Now consider a foliation $\mathcal{F}$ on the primary Hopf surface S induced by $\partial/\partial x$. If $\lambda = 0$, the vector field $x\partial/\partial x$ is invariant by g, and if $\lambda \neq 0$, the vector field $y^n \partial/\partial x$ does: in all cases, we have a vector field tangent to $\mathcal{F}$ invariant by the contraction inducing a holomorphic vector field X on S with nonempty zero set. Let $i \in \{1, 2\}$. Let ω be a section of $K_{\mathcal{F}}^i$. The holomorphic function on S given by $\omega(X^{\otimes i})$ is constant and vanishes along the zero set of X and is, thus, identically zero. This proves that $h^0(K_{\mathcal{F}}^i) = 0$: the foliated affine structure induced by $\partial/\partial x$ is rigid both as an affine and as a projective one. For the associated secondary Hopf surfaces, Kato's classification shows that the corresponding action on $\mathbf{C}^2 \setminus \{0\}$ preserves $\partial/\partial x$ up to constant factors, thus inducing a foliated affine structure. The rigidity of the foliated structures on the primary Hopf surfaces passes down to the secondary ones.

Acknowledgements

We thank ShengYuan Zhao, Serge Cantat, and Volodia Rubtsov for stimulating conversations on this topic, and Omar Antolín, Sébastien Boucksom, Stéphane Druel, Sorin Dumitrescu, Jorge Pereira, and Fernando Sanz for pointing out helpful references. We also wish to thank the anonymous referee, whose suggestions and thorough reading have greatly helped us improve the text. B.D. is grateful to IMPA and Universidad Federal Fluminense for the stimulating working conditions where part of this work was developed. A.G. thanks the École Normale Supérieure de Paris for their hospitality during the sabbatical leave where this work began; he gratefully acknowledges support from grant PAPIIT-IN102518 (UNAM, Mexico).

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Bertrand Deroin bertrand.deroin@cyu.fr
CNRS-Laboratoire AGM-Université de Cergy-Pontoise, 2 Avenue Adolphe Chauvin, 95302 Cergy-Pontoise, France

Adolfo Guillot adolfo.guillot@im.unam.mx
Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad Universitaria 04510, Ciudad de México, Mexico