Canad. Math. Bull. Vol. **49** (1), 2006 pp. 134–143

Real Hypersurfaces in Complex Two-Plane Grassmannians with Vanishing Lie Derivative

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Abstract. In this paper we give a characterization of real hypersurfaces of type A in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which are tubes over totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ in terms of the *vanishing Lie derivative* of the shape operator A along the direction of the Reeb vector field ξ.

0 Introduction

In the geometry of real hypersurfaces there were some characterizations of homogeneous real hypersurfaces of type A_1, A_2 in complex projective space $\mathbb{C}P^m$ and of type A_0, A_1, A_2 in complex hyperbolic space $\mathbb{C}H^m$. As an example, we say that the shape operator A and the structure tensor ϕ commuting with each other, that is $A\phi = \phi A$, is a model characterization of this type hypersurface, which is a tube over a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^m$ (See Okumura [8]), a tube over a totally geodesic $\mathbb{C}H^k$ in $\mathbb{C}H^m$, or a horosphere in $\mathbb{C}H^m$ (See Montiel and Romero [7]).

Now let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$, which is said to be a *complex two-plane Grassmannian*. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ is equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. Then for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ we have considered the two natural geometric conditions that $[\xi] = \mathrm{Span}\{\xi\}$ or $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ be invariant under the shape operator A of M, where $\xi = -JN$ and $J_\nu = -J_\nu N$, $\nu = 1, 2, 3$ for a unit normal vector field N of M in $G_2(\mathbb{C}^{m+2})$. (See the details in [2, 3].)

The first result in this direction is the classification of real hypersurfaces in

$$G_2(\mathbb{C}^{m+2})$$

satisfying both conditions mentioned above. Namely, Berndt and the present author [2] have proved the following:

Theorem A *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

Received by the editors February 1, 2004; revised July 5, 2004.
This work was supported by grant Proj. No. R14-2002-003-01001-0 from the Korea Science & Engineering Foundation.
AMS subject classification: Primary: 53C40; secondary: 53C15.

In [3], Berndt and the present author have given a characterization of real hypersurfaces of type A in Theorem A when the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor ϕ, which is equivalent to the condition that *the Reeb flow on M is isometric*, that is $\mathcal{L}_\xi g = 0$, where $\mathcal{L}$(resp. g) denotes the Lie derivative(resp. the induced Riemannian metric) of M in the direction of the Reeb vector field ξ. Namely, we proved the following:

Theorem B *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

When the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric* in Theorem B, we say that the Reeb vector field ξ on M is Killing. This means that the metric tensor g is invariant under the Reeb flow of ξ on M. In this paper, specifically we assert a characterization of real hypersurfaces of type A in Theorem A by another geometric Lie invariant, that is, the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ is invariant under the Reeb flow on M as follows:

Main Theorem *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb flow on M satisfies $\mathcal{L}_\xi A = 0$ if and only if M is an open part of a tube around some totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

1 Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$

In this section we give basic material about complex two-plane Grassmannians

$$G_2(\mathbb{C}^{m+2}),$$

for details see [2, 3]. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S\big(U(2) \times U(m)\big) \subset G$. Then $G_2\left(\mathbb{C}^{m+2}\right)$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2\left(\mathbb{C}^{m+2}\right)$ becomes analytic.

Now let us denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $\mathrm{Ad}(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $\mathrm{Ad}(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathrm{su}(m) \oplus \mathrm{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathrm{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{tr}(JJ_1) = 0$.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\overline{\nabla}$ of

$\left(G_2(\mathbb{C}^{m+2}), g\right)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\overline{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

2 Some Fundamental Formulas for Real Hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some basic formulae from the Codazzi equation for a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces an almost contact metric structure (ϕ, ξ, η, g) on M. Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the expression for the curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ given in [2] and [3], the Codazzi equation becomes

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ &\quad + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \\ &\quad + \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\ &\quad + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu.\end{aligned}$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations, (see [10, 11]):

$$\begin{aligned} \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad & \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ \phi\xi_\nu = \phi_\nu\xi, \quad & \eta_\nu(\phi X) = \eta(\phi_\nu X), \\ \phi_\nu\phi_{\nu+1}X &= \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ \phi_{\nu+1}\phi_\nu X &= -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}, \end{aligned} \tag{2.1}$$

where the index ν denotes $\nu = 1, 2, 3$.

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{2.2}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.3}$$

$$(\nabla_X\phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX,Y)\xi_\nu. \tag{2.4}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.5}$$

3 Proof of the Main Theorem

Before giving the proof of our theorem, let us determine which of the model hypersurfaces given in Theorem A satisfy the formula $\mathcal{L}_\xi A = 0$. First note that

$$\begin{aligned}(\mathcal{L}_\xi A)X &= \mathcal{L}_\xi(AX) - A\mathcal{L}_\xi X\\ &= \nabla_\xi(AX) - \nabla_{AX}\xi - A(\nabla_\xi X - \nabla_X\xi)\\ &= (\nabla_\xi A)X - \nabla_{AX}\xi + A\nabla_X\xi\\ &= (\nabla_\xi A)X - \phi A^2X + A\phi AX\\ &= 0\end{aligned}$$

for any vector field X on M. Then the assumption $\mathcal{L}_\xi A = 0$ holds if and only if $(\nabla_\xi A)X = \phi A^2X - A\phi AX$. In this section we will show that only a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ satisfies the formula $\mathcal{L}_\xi A = 0$.

Now let us consider a real hypersurfaces of type A, that is, a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. By Proposition A of [11] we know that $\xi\in\mathfrak{D}^\perp$ and that the shape operator A and the structure tensor ϕ commute with each other. Then this implies that ξ is principal, that is, $A\xi = \alpha\xi$, where $\alpha = \sqrt{8}\cot(\sqrt{8}r)$. Differentiating this one, by (2.2) we have

$$\begin{aligned}(\nabla_X A)\xi &= \nabla_X(A\xi) - A\nabla_X\xi\\ &= \alpha\nabla_X\xi - A\nabla_X\xi\\ &= \alpha\phi AX - A\phi AX.\end{aligned}$$

On the other hand, by the equation of Codazzi and the assumption of $\mathcal{L}_\xi A = 0$ we have

$$\begin{aligned}(\nabla_X A)\xi = -\phi X - \sum_\nu\{\eta_\nu(\xi)\phi_\nu X - \eta_\nu(X)\phi_\nu\xi\}\\ - 3\sum_\nu\eta_\nu(\phi X)\xi_\nu + \phi A^2X - A\phi AX,\end{aligned} \tag{3.1}$$

https://doi.org/10.4153/CMB-2006-014-8 Published online by Cambridge University Press

where $\sum_\nu$ denotes the summation from $\nu = 1$ to 3. Then from these two formulas we have

$$\alpha\phi AX - A\phi AX = -\phi X - \phi_1 X + \sum_\nu \eta_\nu(X)\phi_\nu\xi - 3\sum_\nu \eta_\nu(\phi X)\xi_\nu + \phi A^2 X - A\phi AX. \tag{3.2}$$

Now let us check case by case whether two sides in (3.2) are equal to each other as follows:

Case 1. $X = \xi = \xi_1$

In this case it can be easily checked that two sides are equal to each other.

Case 2. $X = \xi_2, \xi_3$

Then we may put $A\xi_2 = \beta\xi_2$, $A\xi_3 = \beta\xi_3$, where $\beta = \sqrt{2}\cot(\sqrt{2}r)$. Then by putting $X = \xi_2$ in (3.2) we have

$$\alpha\beta\phi\xi_2 = -3\xi_3 - 3\sum_\nu \eta_\nu(\phi\xi_2)\xi_\nu + \beta^2\phi\xi_2.$$

From this we know $\beta(\alpha - \beta)\phi\xi_2 = 2\xi_3$, which implies that both sides are equal to $2\xi_3$.

Case 3. $X \in T_\lambda = \{X | X \perp \mathbb{H}\xi, \phi X = \phi_1 X\}$

Then by putting $X \in T_\lambda$, $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ in (3.2) we have

$$\alpha\lambda\phi X = -\phi X - \phi_1 X + \sum_\nu \eta_\nu(X)\phi_\nu\xi - 3\sum_\nu \eta_\nu(\phi X)\xi_\nu + \lambda^2\phi X,$$

from this it implies that $\lambda(\alpha - \lambda)\phi X = -2\phi X$. This gives our assertion.

Case 4. $X \in T_\mu = \{X | X \perp \mathbb{H}\xi, \phi X = -\phi_1 X\}$

By putting $X \in T_\mu$, $\mu = 0$, in (3.2) we know that both sides are all vanishing.

Accordingly we conclude that real hypersurfaces of type A in Theorem A satisfy $\mathcal{L}_\xi A = 0$. In this section we are going to give the complete classification of real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ satisfying $\mathcal{L}_\xi A = 0$.

Now let us take an orthonormal basis $\{e_1, \dots, e_{4m-1}\}$ for the tangent space $T_x M$, $x \in M$, of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$. Then by the equation of Codazzi we may put

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$$(3.3)\quad (\nabla_{e_i}A)X - (\nabla_X A)e_i = \eta(e_i)\phi X - \eta(X)\phi e_i - 2g(\phi e_i, X)\xi$$
$$+ \sum_{\nu=1}^{3}\{\eta_\nu(e_i)\phi_\nu X - \eta_\nu(X)\phi_\nu e_i - 2g(\phi_\nu e_i, X)\xi_\nu\}$$
$$+ \sum_{\nu=1}^{3}\{\eta_\nu(\phi e_i)\phi_\nu\phi X - \eta_\nu(\phi X)\phi_\nu\phi e_i\}$$
$$+ \sum_{\nu=1}^{3}\{\eta(e_i)\eta_\nu(\phi X) - \eta(X)\eta_\nu(\phi e_i)\}\xi_\nu,$$

from which, together with the formulas (2.1) and (2.5) it follows that

$$(3.4)\quad \sum_{i=1}^{4m-1} g((\nabla_{e_i}A)X, \phi e_i)$$
$$= -(4m-2)\eta(X)$$
$$+ \sum_\nu \{g(\phi_\nu X, \phi\xi_\nu) + \eta_\nu(X)\operatorname{Tr}\phi\phi_\nu + 2g(\phi_\nu^2\xi, X)\}$$
$$- \sum_\nu g(\phi_\nu\phi X, \phi\phi_\nu\xi) - \sum_\nu \eta(X)g(\phi\xi_\nu, \phi\xi_\nu)$$
$$= -(4m-2)\eta(X) - 3\eta(X) + \sum_\nu \eta_\nu(\xi)\eta_\nu(X) + \sum_\nu \eta_\nu(X)\operatorname{Tr}\phi\phi_\nu$$
$$- \sum_\nu \eta_\nu(\xi)g(\phi_\nu\phi X, \xi) - 3\eta(X) + \sum_\nu \eta^2(\xi_\nu)\eta(X)$$
$$= -4(m+1)\eta(X) + 2\sum_\nu \eta_\nu(\xi)\eta_\nu(X) + \sum_\nu \eta_\nu(X)\operatorname{Tr}\phi\phi_\nu,$$

where in the second equality we have used the formulas

$$\sum_\nu g(\phi_\nu\phi X, \phi\phi_\nu\xi) = \sum_\nu \eta_\nu(\xi)g(\phi_\nu\phi X, \xi),$$
$$\sum_\nu g(\phi\xi_\nu, \phi\xi_\nu) = 3\eta(X) - \sum_\nu \eta^2(\xi_\nu)\eta(X),$$
$$\sum_\nu g(\phi_\nu^2\xi, X) = -3\eta(X) + \sum_\nu \eta_\nu(\xi)\eta_\nu(X),$$

and respectively in the third equality,

$$-\sum_\nu \eta_\nu(\xi)g(\phi_\nu\phi X, \xi) = \sum_\nu \eta_\nu(\xi)\eta_\nu(X) - \sum_\nu \eta(X)\eta_\nu^2(\xi).$$

Now let us denote by U the vector $\nabla_\xi\xi = \phi A\xi$. Then by (2.2) its derivative can be given by

$$\nabla_{e_i}U = \eta(A\xi)Ae_i - g(Ae_i, A\xi)\xi + \phi(\nabla_{e_i}A)\xi + \phi A\nabla_{e_i}\xi.$$

Then its divergence is given by

$$\text{(3.5)}\qquad \begin{aligned}\operatorname{div} U &= \sum_i g(\nabla_{e_i} U, e_i)\\ &= h\eta(A\xi) - \eta(A^2\xi) - g\big((\nabla_{e_i} A)\xi, \phi e_i\big) - g(\phi A e_i, A\phi e_i),\end{aligned}$$

where h denotes the trace of the shape operator of M in $G_2(\mathbb{C}^{m+2})$.

Now we calculate the squared norm of the following:

$$\text{(3.6)}\qquad \begin{aligned}\|\phi A - A\phi\|^2 &= \sum_i g\big((\phi A - A\phi)e_i, (\phi A - A\phi)e_i\big)\\ &= \sum_{i,j} g\big((\phi A - A\phi)e_i, e_j\big)\, g\big((\phi A - A\phi)e_i, e_j\big)\\ &= \sum_{i,j} \{g(\phi A e_j, e_i) + g(\phi A e_i, e_j)\}\{g(\phi A e_j, e_i) + g(\phi A e_i, e_j)\}\\ &= 2\sum_{i,j} g(\phi A e_j, e_i) g(\phi A e_j, e_i) + 2\sum_{i,j} g(\phi A e_j, e_i) g(\phi A e_i, e_j)\\ &= -2\sum_j g(\phi A e_j, A\phi e_j) + 2\sum_j g(\phi A e_j, \phi A e_j)\\ &= 2\operatorname{Tr} A^2 - 2h\eta(A\xi) + 2\sum_i g\big((\nabla_{e_i} A)\xi, \phi e_i\big) + 2\operatorname{div} U,\end{aligned}$$

where $\sum_i$(resp. $\sum_{i,j}$) denotes the summation from $i = 1$ to $i = 4m-1$(resp. from $i, j = 1$ to $4m-1$) and in the final equality we have used (3.5). From this together with the formula (3.4) it follows that

$$\text{(3.7)}\qquad \begin{aligned}\operatorname{div} U = \frac{1}{2}\|\phi A - A\phi\|^2 - \operatorname{Tr} A^2 + \alpha h\\ + 4(m+1) - 2\sum_\nu \eta_\nu^2(\xi) - \sum_\nu \eta_\nu(\xi) \operatorname{Tr} \phi\phi_\nu.\end{aligned}$$

From this formula, together with the assumption $\mathcal{L}_\xi A = 0$ we want to show that the structure tensor ϕ and the shape operator A commute with each other, that is, $\phi A - A\phi = 0$. Then by Theorem B we are able to assert that M is a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Now let us take an inner product (3.1) with the Reeb vector field ξ. Then we have

$$\text{(3.8)}\qquad \begin{aligned}g\big((\nabla_X A)\xi, \xi\big) &= -\sum_\nu \eta_\nu(\xi)\eta_\nu(\phi X) - 3\sum_\nu \eta_\nu(\phi X)\eta_\nu(\xi) - g(A\phi A X, \xi)\\ &= -4\sum_\nu \eta_\nu(\xi)\eta_\nu(\phi X) + g(AX, U).\end{aligned}$$

On the other hand, by the almost contact structure ϕ we have

$$\phi U = \phi^2 A\xi = -A\xi + \eta(A\xi)\xi = -A\xi + \alpha\xi,$$

where the function α denotes $\eta(A\xi)$. From this, differentiating and using (2.2) gives

$$(\nabla_X\phi)U + \phi\nabla_X U = -(\nabla_X A)\xi - A\nabla_X\xi + (X\alpha)\xi + \alpha\nabla_X\xi - g(AX,U)\xi + \phi\nabla_X U. \tag{3.9}$$

Then by taking an inner product (3.9) with ξ and using $U = \phi A\xi$ we have

$$\begin{aligned} g\big((\nabla_X A)\xi,\xi\big) &= g(AX,U) + X\alpha - g(\nabla_X\xi, A\xi) \\ &= 2g(AX,U) + X\alpha. \end{aligned}$$

From this, together with (3.8), we have

$$g(AX,U) + 4\sum_\nu \eta_\nu(\xi)\eta_\nu(\phi X) + X\alpha = 0. \tag{3.10}$$

Now substituting (3.1) and (3.10) into (3.9) and using (2.2), we have

$$\begin{aligned} 4\sum_\nu \eta_\nu(\xi)\eta_\nu(\phi X) + \phi\nabla_X U = \phi X &+ \sum_\nu\{\eta_\nu(\xi)\phi_\nu X - \eta_\nu(X)\phi_\nu\xi\} \\ &+ 3\sum_\nu \eta_\nu(\phi X)\xi_\nu - \phi A^2 X + \alpha\phi A X. \end{aligned} \tag{3.11}$$

Then the above equation can be rewritten as follows:

$$\begin{aligned} \phi\nabla_X U = \phi X - \phi A^2 X + \alpha\phi A X + \sum_\nu\{\eta_\nu(\xi)\phi_\nu X - \eta_\nu(X)\phi_\nu\xi\} \\ + 3\sum_\nu \eta_\nu(\phi X)\xi_\nu - 4\sum_\nu \eta_\nu(\xi)\eta_\nu(\phi X)\xi. \end{aligned}$$

From this, summing up from 1 to $4m-1$ for an orthonormal basis of T_xM, $x\in M$, we have

$$\begin{aligned} \sum_i g(\phi\nabla_{e_i}U, \phi e_i) &= \operatorname{div} U + \|U\|^2 \\ &= (4m-2) - \operatorname{Tr} A^2 + \eta(A^2\xi) + \alpha\{\operatorname{Tr} A - \alpha\} \\ &\quad - \sum_\nu \eta_\nu(\xi)\operatorname{Tr}\phi\phi_\nu \\ &\quad - \sum_\nu g(\phi_\nu\xi,\phi_\nu\xi) + 3\sum_\nu g(\xi_\nu,\xi_\nu) - 3\sum_\nu \eta^2(\xi_\nu), \end{aligned} \tag{3.12}$$

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where in the first equality we have used the notion of div U. Then it follows that

$$\begin{aligned}(3.13)\qquad \operatorname{div} U &= (4m-2) - \operatorname{Tr} A^2 + \alpha \operatorname{Tr} A - \sum_\nu \eta_\nu(\xi) \operatorname{Tr} \phi\phi_\nu \\ &\quad - \Big\{ 3 - \sum_\nu \eta(\xi_\nu)\eta(\xi_\nu) \Big\} + 9 - 3\sum_\nu \eta^2(\xi_\nu) \\ &= 4(m+1) + \alpha h - \operatorname{Tr} A^2 - \sum_\nu \eta_\nu(\xi) \operatorname{Tr} \phi\phi_\nu - 2\sum_\nu \eta^2(\xi_\nu),\end{aligned}$$

where we have used $\|U\|^2 = \|A\xi\|^2 - \alpha^2$ in (3.12).

Now if we compare (3.7) with the formula (3.13), we finally assert that the squared norm $\|A\phi - \phi A\|^2$ vanishes, that is, the structure tensor ϕ and the shape operator A commute with each other. Then by Theorem B in the introduction we are able to assert that M is a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. This completes the proof of our Main Theorem. ■

Remark 3.1 Let M be a real hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ satisfying $\mathcal{L}_\xi \phi = 0$. Then it is not difficult to show that the conditions $\mathcal{L}_\xi \phi = 0$ and $\mathcal{L}_\xi A = 0$ are equivalent. So we remark here that a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfying $\mathcal{L}_\xi \phi = 0$ is also congruent to a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Remark 3.2 In paper [10] due to the present author we have proved some non-existence properties for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel shape operator $\nabla A = 0$. Also in [11] we have investigated some real hpersurfaces M in $G_2(\mathbb{C}^{m+2})$ when the structure tensors ϕ_ν, $\nu = 1, 2, 3$, commute with the shape operator A of M in $G_2(\mathbb{C}^{m+2})$.

Acknowledgement The author would like to express his sincere gratitude to the referee for careful reading of the manuscript and valuable comments.

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