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Twisted trace Paley–Wiener theorems for special and general linear groups

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Abstract

Let G be a real special or general linear group and σ_0 be the transpose-inverse involution. We characterize the image of $f \mapsto \mathrm{tr}(\pi(f)\pi(\sigma_0))$ for irreducible representations π of $G \rtimes \langle \sigma_0 \rangle$, and K-finite $f \in C_c^\infty(G)$.

1. Introduction

From the perspective of representation theory, the Fourier transform of an integrable function f is given by integration against the irreducible unitary representations π of the real line

$$\pi \mapsto \int_{\mathbb{R}} \pi(x) f(x)\, dx.$$

The classical Paley–Wiener theorem is a characterization of the image under the Fourier transform of the space of smooth compactly supported functions.

If, in place of the real line, one considers a connected real reductive group G, then the 'Fourier transform' of a smooth compactly supported function f can be taken to be integration against the unitary representations π induced from the discrete series or limits of discrete series. The resulting integral is an operator rather than a scalar, but this operator is of trace class so one can consider the transform

$$\pi \mapsto \mathrm{tr}\left(\int_G \pi(x) f(x)\, dx \right) = \mathrm{tr}(\pi(f)).$$

Clozel and Delorme have characterized the image of this transform for smooth compactly supported functions which are finite under the action of a maximal compact subgroup [CD84, CD90]. This constitutes what is known as a *trace* Paley–Wiener theorem for G.

One can continue to generalize by supposing that σ is an automorphism of G of finite order. One can then consider the representations π induced from the discrete series or limits of discrete series of the group $G \rtimes \langle \sigma \rangle$, and the transform

$$\pi \mapsto \mathrm{tr}\left(\int_G \pi(x) f(x)\, dx\, \pi(\sigma) \right) = \mathrm{tr}(\pi(f)\pi(\sigma)).$$

A *twisted* trace Paley–Wiener theorem for G is the characterization of the image of such a transform on the space of smooth compactly supported functions which are finite under the action of a maximal compact subgroup. Delorme has proven a twisted trace Paley–Wiener theorem when G is a complex Lie group and σ is the automorphism provided by complex conjugation. The goal of this work is to prove twisted trace Paley–Wiener theorems for G equal to one of $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}^{\pm}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{R})$.

The principal result is a σ_0-twisted trace Paley–Wiener theorem (Theorem 1) in which σ_0 is the automorphism of $\mathrm{SL}(n,\mathbb{R})$ given by transposing and inverting. Instead of dealing with the group

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$\mathrm{SL}(n,\mathbb{R}) \rtimes \langle\sigma_0\rangle$ and its representations, we work with $\mathrm{SL}(n,\mathbb{R})$ and its σ_0-*stable* representations. The two classes of representations are essentially the same (cf. [Del91, § 1]). Sections 3–5 are concerned with the identification of the σ_0-stable representations and § 6 handles their associated intertwining operators. There are two families of intertwining operators, each of which is distinguished by a Weyl group which either includes or ignores an action of R groups. These two families lead to a dichotomy in the σ_0-twisted trace Paley–Wiener theorems of § 7. In § 8 we set forth a necessary compatibility condition and a conjecture, which together eliminate this dichotomy.

The final section is concerned with applications of the σ_0-twisted trace Paley–Wiener theorems. We begin by indicating how to obtain further twisted trace Paley–Wiener theorems for the groups $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{R})$. We conclude with an application needed for the twisted invariant Arthur–Selberg trace formula. This application depends on the conjecture of § 8 if the underlying group is $\mathrm{SL}(n,\mathbb{R})$ and n is even. In all other cases, the application follows unconditionally.

2. Preliminaries

The reader is assumed to be familiar with the basic theory of Lie groups and their representations. We shall therefore set up our notation without references.

Set $G = \mathrm{SL}(n,\mathbb{R})$. The subgroup of upper-triangular matrices P_0 is a minimal parabolic subgroup of G. A parabolic subgroup $P \subset G$ is said to be *standard* if $P \supset P_0$. Suppose P is a standard parabolic subgroup. Then it has a *Levi decomposition* $P = MU_M$, where U_M is its unipotent radical and M is a Levi subgroup containing M_0, the diagonal subgroup. The subgroup P also has a *Langlands decomposition* $P = M^1A_MU_M$, in which A_M is the connected component of the centre of M and M^1 is the subgroup formed by the elements of M having determinant ±1. The subgroup M^1 is isomorphic to

$$\left\{(x_1,\dots,x_\ell) \in \mathrm{SL}^{\pm}(n_1,\mathbb{R}) \times \cdots \times \mathrm{SL}^{\pm}(n_\ell,\mathbb{R}) : \prod_{j=1}^{\ell} \det(x_j) = 1\right\},$$

for some positive integers $n_1,\dots,n_\ell$ such that $\sum_{j=1}^{\ell} n_j = n$. In other words, M^1 (and M) are block diagonal subgroups of G. We shall refer to the integers $n_1,\dots,n_\ell$ as the *block sizes* of M^1 or M. We shall say that an element $x \in G$ *permutes the blocks of* M if $x^{-1}Mx$ is a Levi subgroup containing M_0, and has the same block sizes as M.

These notions make sense for arbitrary parabolic subgroups, but we shall only require them for standard parabolic subgroups. With this in mind, for any Levi subgroup L containing M_0 define P_L to be the unique standard parabolic subgroup whose Langlands decomposition is $P_L = L^1A_LU_L$. In particular, for P and M as above we have $P = P_M$.

We denote the Lie algebra of A_M by $\mathfrak{a}_M$. Its dual is denoted by $\mathfrak{a}_M^*$. The complexification of a vector space shall be denoted by a subscript $\mathbb{C}$. For example, the complex dual of $\mathfrak{a}_M$ is $\mathfrak{a}_{M,\mathbb{C}}^*$. There is a canonical embedding of $\mathfrak{a}_M$ into $\mathfrak{a}_{M_0}$. The Killing form provides an inner product on $\mathfrak{a}_{M_0}$ which by restriction is an inner product on $\mathfrak{a}_M$. Given two Levi subgroups $L \subset M$ we write $\mathfrak{a}_L^M$ for the orthogonal dual of $\mathfrak{a}_M$ in $\mathfrak{a}_L$. We extend this notation to the duals and complex duals in the obvious way.

The set of (equivalence classes of) discrete series or non-degenerate limit of discrete series representations of M^1 is denoted by $(\hat{M})_{\mathrm{lds}}$. Given $\delta \in (\hat{M})_{\mathrm{lds}}$ and $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ we can form the representation $\delta \otimes e^\lambda$ of M by defining

$$(\delta \otimes e^\lambda)(m,a) = \lambda(\log(a)) \cdot \delta(m), \quad m \in M^1,\ a \in A_M.$$

We can extend this representation trivially to P and induce to G in order to obtain the representation

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$\mathrm{ind}_P^G(\delta \otimes e^{\lambda})$. Such induced representations may of course be formed for any parabolic subgroup of G, standard or not. The above induction is assumed to be normalized so that $\mathrm{ind}_P^G(\delta \otimes e^{\lambda})$ is unitary whenever $\lambda \in i\mathfrak{a}_M^*$.

Set $K = \mathrm{SO}(n, \mathbb{R})$. It is a maximal compact subgroup of G. We shall adopt the compact picture of induction so that the vectors in the space of $\mathrm{ind}_P^G(\delta \otimes e^{\lambda})$ are functions on K.

Suppose $x \in G$ and H is a subgroup of G. If π is a representation of $x^{-1}Hx$ define

$$x\pi(h) = \pi(x^{-1}hx), \quad h \in H.$$

Similarly, if $x^{-1}Mx$ is a Levi subgroup of G and $\lambda \in \mathfrak{a}^*_{x^{-1}Mx,\mathbb{C}}$ then set

$$x\lambda(X) = \lambda(\mathrm{Ad}(x)^{-1}(X)), \quad X \in \mathfrak{a}_M.$$

Here, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{a}_M)$ is the adjoint homomorphism.

As usual we define the Weyl group $W(A_M : G)$ as the quotient of the normalizer of $\mathfrak{a}_M$ by the centralizer of $\mathfrak{a}_M$ with respect to the adjoint action. If $w \in W(A_M : G)$, π is a representation of M^1, and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, then we define $w\pi$ and $w\lambda$ by choosing a representative for w and following the previous scheme. Clearly, the equivalence class of $w\pi$ is independent of the choice of representative. We shall often confuse an element of a Weyl group with one of its representatives.

Suppose H is a group and $h \in H$. We define the automorphism σ_h of H by

$$\sigma_h(x) = h^{-1}xh, \quad x \in H.$$

Given an automorphism σ and a representation π of H set

$$\pi^{\sigma}(x) = \pi(\sigma(x)), \quad x \in H.$$

Notice that $\pi^{\sigma_h} = h\pi$ when $H = \mathrm{SL}(n, \mathbb{R})$. The representation π is said to be σ-*stable* if it is equivalent to π^{σ}. The representation π^{σ} is called the σ-*conjugate* of π.

The automorphism of principal interest to us is the involution σ_0 of $\mathrm{SL}(n, \mathbb{R})$ given by taking the transpose and inverse of a matrix. By the definition of $K = \mathrm{S0}(n, \mathbb{R})$, σ_0 fixes K pointwise. The differential of σ_0 is easily seen to send $X \in \mathfrak{a}_{M_0}$ to $-X$. This induces the map $\lambda \mapsto -\lambda$ on $\mathfrak{a}^*_{M_0,\mathbb{C}}$.

Given a real number $c > 0$, we define $\lfloor c \rfloor$ to be the greatest integer less than or equal to c.

3. Necessary conditions for σ_0-stable representations

Suppose $P_M = M^1 A_M U_M$ is a standard parabolic subgroup of G, ρ is an irreducible tempered representation of M^1, and $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ with its real part lying in the open Weyl chamber of $\mathfrak{a}^*_M$ determined by P_M. The Langlands classification of irreducible admissible representations [Kna86, Theorem 8.54] tells us that $\mathrm{ind}_{P_M}^G(\rho \otimes e^{\lambda})$ has a unique irreducible admissible quotient $J(P_M, \rho, \lambda)$, the *Langlands quotient*. Moreover, it tells us that every irreducible admissible representation of G is equivalent to some Langlands quotient. As the composition of $\mathrm{ind}_{P_M}^G(\rho \otimes e^{\lambda})$ with σ_0 preserves subquotients, the representation $(J(P_M, \rho, \lambda))^{\sigma_0}$ is the unique irreducible quotient of $(\mathrm{ind}_{P_M}^G(\rho \otimes e^{\lambda}))^{\sigma_0}$. In the compact picture, the vector spaces of $(\mathrm{ind}_{P_M}^G(\rho \otimes e^{\lambda}))^{\sigma_0}$ and $\mathrm{ind}_{\sigma_0 P_M}^G(\rho^{\sigma_0} \otimes e^{-\lambda})$ are identical, for the elements of K are fixed by σ_0. A simple computation shows that their actions on this vector space are also identical; that is

$$(\mathrm{ind}_{P_M}^G(\rho \otimes e^{\lambda}))^{\sigma_0} = \mathrm{ind}_{\sigma_0 P_M}^G(\rho^{\sigma_0} \otimes e^{-\lambda}).$$

The real part of the element $-\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ lies in the open Weyl chamber determined by $\sigma_0 P_M$, which is opposite to P_M, so the representation on the right has a unique Langlands quotient $J(\sigma_0 P_M, \rho^{\sigma_0}, -\lambda)$. As a result, $(J(P_M, \rho, \lambda))^{\sigma_0}$ is equal to $J(\sigma_0 P_M, \rho^{\sigma_0}, -\lambda)$.

Now suppose that $J(P_M, \rho, \lambda)$ is equivalent to $(J(P_M, \rho, \lambda))^{\sigma_0}$. Then the uniqueness statement of Langlands' classification [Lan89, Lemma 3.14] implies the existence of an element w_0, belonging to the normalizer of A_M in G, such that

$$\sigma_0 P_M = w_0^{-1} P_M w_0, \tag{1}$$

$$-w_0 \lambda = \lambda, \tag{2}$$

$$w_0(\rho^{\sigma_0}) \cong \rho. \tag{3}$$

If we translate w_0 on the right by an element in M in this statement then the same conclusions still hold. Indeed, M normalizes $\sigma_0 P_M$ and is equal to the centralizer of A_M in G. In other words, the conclusions depend only on the class of w_0 in $W(A_M : G)$.

Equation (1) implies that P_M contains $w_0 \sigma_0 P_0 w_0^{-1}$, as P_M is standard. It is well-known that Borel subgroups of linear algebraic groups are conjugate. Therefore, there exists an element $m \in P_M$ such that

$$m^{-1} P_0 m = w_0 \sigma_0 P_0 w_0^{-1}.$$

It is easy to see that we may actually take m to belong to M. This means that $(mw_0)^{-1} P_0 m w_0$ is the parabolic subgroup of G opposite to P_0. This fact and the invariance under M described in the previous paragraph allow us to assume that w_0 is equal to the unique skew-diagonal matrix in G which also lies in

$$\left\{ \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & 1 & 0 \\ 0 & 1 & & \vdots \\ \pm 1 & 0 & \cdots & 0 \end{pmatrix} \right\}.$$

Having ascertained the class of w_0 in $W(A_M : G)$, we may compute directly from Equation (1) that the block sizes $n_1, \ldots, n_\ell$ of M satisfy $n_j = n_{\ell+1-j}$ for $1 \leqslant j \leqslant \ell$. We may also compute directly from Equation (2) that λ is restricted to a subspace of $\mathfrak{a}^*_{M,\mathbb{C}}$ of complex dimension $\lfloor (\ell+1)/2 \rfloor$.

With regard to ρ we know [Kna86, Theorem 14.91] that there exists a standard parabolic subgroup P_L contained in P_M with Langlands decomposition $L^1 A_L U_L$, a representation $\delta \in (\hat{L})_{\mathrm{lds}}$, and $\nu \in i(\mathfrak{a}_L^M)^*$ such that

$$\rho = \mathrm{ind}_{P_L \cap M^1}^{M^1} (\delta \otimes e^{\nu}).$$

We shall use this expansion to describe equivalence (3). We do so in three steps. First, observe that the earlier argument for $(\mathrm{ind}_{P_M}^{G}(\rho \otimes e^{\lambda}))^{\sigma_0}$ can be mimicked to conclude that $(\mathrm{ind}_{P_L \cap M^1}^{M^1}(\delta \otimes e^{\nu}))^{\sigma_0}$ is equal to $\mathrm{ind}_{\sigma_0 P_L \cap M^1}^{M^1}(\delta^{\sigma_0} \otimes e^{-\nu})$.

Second, the only Levi subgroups of G supporting discrete series or limit of discrete series representations are those whose blocks are of rank one or two. Observe that the involution σ_0 acts on $\mathrm{SL}^{\pm}(1, \mathbb{R}) = \{1, -1\}$ as the identity, and on $\mathrm{SL}^{\pm}(2, \mathbb{R})$ as conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. It follows that there exists a permutation matrix $w_L \in L^1$, which depends solely on the block sizes of L, and satisfies $\delta^{\sigma_0} = w_L \delta$. This implies that left multiplication by the operator $\delta(w_L)$ intertwines $\mathrm{ind}_{\sigma_0 P_L \cap M^1}^{M^1}(\delta^{\sigma_0} \otimes e^{-\nu})$ with $\mathrm{ind}_{\sigma_0 P_L \cap M^1}^{M^1}(\delta \otimes e^{-\nu})$.

Third, it is a simple exercise to show that the operator defined by

$$(A(w_0)\varphi)(x) = \varphi(w_0^{-1} x w_0), \quad \varphi \in \mathrm{ind}_{\sigma_0 P_L \cap M^1}^{M^1}(\delta \otimes e^{-\nu}),$$

intertwines $\mathrm{ind}_{\sigma_0 P_L \cap M^1}^{M^1}(\delta \otimes e^{-\nu})$ with $\mathrm{ind}_{w_0 \sigma_0 P_L w_0^{-1} \cap M^1}^{M^1}(w_0 \delta \otimes e^{-w_0 \nu})$.

These three equivalences taken together with (3) imply that

$$(\mathrm{ind}_{P_L \cap M^1}^{M^1}(\delta \otimes e^{\nu}))^{\sigma_0} \cong \mathrm{ind}_{w_0 \sigma_0 P_L w_0^{-1} \cap M^1}^{M^1}(w_0 \delta \otimes e^{-w_0 \nu}).$$

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According to [Kna86, Theorem 14.91] there exists an element $w_M \in M$ such that $w_M w_0 A_L w_0^{-1} w_M^{-1} = A_L$, $-w_M w_0 \nu = \nu$, and $w_M w_0 \delta = \delta$. Among other things, we have proved the following lemma.

Lemma 1. *Suppose $P_M = M^1 A_M U_M \supset P_L = L^1 A_L U_L$ are the Langlands decompositions of standard parabolic subgroups. Suppose $\delta \in (\hat{L})_{\mathrm{lds}}$ and $\nu \in i(\mathfrak{a}_L^M)^*$ such that $\mathrm{ind}_{P_L \cap M^1}^{M^1}(\delta \otimes e^\nu)$ is irreducible. Finally, suppose the real part of $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ lies in the open positive Weyl chamber determined by P_M, and $J(P_M, \mathrm{ind}_{P_L \cap M^1}^{M^1}(\delta \otimes e^\nu), \lambda)$ is σ_0-stable. Then the block sizes, $n_1, \dots, n_\ell$, of M satisfy*

$$n_j = n_{\ell+1-j}, \quad 1 \leqslant j \leqslant \ell,$$

$\sigma_0 P_M = w_0 P_M w_0^{-1}$, and $w_0 \lambda = -\lambda$. Furthermore, there exists $w_M \in M$ such that $w_M w_0 A_L (w_M w_0)^{-1} = A_L$, $w_M w_0 \delta = \delta$, and $w_M w_0 \nu = -\nu$.

4. The construction of some σ_0-stable representations

We wish to produce a set of Levi subgroups and representations which furnish σ_0-stable representations under parabolic induction. Our approach here shall be quite concrete in that we shall provide explicit intertwining operators demonstrating σ_0-stability.

Suppose $P_M = M^1 A_M U_M \supset P_L = L^1 A_L U_L$ and w_0 are as in the conclusion of Lemma 1. Notice that σ_0 sends each of these parabolic subgroups to its opposite parabolic subgroup. Conjugation by w_0 has the same effect on P_M. However, conjugation by w_0 does not necessarily send P_L to its opposite parabolic subgroup. In fact, unless we place some restrictions on L, the element w_0 need not even belong to the normalizer of L. This suggests that we assume L satisfies the conditions of the next few paragraphs.

Let us begin with the description of L. The subgroup L^1 is isomorphic to

$$\left\{(x_1, \dots, x_k) \in \mathrm{SL}^{\pm}(r_1, \mathbb{R}) \times \cdots \times \mathrm{SL}^{\pm}(r_k, \mathbb{R}) : \prod_{j=1}^{k} \det(x_j) = 1\right\}, \tag{4}$$

for some $r_1, \dots, r_k = 1, 2$ satisfying $\sum_{j=1}^k r_j = n$. Obviously, this subgroup is completely determined by the integers $r_1, \dots, r_k$. Suppose $0 \leqslant t \leqslant \lfloor n/2 \rfloor$ is an integer and I_j is the identity matrix of rank j. If n is even let w_t be the unique element in

$$\left\{\begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & 0 & I_{2t} & 0 & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ \pm 1 & 0 & \cdots & 0 & 0 \end{pmatrix}\right\}$$

which belongs to G. If n is odd let w_t be the unique element in

$$\left\{\begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \vdots & 0 & I_{2t+1} & 0 & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ \pm 1 & 0 & \cdots & 0 & 0 \end{pmatrix}\right\}$$

which belongs to G. Note that this definition is consistent with the definition of w_0 in § 3. The element w_t is meant to fulfill the role of $w_M w_0$ as in Lemma 1.

Let $\mathcal{L}(t)$ be the set of Levi subgroups $L \supset M_0$ for which L^1 is of the form (4) and satisfies the following additional requirements:

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1) $w_t L w_t^{-1} = L$;
2) there is an integer $1 \leqslant s \leqslant k$ such that $\sum_{j=1}^{s} r_j = t$;
3) at most two of the block sizes in the sequence, $r_{s+1}, \dots, r_{k-s}$, are equal to one.

For the remainder of this section suppose that L belongs to $\mathcal{L}(t)$. The element w_t has been chosen so that $w_t \sigma_0 P_L w_t^{-1}$ is a parabolic subgroup with Levi component L. Unfortunately, $w_t \sigma_0 P_L w_t^{-1}$ might not be a standard parabolic subgroup. After all, w_t does nothing at all to the 'middle' blocks of L, whereas σ_0 sends P_L to its opposite parabolic subgroup. To correct this discrepancy, we shall eventually turn to some intertwining operators of Knapp and Stein. Before this, we present the relevant parabolic subgroups.

In keeping with the earlier definitions, we specify a Levi subgroup $M_{L,t} \supset L$ by referring to its block sizes $r'_1, \dots, r'_{k'}$. Working under the assumptions of the previous paragraph, set $M_{L,t} = L$ if $k = 2s$. Otherwise set

$$r'_j = r_j,\ 1 \leqslant j \leqslant s, \quad r'_{s+1} = \sum_{j=s+1}^{k-s} r_j, \quad r'_{2s+2-j} = r'_j = r_j,\ 1 \leqslant j \leqslant s.$$

Clearly, the standard parabolic subgroup $P_{M_{L,t}} = M^1_{L,t} A_{M_{L,t}} U_{M_{L,t}}$ contains P_L. It is easy to see that $U_{M_{L,t}}$ is contained in U_L, and it is left as an exercise to the reader to show that

$$w_t \sigma_0 P_L w_t^{-1} = (\sigma_0 P_L \cap M_{L,t}) U_{M_{L,t}}. \tag{5}$$

We now list sets of representations that are attached to $L \in \mathcal{L}(t)$ and w_t in terms of Weyl groups. Let $\mathfrak{g}$ be the Lie algebra of G. The set of useful roots Δ_L of $(\mathfrak{g}, \mathfrak{a}_L)$ forms a root system [Kna86, Theorem 14.39]. We fix a set of positive roots Δ_L^+ with respect to the parabolic subgroup P_L. Given an irreducible tempered representation δ of L^1, set W_δ to be the subgroup of the Weyl group of Δ_L which stabilizes the equivalence class of δ. The group W_δ is the semidirect product of two abstract Weyl groups [Kna86, ch. XIV, § 9]. One of them is the R group R_δ and is, as we shall soon see, isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The R group normalizes the other subgroup $W_\delta^0 \subset W_\delta$, which is the Weyl group of a root system $\Delta_\delta^0 \subset \Delta_L$. We fix the set of positive roots of Δ_δ^0 to be $\Delta_\delta^0 \cap \Delta_L^+$.

By applying a well-known property of Weyl groups [Kna86, ch. IV, § 4, property (2)], one can show that the subgroup $Z_{W_\delta}(R_\delta)$ of elements in W_δ which are centralized by R_δ is an abstract Weyl group. In fact, its root system is the direct product of the root system of R_δ and the subset of roots in Δ_δ^0 which are fixed by R_δ. If R_δ is non-trivial then Δ_L^+ determines a unique positive root α_R in the root system of R_δ [Kna86, Theorem 14.64]. We may therefore fix a set of positive roots for the above direct product by taking the union of α_R with the intersection of Δ_L^+ and the second root system in the product.

Define $(\hat{L})_{\mathrm{lds},t}$ to be the subset of $(\hat{L})_{\mathrm{lds}}$ given by those $\delta \in (\hat{L})_{\mathrm{lds}}$ such that w_t is (a representative of) the longest element in either W_δ^0 or $Z_{W_\delta}(R_\delta)$.

Given an element $w \in W(A_L : G)$, define $\mathfrak{a}_L^w$ to be the subspace of $\mathfrak{a}_L$ spanned by the elements $X \in \mathfrak{a}_L$ which satisfy $\mathrm{Ad}(w)(X) = -X$. We denote the relevant dual spaces and their complexifications as we do for $\mathfrak{a}_L$.

Proposition 1. *Suppose δ is (a representative of a class) in $(\hat{L})_{\mathrm{lds},t}$ and $\nu \in (\mathfrak{a}_{L,\mathbb{C}}^{w_t})^*$. Then $\mathrm{ind}_{P_L}^G(\delta \otimes e^\nu)$ is σ_0-stable.*

Proof. We shall define an operator $T(\delta)$ intertwining $(\mathrm{ind}_{P_L}^G(\delta \otimes e^\nu))^{\sigma_0}$ with $\mathrm{ind}_{P_L}^G(\delta \otimes e^\nu)$. As in earlier considerations, we see that $(\mathrm{ind}_{P_L}^G(\delta \otimes e^\nu))^{\sigma_0}$ is equal to $\mathrm{ind}_{\sigma_0 P_L}^G(\delta^{\sigma_0} \otimes e^{-\nu})$. The latter representation is equivalent to

$$\mathrm{ind}_{w_t \sigma_0 P_L w_t^{-1}}^G(w_t(\delta^{\sigma_0}) \otimes e^{-w_t \nu}) = \mathrm{ind}_{w_t \sigma_0 P_L w_t^{-1}}^G(w_t(\delta^{\sigma_0}) \otimes e^{\nu})$$

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by virtue of the intertwining operator

$$(A(w_t)\varphi)(x) = \varphi(w_t^{-1} x w_t), \quad \varphi \in \mathrm{ind}_{\sigma_0 P_L}^G(\delta^{\sigma_0} \otimes e^{-\nu}).$$

The intertwining operator $\delta(w_L)$ (which depends only on L) defined in § 3 intertwines the representation on the right with

$$\mathrm{ind}_{w_t \sigma_0 P_L w_t^{-1}}^G(w_t\delta \otimes e^{\nu}).$$

As $w_t\delta \cong \delta$, [Kna86, Theorem 14.91] provides an element $w_\delta \in L \cap K$ such that left multiplication by $\delta(w_\delta)$ intertwines the above representation with

$$\mathrm{ind}_{w_t \sigma_0 P_L w_t^{-1}}^G(\delta \otimes e^{\nu}). \tag{6}$$

Knapp and Stein have defined a normalized intertwining operator

$$\mathcal{A}(P_L : w_t\sigma_0 P_L w_t^{-1} : \delta : \nu)$$

which intertwines $\mathrm{ind}_{w_t \sigma_0 P_L w_t^{-1}}^G(\delta \otimes e^{\nu})$ with $\mathrm{ind}_{P_L}^G(\delta \otimes e^{\nu})$, for ν in a dense open subset of $(\mathfrak{a}_{L,\mathbb{C}}^M)^*$ [Kna86, § 6, XIV]. It follows from Equation (5) and the inductive definition of this operator that it is induced from an intertwining operator

$$\mathcal{A}(P_L \cap M_{L,t}^1 : \sigma_0 P_L \cap M_{L,t}^1 : \delta : \nu_{M_{L,t}}),$$

where $\nu_{M_{L,t}}$ is the orthogonal projection of ν to $(\mathfrak{a}_{L,\mathbb{C}}^{M_{L,t}})^*$. It is an immediate consequence of the definitions that

$$w_t\nu_1 = \nu_1, \quad \nu_1 \in (\mathfrak{a}_{L,\mathbb{C}}^{M_{L,t}})^*.$$

From this equation and the fact that $w_t\nu = -\nu$ it follows that $\nu_{M_{L,t}} = 0$. By [Kna86, Proposition 14.20(d)], the operator $\mathcal{A}(P_L \cap M_{L,t}^1 : \sigma_0 P_L \cap M_{L,t}^1 : \delta : 0)$ is defined. We may therefore use it to intertwine representation (6) with $\mathrm{ind}_{P_L}^G(\delta \otimes e^{\nu})$. For convenience, we shall identify $\mathcal{A}(P_L \cap M_{L,t}^1 : \sigma_0 P_L \cap M_{L,t}^1 : \delta : 0)^{-1}$ with its corresponding induced operator on $\mathrm{ind}_{w_t \sigma_0 P_L w_t^{-1}}^G(\delta \otimes e^{\nu})$.

After working through these equivalences in reverse, we find that

$$T(\delta) = \mathcal{A}(P_L \cap M_{L,t}^1 : \sigma_0 P_L \cap M_{L,t}^1 : \delta : 0)^{-1}\delta(w_\delta^{-1})\delta(w_L^{-1})A(w_t^{-1})$$

is an intertwining operator between $\mathrm{ind}_{P_L}^G(\delta \otimes e^{\nu})$ and $(\mathrm{ind}_{P_L}^G(\delta \otimes e^{\nu}))^{\sigma_0}$. □

We can simplify the notation in the definition of $T(\delta)$ somewhat by defining operators $\mathrm{L}(w)$ and $\mathrm{R}(w)$ on $\varphi \in \mathrm{ind}_{P_L}^G(\delta \otimes e^{\nu})$ by

$$(\mathrm{L}(w)\varphi)(x) = \varphi(w^{-1}x), \quad (\mathrm{R}(w)\varphi)(x) = \varphi(xw), \quad w \in K.$$

Unraveling the definitions, and noting that

$$\mathcal{A}(P_L \cap M_{L,t} : \sigma_0 P_L \cap M_{L,t} : \delta : 0)^{-1} = \mathcal{A}(\sigma_0 P_L \cap M_{L,t} : P_L \cap M_{L,t} : \delta : 0)$$

[Kna86, Lemma 14.18], we see that $T(\delta)$ reduces to

$$T(\delta) = \mathcal{A}(\sigma_0 P_L \cap M_{L,t} : P_L \cap M_{L,t} : \delta : 0)\mathrm{L}(w_\delta^{-1} w_L^{-1} w_t)\mathrm{R}(w_t^{-1}). \tag{7}$$

It is valuable to note that $T(\delta)$ is independent of $\nu \in (\mathfrak{a}_{L,\mathbb{C}}^{w_t})^*$.

5. The exhaustion of the σ_0-stable representations

Our goal here is to show that every irreducible σ_0-stable admissible representation is equivalent to a quotient of $\mathrm{ind}_{P_L}^G(\delta' \otimes e^{\lambda'})$, where $L \in \mathcal{L}(t)$, $\delta' \in (\hat{L}')_{\mathrm{lds},t}$, and $\lambda' \in (\mathfrak{a}_{L',\mathbb{C}}^{w_t})^*$ for some $0 \leqslant t \leqslant \lfloor n/2 \rfloor$. This is not a classification of the irreducible σ_0-stable representations for we have provided neither a uniqueness assertion nor a description of the quotients of $\mathrm{ind}_{P_L}^G(\delta' \otimes e^{\lambda'})$. These additional

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features may be derived from a careful application of the Langlands classification and [Kna86, Theorem 14.91]. (For a better impression of these issues, consult the example at the beginning of § 6.) In any case, the characters of irreducible σ_0-stable representations may be recovered in a prescribed fashion from the characters of the representations $\mathrm{ind}_{P_L}^G(\delta' \otimes e^{\lambda'})$ as above (cf. [Art89, § 5]).

In light of Lemma 1, we will achieve the above goal if we prove the following proposition.

Proposition 2. *Suppose P_M, P_L, δ, and ν are as in the conclusion of Lemma 1. Set $n' = n_{(\ell+1)/2}$ if ℓ is odd and $n' = 0$ otherwise. Then there exist $0 \leqslant t \leqslant n'$, $L' \in \mathcal{L}(t)$, $\delta' \in (\hat{L}')_{\mathrm{lds},t}$, and $\nu' \in i(\mathfrak{a}_{L'}^{w_t})^*$ such that $P_{L'} \subset P_M$ and $\mathrm{ind}_{P_L \cap M^1}^{M^1}(\delta \otimes e^{\nu})$ is equivalent to $\mathrm{ind}_{P_{L'} \cap M^1}^{M^1}(\delta' \otimes e^{\nu'})$.*

Indeed, the integer n' of this proposition has been chosen so that $w_t\lambda = w_0\lambda$ whenever $\lambda \in (\mathfrak{a}_{M,\mathbb{C}}^{w_0})^*$. Therefore, one may take the λ' occurring in the above representation $\mathrm{ind}_{P_L}^G(\delta' \otimes e^{\lambda'})$ equal to $\lambda + \nu' \in (\mathfrak{a}_{L',\mathbb{C}}^{w_t})^*$. The proof of this proposition will consume this entire section.

Proof of Proposition 2. Suppose that $w \in M$ permutes the blocks of L. Then the operator

$$S(w,\delta,\nu) = \mathcal{A}(P_{wLw^{-1}} \cap M^1 : wP_Lw^{-1} \cap M^1 : w\delta : w\nu)L(w) \tag{8}$$

(cf. § 4 with apologies for the double usage of L) is invertible and intertwines $\mathrm{ind}_{P_L}^G(\delta \otimes e^{\nu})$ with $\mathrm{ind}_{P_{wLw^{-1}}}^G(w\delta \otimes e^{w\nu})$ for any $\nu \in i(\mathfrak{a}_L)^*$ [Kna86, Proposition 14.20(d)]. The upshot of this observation is that in order to prove Proposition 2 it suffices to find an element $w \in M$ which permutes the blocks of L such that the following properties are satisfied:

i) $wLw^{-1} = L'$ for some $L' \in \mathcal{L}(t)$, $0 \leqslant t \leqslant n'$;

ii) $w\delta \in (\hat{L}')_{\mathrm{lds},t}$;

iii) $w\nu \in i(\mathfrak{a}_{L'}^{w_t})^*$.

We shall prove these three properties by portraying δ as a k-tuple of representations of $\mathrm{SL}^{\pm}(r_j,\mathbb{R})$, $r_j = 1, 2$, and providing permutations of these representations which will also place δ and ν into the desired form. Suppose δ and ν are as in Lemma 1 and L^1 is of the form (4). Then there exists an irreducible representation $\delta^{\pm}$ of

$$L^{\pm} \cong \mathrm{SL}^{\pm}(r_1,\mathbb{R}) \times \cdots \times \mathrm{SL}^{\pm}(r_k,\mathbb{R}),$$

whose restriction $\delta^{\pm}_{|L^1}$ to L^1 contains δ as one of its two possible subrepresentations. Indeed, by Frobenius reciprocity $\delta^{\pm}$ can be taken to be an irreducible subrepresentation of the induced representation of δ to $L^{\pm}$. Clearly, we may express $\delta^{\pm}$ uniquely as a k-tuple $((\delta^{\pm})_1, \ldots, (\delta^{\pm})_k)$ of irreducible representations of $\mathrm{SL}^{\pm}(r_1,\mathbb{R}) \times \cdots \times \mathrm{SL}^{\pm}(r_k,\mathbb{R})$. It is important to note that there may be another inequivalent representation of $L^{\pm}$ whose restriction to L^1 contains δ. The following lemma fully describes this possibility in terms of the non-trivial character sgn of $\mathrm{SL}^{\pm}(1,\mathbb{R}) = \{1,-1\}$.

Lemma 2. *Suppose $\delta^{\pm} = ((\delta^{\pm})_1, \ldots, (\delta^{\pm})_k)$ and $\delta'^{\pm} = ((\delta'^{\pm})_1, \ldots, (\delta'^{\pm})_k)$ are irreducible representations of $L^{\pm}$ such that $\delta^{\pm}_{|L^1}$ and $\delta'^{\pm}_{|L^1}$ contain δ as a subrepresentation. Then $\delta^{\pm}_j = \delta'^{\pm}_j$, for $1 \leqslant j \leqslant k$ such that $r_j = 2$. Furthermore, either $(\delta^{\pm})_j = (\delta'^{\pm})_j$, for all $1 \leqslant j \leqslant k$ such that $r_j = 1$, or $(\delta^{\pm})_j = \mathrm{sgn} \cdot (\delta'^{\pm})_j$, for all $1 \leqslant j \leqslant k$ such that $r_j = 1$.*

Proof. Suppose $1 \leqslant j \leqslant k$ and $r_j = 2$. Then, according to (and in the notation of) [Kna79, § 2], there exist positive integers d_j, d'_j such that $(\delta^{\pm})_j$ and $(\delta'^{\pm})_j$ are the respective induced representations of $\mathcal{D}^+_{d_j}$ and $\mathcal{D}^+_{d'_j}$. Moreover, we have $((\delta^{\pm})_j)_{|\mathrm{SL}(2,\mathbb{R})} = \mathcal{D}^+_{d_j} \oplus \mathcal{D}^-_{d_j}$ and $((\delta'^{\pm})_j)_{|\mathrm{SL}(2,\mathbb{R})} = \mathcal{D}^+_{d'_j} \oplus \mathcal{D}^-_{d'_j}$.

It follows that $d_j = d'_j$ and the first assertion of the lemma is proven. Now set

$$H = \left\{(x_1, \ldots, x_k) \in L^1 : x_j = 1, \text{ if } r_j = 2, \text{ and } \prod_{j=1}^{k} \det(x_j) = 1\right\}.$$

It is a simple exercise to show that if the one-dimensional representation $\delta^{\pm}_{|H}$ is trivial then either $(\delta^{\pm})_j$ is the trivial character for all $1 \leqslant j \leqslant k$ such that $r_j = 1$, or $(\delta^{\pm})_j = \operatorname{sgn}$ for all $1 \leqslant j \leqslant k$ such that $r_j = 1$. The lemma now follows by replacing $\delta^{\pm}_{|H}$ in this exercise with $\delta^{\pm}_{|H}(\delta'^{\pm}_{|H})^{-1}$. $\square$

We are assuming that $w_M w_0 \delta \cong \delta$ as in Lemma 1. This implies that $w_M w_0 \delta^{\pm}_{|L^1}$ contains δ as a subrepresentation. Therefore, Lemma 2 tells us that either $w_M w_0 \delta^{\pm}$ is equivalent to $\delta^{\pm}$, or both the trivial and sign characters of $\mathrm{SL}^{\pm}(1, \mathbb{R})$ occur in $((\delta^{\pm})_1, \ldots, (\delta^{\pm})_k)$ an equal number of times and $w_M w_0$ sends each trivial character to a sign character. We shall prove the three properties (i)–(iii) by considering each of these two cases separately.

First we fix some more notation for the representations occurring in $\delta^{\pm}$. Let $\tau_1, \ldots, \tau_b$ be mutually inequivalent representations of $\mathrm{SL}^{\pm}(2, \mathbb{R})$ or $\mathrm{SL}^{\pm}(1, \mathbb{R})$ such that every representation occurring in the expansion $((\delta^{\pm})_1, \ldots, (\delta^{\pm})_k)$ of $\delta^{\pm}$ is equivalent to some τ_j, $1 \leqslant j \leqslant b$, and *vice versa*. Let a_j be the number of representations in $((\delta^{\pm})_1, \ldots, (\delta^{\pm})_k)$ which are equivalent to τ_j, $1 \leqslant j \leqslant b$.

We now prove properties (i)–(iii) under the assumption that $w_M w_0 \delta^{\pm}$ is equivalent to $\delta^{\pm}$. Recall that M is comprised of ℓ blocks. We define $M^{\pm} \supset L^{\pm}$ in the obvious way. If $(\delta^{\pm})_j$ is a representation of a block of $L^{\pm}$ contained in the ith block of $M^{\pm}$, then $w_M w_0 \delta^{\pm} \cong \delta^{\pm}$ implies that it also occurs as a representation of a block of $L^{\pm}$ contained in the $(\ell + 1 - i)$th block of $M^{\pm}$. It follows that when ℓ is even the integers $a_1, \ldots, a_b$ are all even and that there exists $w \in M^1$ which permutes the blocks of L such that $w_0 w \delta^{\pm} \cong w\delta^{\pm}$.

On the other hand, suppose that ℓ is odd. Let t be the number of integers in $a_1, \ldots, a_b$ which are odd. Using the same reasoning as in the case that ℓ is even, it is not difficult to see that there exists $w \in M^1$ which permutes the blocks of L in such a way that each representation of $\tau_1, \ldots, \tau_b$ occurring an odd number of times in $((\delta^{\pm})_1, \ldots, (\delta^{\pm})_k)$ occurs an odd number of times as a representation of a block contained in $M_{(\ell+1)/2}$. In particular, the integer t is no greater than the block size $n_{(\ell+1)/2}$. Moreover, we may assume that w satisfies $w_t w \delta^{\pm} = w\delta^{\pm}$.

Bearing in mind our observation concerning the intertwining operator (8) and permutations of the blocks of L, we may now assume that $\delta^{\pm}$ satisfies $w_t \delta^{\pm} = \delta^{\pm}$ for some $0 \leqslant t \leqslant n'$. This implies that $L \in \mathcal{L}(t)$ and that (the class of) w_t belongs to W_δ by restricting $\delta^{\pm}$ to L^1. That is to say, we may assume that property (i) in our proof holds with $L = L'$.

To obtain property (ii) we wish to show that w_t is (a representative of) the longest element in W^0_δ. For this we require a better understanding of R_δ. Apparently, the Weyl group $W(A_L : G)$ is the direct product of the permutation group W_e, of the blocks of rank two occurring in L, and the permutation group S, of the blocks of rank one occurring in L. (The subgroup W_e is generated by reflections of the even roots of Δ_L (for definitions see [Kna86, § 10, XIV]).) [Kna86, Corollary 14.50] and [Kna86, Theorem 14.59] together imply that R_δ is contained in S. In view of Lemma 2, an element $\tilde{w} \in S \cap W_\delta$ can have either one of two possible effects: either it sends the trivial characters of $\delta^{\pm}$ to trivial characters and the sign characters of $\delta^{\pm}$ to sign characters; or the trivial and sign characters of $\delta^{\pm}$ appear an equal (positive) number of times, and $\tilde{w}$ sends each trivial character to a sign character and *vice versa*. We can reduce the question of whether $\tilde{w}$ lies in R_δ to the case $G = \mathrm{SL}(2, \mathbb{R})$ by applying [Kna86, Theorem 14.43(b)]. Combining this theorem with our knowledge of R groups for $\mathrm{SL}(2, \mathbb{R})$, we can deduce that $\tilde{w}$ lies in R_δ only if the second possibility for $\tilde{w}$ holds. As $w_t \delta^{\pm} \cong \delta^{\pm}$, the second possibility does not hold for the action of w_t on the blocks of rank one. Therefore, w_t belongs to W^0_δ. In addition, W^0_δ is isomorphic to $W_1 \times \cdots \times W_b$, where W_j is the permutation group of the representations in $((\delta^{\pm})_1, \ldots, (\delta^{\pm})_k)$ equivalent to τ_j, $1 \leqslant j \leqslant b$. It follows that the simple reflections of W_δ^0 are those which transpose the ith block of L with the jth block, where j is the least number greater than i such that $(\delta^{\pm})_i \cong (\delta^{\pm})_j$. The action of w_t on Δ_δ^0 sends each simple root to a negative root. The longest element of W_δ^0 is therefore equal to (the class of) w_t [Hum72, Lemma 10.3A]. This completes property the proof of property (ii)

To achieve property (iii) we require some more information about $w_M w_0$. Since we are assuming that $w_M w_0$ fixes $\delta^{\pm}$, we know from our description of W_δ that it belongs to W_δ^0. We may therefore suppose that $\nu \in i(\mathfrak{a}_L^M)^*$ such that $w\nu = -\nu$ for $w = w_M w_0 \in W_\delta^0$. Our goal is to show that there exists $w' \in W_\delta^0$ such that $w_t w'\nu = -w'\nu$, that is, that $w'\nu \in i(\mathfrak{a}_L^{w_t})^*$; for then we may set $\delta' = w'\delta = \delta$, $\nu' = w'\nu$, and the conditions of Proposition 2 are fulfilled. We may choose w' to be an element of W_δ^0 such that $w'\nu$ lies in the closure of the positive Weyl chamber determined by $\Delta_\delta^0 \cap \Delta_L^+$. After all, W_δ^0 acts simply transitively on the Weyl chambers, and our assumption that $w \in W_\delta^0$ implies that ν is in the real linear span of $i\Delta_\delta^0$. The element $w_t(w'w(w')^{-1})w'\nu$ is also in the closure of the positive Weyl chamber, as $(w'w(w')^{-1})w'\nu = -w'\nu$ lies in the closure of the chamber opposite to the positive chamber and w_t sends this latter chamber back to the positive one. By [Hum72, Lemma 10.3B], we have $w_t(w'w(w')^{-1})w'\nu = w'\nu$, whence

$$w_t w'\nu = (w_t)^{-1} w'\nu = (w'w(w')^{-1})w'\nu = -w'\nu,$$

and property (iii) is satisfied.

We now prove Proposition 2 in the case that $w_M w_0 \delta \cong \delta$, but $w_M w_0 \delta^{\pm}$ is *not* equivalent to $\delta^{\pm}$. According to our remarks concerning the nature of the R group, this implies that R_δ is non-trivial and that (the class of) $w_M w_0$ does not belong to W_δ^0. We have also pointed out that $R_\delta \neq \{1\}$ implies that the trivial and sign characters of $\mathrm{SL}^{\pm}(1, \mathbb{R})$ appear in $((\delta^{\pm})_1, \ldots, (\delta^{\pm})_k)$ and that they do so an equal number of times. We shall assume that τ_{b-1} is the trivial character and τ_b is the sign character, so that $a_{b-1} = a_b$. Our assumption and Lemma 2 imply that if $(\delta^{\pm})_j$ is a representation of a *rank-two* block of $L^{\pm}$ contained in the ith block of $M^{\pm}$, then it also occurs as a representation of a block of $L^{\pm}$ contained in the $(\ell+1-i)$th block of $M^{\pm}$. In contrast, if $(\delta^{\pm})_j$ is a representation of a *rank-one* block of $L^{\pm}$ contained in the ith block of $M^{\pm}$, then $\mathrm{sgn} \cdot (\delta^{\pm})_j$ occurs as a representation of a block of $L^{\pm}$ contained in the $(\ell+1-i)$th block of $M^{\pm}$. Let t be the number of representations among $\tau_1, \ldots, \tau_b$ which are not representations of $\mathrm{SL}^{\pm}(1, \mathbb{R})$ and which appear an odd number of times in $((\delta^{\pm})_1, \ldots, (\delta^{\pm})_k)$. It should be clear that there is an element $w \in M^1$, permuting the blocks of L, such that

$$(w_t w \delta^{\pm})_j = \begin{cases} (w\delta^{\pm})_j, & \text{if } (w\delta^{\pm})_j \text{ is a representation of } \mathrm{SL}^{\pm}(2, \mathbb{R}) \\ \mathrm{sgn} \cdot (w\delta^{\pm})_j, & \text{otherwise} \end{cases} \tag{9}$$

for all $1 \leqslant j \leqslant k$.

As in the previous case, we may now assume that $L \in \mathcal{L}(t)$ and $w_t \delta = \delta$, so that property (i) holds with $L = L'$. We wish to show property (ii) by proving that w_t may be taken to be (a representative of) the longest element in $Z_{W_\delta}(R_\delta)$. Let us examine the structure of R_δ is some more detail. Recall that a non-trivial element r of R_δ sends each trivial character of $\delta^{\pm}$ to a sign character of $\delta^{\pm}$ and *vice versa*. Moreover, it leaves the remaining representations occurring in $((\delta^{\pm})_1, \ldots, (\delta^{\pm})_k)$ unaffected. By definition, r also stabilizes the set $\Delta_\delta^0 \cap \Delta_L^+$. Consequently, there is an isomorphism between W_δ and

$$W_1 \times \cdots \times W_{b-2} \times ((W_{b-1} \times W_b) \rtimes R_\delta)$$

in which r acts on $W_{b-1} \times W_b$ by transposition. In particular, R_δ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The longest element of $Z_{W_\delta}(R_\delta)$ is seen to be the product of r and the longest element of W_δ^0. This element is distinguished by the property that it sends each positive simple root to a negative root and that it transposes each trivial representation in $\delta^{\pm}$ with a sign representation. According to (9), w_t has

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the latter property. The former property is deduced for w_t just as in the case that $w_M w_0 \delta^{\pm} \cong \delta^{\pm}$. Consequently, w_t is the longest element in $Z_{W_\delta}(R_\delta)$.

We have just shown that property (ii) in the proof of Proposition 2 is satisfied. To obtain property (iii), we may suppose that $\nu \in i(\mathfrak{a}_L^M)^*$ such that $rw\nu = -\nu$ for some $w \in W_\delta^0$. In the notation of § 4, this equation implies that there exist ν_0 in the real linear span of $i\Delta_\delta^0$, and ν_R, a real multiple of $i\alpha_R$, such that $\nu = \nu_0 + \nu_R$. Since the elements of Δ_δ^0 are orthogonal to α_R, this decomposition is unique and elements of W_δ^0 act as the identity on ν_R. Once again, we aim to prove Proposition 2 by showing that there exists $w' \in W_\delta^0$ such that $w_t w'\nu = -w'\nu$. We choose $w' \in W_\delta^0$ such that $w'\nu_0$ lies in the closure of the positive Weyl chamber determined by $\Delta_\delta^0 \cap \Delta_L^+$. The element $(w'rw(w')^{-1})w'\nu_0 = -w'\nu_0$ clearly lies in the closure of the opposite Weyl chamber. It is then immediate from the definition of R_δ that $r(w'rw(w')^{-1})w'\nu_0$ lies in the closure of the opposite chamber as well. It is readily verified that $w_t r$ is the longest element of W_δ^0, and so the element

$$w_t(w'rw(w')^{-1})w'\nu_0 = (w_t r)(r(w'rw(w')^{-1}))w'\nu_0$$

belongs once again to the closure of the positive chamber. [Hum72, Lemma 10.3B] then implies that $w_t(w'rw(w')^{-1})w'\nu_0 = w'\nu_0$ and in turn that $w_t w'\nu_0 = -w'\nu_0$. Therefore

$$w_t w'\nu = -w'\nu_0 + (w_t r)r\nu_R = -w'\nu_0 - \nu_R = -w'\nu,$$

and property (iii) is satisfied. This concludes the proof of Proposition 2. □

6. Some more intertwining operators

Proposition 2 gives us a method of associating any σ_0-stable representation to one of the representations in Proposition 1 by way of a permutation matrix. We shall use this method to define intertwining operators for any $\delta \in (\hat{L})_{\mathrm{lds}}$ and Levi subgroup $L \supset M_0$. In order to apply the existing trace Paley–Wiener theorems to our context it is important that these intertwining operators satisfy two properties.

The first property is one of compatibility for representations which are affiliated by induction. More precisely, let

$$P_{L_i} = L_i^1 A_{L_i} U_{L_i}, \quad 1 \leqslant i \leqslant s,$$

be the finite set of standard parabolic subgroups of G. We say that $\delta' \in (\hat{L_j})_{\mathrm{lds}}$ is *affiliated* to $\delta \in (\hat{L_i})_{\mathrm{lds}}$, if $P_{L_i} \subset P_{L_j}$ and δ' is a subrepresentation of $\mathrm{ind}_{P_{L_i} \cap L_j^1}^{L_j^1}(\delta \otimes e^0)$ (cf. [CD90, Définition 2]). Given $L' \in \mathcal{L}(t')$, $L \in \mathcal{L}(t)$, and $\delta' \in (\hat{L}')_{\mathrm{lds},t'}$ affiliated to $\delta \in (\hat{L})_{\mathrm{lds},t}$, we have the intertwining operators $T(\delta')$ and $T(\delta)$ of § 4 which intertwine $\mathrm{ind}_{P_{L'}}^G(\delta \otimes e^0)$ and $\mathrm{ind}_{P_L}^G(\delta \otimes e^0)$ with their respective σ_0-conjugates. It is immediate that the space of $\mathrm{ind}_{P_{L'}}^G(\delta' \otimes e^0)$ is a subspace of $\mathrm{ind}_{P_L}^G(\delta \otimes e^0)$, so the restriction of $T(\delta)$ to this subspace is also an intertwining operator between $\mathrm{ind}_{P_{L'}}^G(\delta' \otimes e^0)$ and its σ_0-conjugate. This restricted operator might be different from $T(\delta')$, and this is the type of incompatibility we would like to rule out in our subsequent definitions.

The second property we wish our intertwining operators to satisfy is that they intertwine σ_0-stable representations in a manner that is invariant under conjugation by permutation matrices. To attain this property we must define two types of intertwining operators. The reason for this is illustrated by the following example.

Recall the notation of § 5 and consider $G = \mathrm{SL}(4, \mathbb{R})$ and the representation $\delta^{\pm} = (\mathbf{1}, \mathbf{1}, \mathrm{sgn}, \mathrm{sgn})$ of $M_0^{\pm}$ in which $\mathbf{1}$ denotes the trivial representation of $\mathrm{SL}^{\pm}(1, \mathbb{R})$. The Levi subgroup M_0 belongs to $\mathcal{L}(0)$ and δ belongs to $(\hat{M}_0)_{\mathrm{lds},0}$. It is clear that R_δ is non-trivial and that w_0 is a representative

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of the longest element in $Z_{W_\delta}(R_\delta)$. According to Proposition 1, $T(\delta)$ is an intertwining operator between $\mathrm{ind}_{P_{M_0}}^G(\delta\otimes e^\nu)$ and its σ_0-conjugate for all $\nu\in(\mathfrak{a}_{M_0,\mathbb{C}}^{w_0})^*$. The operator $T(\delta)$ might appear to be all we need, but it does not intertwine some other σ_0-stable representations attached to δ. Specifically, suppose ν is given by the 4-tuple $(\nu_1,-\nu_1,\nu_2,-\nu_2)$ via the obvious embedding of $\mathfrak{a}_{M_0,\mathbb{C}}^*$ into $\mathbb{C}^4$. Suppose further that ν_1 and ν_2 are non-zero imaginary numbers and $\nu_1\neq\pm\nu_2$. Then one can show that $\mathrm{ind}_{P_{M_0}}^G(\delta\otimes e^\nu)$ is a σ_0-stable irreducible tempered representation. However,

$$-w_0\nu=(\nu_2,-\nu_2,\nu_1,-\nu_1)\neq\nu,$$

so $\nu\notin(\mathfrak{a}_{M_0,\mathbb{C}}^{w_0})^*$ and $T(\delta)$ does not intertwine $\mathrm{ind}_{P_{M_0}}^G(\delta\otimes e^\nu)$ with its σ_0-conjugate.

To find a different operator which *does* intertwine these representations, we can associate δ to the representation $\delta'=(\mathbf{1},\mathrm{sgn},\mathrm{sgn},\mathbf{1})\in(\hat{M}_0)_{\mathrm{lds},0}$ by a permutation as in § 5. Observe that in this case w_0 is a representative for the longest element in $W_{\delta'}^0$, not $Z_{W_{\delta'}}(R_{\delta'})$. We can then conjugate $T(\delta')$ by an operator of the form (8) to obtain the desired equivalence between $\mathrm{ind}_{P_{M_0}}^G(\delta\otimes e^\nu)$ and its σ_0-conjugate. We emphasize that the above two intertwining operators, $T(\delta)$ and $T(\delta')$, are distinguished by the two distinct types of Weyl groups, W_δ^0 and $Z_{W_{\delta'}}(R_{\delta'})$.

We shall begin our definitions by considering $Z_{W_\delta}(R_\delta)$ and defining an intertwining operator $T_1^\sigma(\delta,\nu)$ for $\delta\in(\hat{L}_i)_{\mathrm{lds}}$, $1\leqslant i\leqslant s$, and ν in a dense open subset of $\mathfrak{a}_{L_i,\mathbb{C}}^*$. Suppose that $\delta\in(\hat{L}_i)_{\mathrm{lds}}$ lies in the discrete series. Define $\{\delta\}$ to be the equivalence class of all discrete series representations $\delta'\in(\hat{L}_j)_{\mathrm{lds}}$ such that $\delta=w\delta'$ for some $w\in K$. If $\delta'\in(\hat{L}_j)_{\mathrm{lds}}$ belongs to $\{\delta\}$ then the set of minimal K types (cf. [Kna86, ch. XV]) of $\mathrm{ind}_{P_{L_j}}^G(\delta'\otimes e^0)$ is equal to the set of minimal K types of $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^0)$. On the other hand, if two such representations belong to two different equivalence classes then their corresponding sets of minimal K types are disjoint [CD84, Proposition 2]. This means that to each such equivalence class $\{\delta\}$ we can attach a minimal K type $\mu_{\{\delta\}}$ which determines it.

Suppose $\delta\in(\hat{L}_i)_{\mathrm{lds}}$ lies in the discrete series and L_i belongs to $\mathcal{L}(t)$ for some $0\leqslant t\leqslant\lfloor n/2\rfloor$. The restriction of the operator $T(\delta)$ of (7) to the minimal K type $\mu_{\{\delta\}}$ is a self-intertwining operator, as σ_0 fixes K pointwise and $\mu_{\{\delta\}}$ is a K type of multiplicity one in $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^0)$ [Kna86, ch. XV, § 1, equation (1)]. This restriction is therefore given by a non-zero scalar a_δ. Define

$$T_1^{\sigma_0}(\delta,\nu)=a_\delta^{-1}T(\delta),\quad \nu\in(\mathfrak{a}_{L_i,\mathbb{C}}^{w_t})^*.$$

The operator $T_1^{\sigma_0}(\delta,\nu)$ intertwines $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)$ with its σ_0-conjugate for any $\nu\in(\mathfrak{a}_{L_i,\mathbb{C}}^{w_t})^*$, and its restriction to $\mu_{\{\delta\}}$ is the identity operator.

Suppose now that $\delta\in(\hat{L}_i)_{\mathrm{lds}}$ lies in the discrete series, where $1\leqslant i\leqslant s$ is arbitrary. Then δ satisfies the properties of Lemma 1 with $M=G$, $L=L_i$, and $\nu=0$. Therefore, by the arguments of § 5, there is an integer $0\leqslant t\leqslant\lfloor n/2\rfloor$ and a permutation matrix $w\in G$ such that $wL_iw^{-1}\in\mathcal{L}(t)$, $w\delta\in(\widehat{wL_iw^{-1}})_{\mathrm{lds},t}$, and w_t is the longest element in $Z_{W_{w\delta}}(R_{w\delta})$. One can then compute that $w^{-1}w_tw$ is the longest element of $Z_{W_\delta}(R_\delta)$ with respect to the positive roots determined by $w^{-1}P_{wL_iw^{-1}}w$. As the Weyl group acts simply transitively on the Weyl chambers, there exists a unique element w'' in $Z_{W_\delta}(R_\delta)$ such that $(ww'')^{-1}w_tww''$ is the longest element of $Z_{W_\delta}(R_\delta)$ with respect to the positive roots determined by P_{L_i}. We may therefore assume that $w^{-1}w_tw$ is the longest element of $Z_{W_\delta}(R_\delta)$ with respect to the positive basis determined by P_{L_i}.

We know that there exists an invertible operator $S(w,\delta,\nu)$ intertwining $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)$ with $\mathrm{ind}_{P_{wL_iw^{-1}}}^G(w\delta\otimes e^{w\nu})$ for ν in a dense open subset of $\mathfrak{a}_{L_i,\mathbb{C}}^*$ (cf. (8)). In fact $\nu\mapsto S(w,\delta,\nu)$ is a meromorphic map on $\mathfrak{a}_{L_i,\mathbb{C}}^*$. Define $T_1^{\sigma_0}(\delta,\nu)$ to be $S(w,\delta,\nu)^{-1}T(w\delta,w\nu)S(w,\delta,\nu)$. Clearly, $T_1^{\sigma_0}(\delta,\nu)$ intertwines $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)$ with $(\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^{-w^{-1}w_tw\nu}))^{\sigma_0}$ whenever $S(w,\delta,\nu)$ is defined, and its restriction to $\mu_{\{\delta\}}$ is the identity operator.

Lemma 3. *Suppose $\delta \in (\hat{L}_i)_{\mathrm{lds}}$ lies in the discrete series for some $1 \leqslant i \leqslant s$. Then $T_1^{\sigma_0}(\delta,\nu)$ is well-defined on a dense open subset of $\mathfrak{a}^*_{L_i,\mathbb{C}}$.*

Proof. Suppose w' is another element of G such that $w' L_i (w')^{-1} \in \mathcal{L}(t')$, $w'\delta \in (w'\widehat{L_i(w')}^{-1})_{\mathrm{lds},t'}$, $w_{t'}$ is the longest element in $Z_{W_{w'\delta}}(R_{w'\delta})$, and $(w')^{-1}w_{t'}w'$ is the longest element in $Z_{W_\delta}(R_\delta)$. It is immediate that $(w')^{-1}w_{t'}w' = w^{-1}w_t w$ for w as above, and so the operator

$$S(w',\delta,\nu)^{-1}T(w'\delta,w'\nu)S(w',\delta,\nu)$$

intertwines $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^{\nu})$ with $(\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^{-w^{-1}w_{t'}w\nu}))^{\sigma_0}$ for ν in a dense open subset of $\mathfrak{a}^*_{L_i,\mathbb{C}}$. According to [SV80], $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^{\nu})$ is irreducible for ν in an open subset of $\mathfrak{a}^*_{L_i,\mathbb{C}}$. Schur's lemma therefore implies that, for ν in an open subset of $\mathfrak{a}^*_{L_i,\mathbb{C}}$, $T_1^{\sigma_0}(\delta,\nu)$ is a scalar multiple of $S(w',\delta,\nu)^{-1}$ $T(w'\delta,w'\nu)S(w',\delta,\nu)$. The traces of the restrictions of these two operators to $\mu_{\{\delta\}}$ are equal (to the degree of $\mu_{\{\delta\}}$). Therefore, the scalar multiple must be 1 and $T_1^{\sigma_0}(\delta,\nu)$ equals $S(w',\delta,\nu)^{-1}T(w'\delta)$ $S(w',\delta,\nu)$ for ν in an open subset of $\mathfrak{a}^*_{L_i,\mathbb{C}}$. By analytic continuation the equality holds for all $\nu\in\mathfrak{a}^*_{L_i,\mathbb{C}}$. ∎

Suppose now that $1\leqslant j\leqslant s$ and $\delta'\in(\hat{L}_j)_{\mathrm{lds}}$ is a non-degenerate limit of discrete series. It is affiliated to some $\delta\in(\hat{L}_i)_{\mathrm{lds}}$ lying in the discrete series for some $1\leqslant i\leqslant s$ [Kna86, Theorem 14.71]). The operator $T_1^{\sigma_0}(\delta,\nu)$ is defined for ν in the positive Weyl chamber of $\mathfrak{a}^*_{L_j,\mathbb{C}}$ (cf. the proof of Lemma 14.1 in [Kna86]). Since $T_1^{\sigma_0}(\delta,\nu)$ is meromorphic in ν, it extends to a meromorphic function on all of $\mathfrak{a}^*_{L_j,\mathbb{C}}$. We define $T_1^{\sigma_0}(\delta',\nu)$ to be the restriction of $T_1^{\sigma_0}(\delta,\nu)$ to $\mathrm{ind}_{P_{L_j}}^G(\delta'\otimes e^{\nu})$ for ν in the dense open subset of $\mathfrak{a}^*_{L_j,\mathbb{C}}$ upon which $T_1^{\sigma_0}(\delta',\nu)$ is defined.

Lemma 4. *Suppose $1\leqslant j\leqslant s$ and $\delta'\in(\hat{L}_j)_{\mathrm{lds}}$ is a non-degenerate limit of discrete series. Then $T_1^{\sigma_0}(\delta',\nu)$ is well-defined on a dense open subset of $\mathfrak{a}^*_{L_j,\mathbb{C}}$.*

Proof. Suppose δ' is affiliated to another representation $\delta''\in(\hat{L}_{i'})_{\mathrm{lds}}$ lying in the discrete series for some $1\leqslant i'\leqslant s$. Then, by the Langlands disjointness theorem [Kna86, Theorem 14.90], there exists $w\in K\cap L_j$ such that $w\delta''=\delta$. We have the intertwining operator

$$S=\mathcal{A}(P_{L_i}\cap L_j^1 : wP_{L_{i'}}w^{-1}\cap L_j^1 : \delta : 0)\mathrm{L}(w)$$

from $\mathrm{ind}_{P_{L_{i'}}\cap L_j^1}^{L_j^1}(\delta''\otimes e^0)$ to $\mathrm{ind}_{P_{L_i}\cap L_j^1}^{L_j^1}(\delta\otimes e^0)$. The operator S also induces an intertwining operator from $\mathrm{ind}_{P_{L_{i'}}}^G(\delta''\otimes e^{\nu})$ to $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^{\nu})$, by virtue of the equivalence

$$\mathrm{ind}_{P_{L_j}}^G(\mathrm{ind}_{P_{L_{i'}}\cap L_j}^{L_j}(\delta'')\otimes e^{\nu})\cong\mathrm{ind}_{P_{L_{i'}}}^G(\delta''\otimes e^{\nu}).$$

We also denote this induced operator by S. Since δ' occurs in $\mathrm{ind}_{P_{L_i}\cap L_j^1}^{L_j^1}(\delta\otimes e^0)$ with multiplicity one [Kna86, Corollary 14.66] and $S^{-1}\delta' S=\delta'$, the restriction of $S^{-1}T_1^{\sigma_0}(\delta,\nu)S$ to $\mathrm{ind}_{P_{L_j}}^G(\delta'\otimes e^{\nu})$ is equal to the restriction of $T_1^{\sigma_0}(\delta,\nu)$ to the same space. Obviously, $S^{-1}T_1^{\sigma_0}(\delta,\nu')S$ and $T_1^{\sigma_0}(\delta'',\nu')$ both intertwine $\mathrm{ind}_{P_{L_{i'}}}^G(\delta''\otimes e^{\nu'})$ with $(\mathrm{ind}_{P_{L_{i'}}}^G(\delta''\otimes e^{-w''\nu'}))^{\sigma_0}$ for w'', the longest element in $Z_{W_\delta}(R_\delta)$, and ν' in an open subset of $\mathfrak{a}^*_{L_{i'},\mathbb{C}}$.

We shall finish the proof by showing that $S^{-1}T_1^{\sigma_0}(\delta,\nu')S$ is equal to $T_1^{\sigma_0}(\delta'',\nu')$. By [SV80], $\mathrm{ind}_{P_{L_{i'}}}^G(\delta''\otimes e^{\nu'})$ is irreducible for ν' in an open subset of $\mathfrak{a}^*_{L_{i'},\mathbb{C}}$. This implies that $S^{-1}T_1^{\sigma_0}(\delta,\nu')S$ is a scalar multiple of $T_1^{\sigma_0}(\delta'',\nu')$. The traces of the restrictions of these operators to $\mu_{\{\delta\}}$ are equal. In consequence the scalar multiple must be one and the operators are equal. ∎

Now that we have defined $T_1^{\sigma_0}(\delta',\nu)$ we ought to ensure that it is an operator intertwining the desired representations.

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Lemma 5. *Suppose $1 \leqslant j \leqslant s$, $\delta' \in (\hat{L}_j)_{\mathrm{lds}}$ is a non-degenerate limit of discrete series, and w is the longest element of $Z_{W_{\delta'}}(R_{\delta'})$. Then $T_1^{\sigma_0}(\delta',\nu)$ intertwines $\mathrm{ind}_{P_{L_j}}^{G}(\delta' \otimes e^{\nu})$ with its σ_0-conjugate for ν in a dense open subset of $(\mathfrak{a}^{w}_{L_i,\mathbb{C}})^*$.*

Proof. We suppose first that $L_j \in \mathcal{L}(t)$ and $w = w_t$ is the longest element in $Z_{W_{\delta'}}(R_{\delta'})$ for some $0 \leqslant t \leqslant \lfloor n/2 \rfloor$. Suppose also that δ' is affiliated to a discrete series representation $\delta \in (\hat{L}_i)_{\mathrm{lds}}$ as above, $\nu \in (\mathfrak{a}^{w_t}_{L_j,\mathbb{C}})^*$, and that $(\delta')^{\pm}$ is a representation of $L_j^{\pm}$ as in § 5. In this case, $T_1^{\sigma_0}(\delta,\nu)$ is defined.

Since the only limit of discrete series representations occurring in the expansion of $(\delta')^{\pm}$ are induced from a Borel subgroup B and are of the form

$$\mathrm{ind}_B^{\mathrm{SL}^{\pm}(2,\mathbb{R})}((\mathbf{1} \otimes \mathrm{sgn}) \otimes e^0) = \mathrm{ind}_B^{\mathrm{SL}^{\pm}(2,\mathbb{R})}((\mathrm{sgn} \otimes \mathbf{1}) \otimes e^0)$$

(cf. [Kna79, § 2]), we know that the expansion of $\delta^{\pm}$ is given by replacing some of these representations in $(\delta')^{\pm}$ with $(\mathbf{1}, \mathrm{sgn})$ or $(\mathrm{sgn}, \mathbf{1})$. A simple reordering of the latter representations implies the existence of $w' \in L_j$, which normalizes L_i and satisfies $w'\delta \in (\hat{L}_i)_{\mathrm{lds},t}$ with w_t as the longest element in $Z_{W_\delta}(R_\delta)$. In addition, since

$$T_1^{\sigma_0}(\delta,\nu) = T_1^{\sigma_0}(w'\delta, w'\nu) = T_1^{\sigma_0}(w'\delta,\nu),$$

we may assume that $w' = 1$. Thus, the restriction of $T_1^{\sigma_0}(\delta,\nu)$ to $\mathrm{ind}_{P_{L_j}}^{G}(\delta' \otimes e^{\nu})$ intertwines this representation with its σ_0-conjugate.

Dropping the assumption that $L_j \in \mathcal{L}(t)$, the normalization of [Kna86, Lemma 14.1] still ensures that $T_1(\delta',\nu)$ is defined for ν in a dense open subset (the positive Weyl chamber) of $(\mathfrak{a}^{w}_{L_j,\mathbb{C}})^*$. We may therefore use the previous arguments to prove the lemma in this case as well. $\square$

Thus far we have defined an operator $T_1(\delta,\nu)$ which intertwines $\mathrm{ind}_{P_{L_i}}^{G}(\delta \otimes e^{\nu})$ with its σ_0-conjugate for every $\delta \in (\hat{L}_i)_{\mathrm{lds}}$, $1 \leqslant i \leqslant s$, and ν in a dense open subset of $(\mathfrak{a}^{w}_{L_i,\mathbb{C}})^*$, where w is the longest element in $Z_{W_\delta}(R_\delta)$. It is apparent from the definitions that these intertwining operators are compatible with respect to affiliation.

The example at the beginning of this section illustrates the need for an operator $T_2^{\sigma_0}(\delta,\nu)$ for $\delta \in (\hat{L}_i)_{\mathrm{lds}}$, $1 \leqslant i \leqslant s$, and ν in a dense open subset of $(\mathfrak{a}^{w}_{L_i,\mathbb{C}})^*$ where w is now the longest element in W_δ^0. We define this second type of intertwining operator by following the definition of $T_1^{\sigma_0}(\delta,\nu)$ and replacing $Z_{W_\delta}(R_\delta)$ everywhere with W_δ^0.

It is important to realize that if n is odd then $T_1^{\sigma_0}(\delta,\nu)$ is equal to $T_2^{\sigma_0}(\delta,\nu)$. Indeed, we know from § 5 that the only way the R group of δ can be non-trivial is if the trivial character and sign character appear in $\delta^{\pm}$ an equal number of times. This is only possible if n is even.

7. σ_0-twisted trace Paley–Wiener theorems

Our strategy in proving a σ_0-twisted trace Paley–Wiener theorem for $\mathrm{SL}(n,\mathbb{R})$ is to follow [CD84] and [CD90]. We first prove an analogue of their Proposition 1 [CD84], which deals with individual representations in $(\hat{L})_{\mathrm{lds},t}$ for some $L \in \mathcal{L}(t)$ and $0 \leqslant t \leqslant \lfloor n/2 \rfloor$. Then we prove a σ_0-twisted analogue of their trace Paley–Wiener theorems for two different cases, depending on some R groups.

To refer to the proof of Proposition 1 [CD84] we need some notation. Let N be a positive real number. Suppose $L = L_i$ for some $1 \leqslant i \leqslant s$ and define $\mathcal{PW}(\mathfrak{a}_L)_N$ to be the image under the Fourier transform of the smooth functions on $\mathfrak{a}_L$ with support in the closed ball of radius N about the origin. The classical Paley–Wiener theorem tells us that functions ϕ in $\mathcal{PW}(\mathfrak{a}_L)_N$ are entire on

$\mathfrak{a}^*_{L,\mathbb{C}}$ and satisfy a growth condition,

$$\sup_{\lambda\in\mathfrak{a}^*_{L,\mathbb{C}}} \{|\phi(\lambda)|e^{-N|\mathrm{Re}(\lambda)|}(1+|\mathrm{Im}(\lambda)|)^k t\} < \infty,$$

for every integer k. If W is a group acting on $\mathfrak{a}_L$ we denote the W-invariant subspace of $\mathcal{PW}(\mathfrak{a}_L)_N$ by $\mathcal{PW}(\mathfrak{a}_L)^W_N$.

Suppose now that $L \in \mathcal{L}(t)$, $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{t} \subset \mathfrak{k}$ is the Lie algebra of a compact Cartan subgroup of L^1. Then

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}_L = \mathfrak{t} \oplus \mathfrak{a}^{w_t}_L \oplus (\mathfrak{a}^{w_t}_L)^\perp$$

is a Cartan subalgebra of $\mathfrak{g}$, the Lie algebra of G. Let $W(\mathfrak{h})$ be the Weyl group of $\mathfrak{h}$, $S(\mathfrak{h})$ be the polynomial algebra on $\mathfrak{h}^*_{\mathbb{C}}$, and $\mathcal{PW}(\mathfrak{h})$ be the Paley–Wiener space determined by $\mathfrak{h}$. As before, we denote the W-invariant subspaces of $S(\mathfrak{h})$ and $\mathcal{PW}(\mathfrak{h})$ by $S(\mathfrak{h})^W$ and $\mathcal{PW}(\mathfrak{h})^W$, respectively, for any group W acting on $\mathfrak{h}$.

Define $C^\infty_c(G,K)_N$ to be the space of smooth K-finite functions of G with support in $K\exp(\mathfrak{a}(N))K$, where $\mathfrak{a}(N)$ is the closed ball of radius N about the origin in $\mathfrak{a}_{M_0}$.

Proposition 3. *Suppose $k = 1, 2$, $0 \leqslant t \leqslant \lfloor n/2\rfloor$, $L \in \mathcal{L}(t)$, $\delta \in (\hat{L})_{\mathrm{lds},t}$, μ is a minimal K type of $\mathrm{ind}^G_{P_L}(\delta\otimes e^0)$, and F is a complex function of $(\mathfrak{a}^{w_t}_{L,\mathbb{C}})^*$. Suppose further that*

$$W = \{w \in W_\delta : w\mathfrak{a}^{w_t}_L \subset \mathfrak{a}^{w_t}_L\}.$$

Then there exists $f \in C^\infty_c(G,K)_N$ of type (μ,μ) such that

$$F(\nu) = \mathrm{tr}(\mathrm{ind}^G_{P_L}(\delta\otimes e^\nu)(f)T^{\sigma_0}_k(\delta,\nu))$$

if and only if F belongs to $\mathcal{PW}(\mathfrak{a}^{w_t}_L)^W_N$.

Proof. Suppose $F \in \mathcal{PW}(\mathfrak{a}^{w_t}_L)^W_N$. By the Corollary of [Cow86], F extends to a function $F' \in \mathcal{PW}(\mathfrak{h})^W_N$ such that

$$F'(\lambda_\delta+\nu) = F(\nu), \quad \nu \in (\mathfrak{a}^{w_t}_{L,\mathbb{C}})^*$$

for the Harish–Chandra parameter $\lambda_\delta \in \mathfrak{t}^*_{\mathbb{C}}$ of δ (cf. [Kna86, § 7, ch. IX]). By a theorem of Raïs (cf. [CD84, Lemme 8]), it follows that

$$S(\mathfrak{h})^W \mathcal{PW}(\mathfrak{h})^{W(\mathfrak{h})}_N = \mathcal{PW}^W(\mathfrak{h})_N.$$

We therefore write $F'(\Lambda) = \sum P_i(\Lambda)F'_i(\Lambda)$, where $P_i \in S(\mathfrak{h})^W$ and $F'_i \in \mathcal{PW}(\mathfrak{h})^{W(\mathfrak{h})}_N$. Since the involution given by

$$\lambda \mapsto -w_t\lambda, \quad \lambda \in \mathfrak{a}^*_L$$

stabilizes $\Delta^0_\delta \cap \Delta^+_L$, [CD90, Proposition A.1] tells us that there exists a function $P''_i \in S(\mathfrak{h})^{W^0_\delta}$ which agrees with P_i on $\lambda_\delta + (\mathfrak{a}^{w_t}_{L,\mathbb{C}})^*$. As $R_\delta \subset W$, the function

$$P'_i(\lambda) = |R_\delta|^{-1}\sum_{r\in R_\delta} P''_i(r\Lambda), \quad \Lambda \in \mathfrak{h}^*_{\mathbb{C}},$$

is a function in $S(\mathfrak{h})^{W_\delta}$ which agrees with P_i on $\lambda_\delta + (\mathfrak{a}^{w_t}_{L,\mathbb{C}})^*$. By [CD84, Theorem 2] there exists a $\mathfrak{k}$-invariant element u_i in the universal enveloping algebra of $\mathfrak{g}$ which acts on the minimal K type μ of $\mathrm{ind}^G_{P_L}(\delta\otimes e^\lambda)$ as multiplication by

$$P'_i(\lambda_\delta+\lambda) = P_i(\lambda_\delta+\lambda), \quad \lambda \in \mathfrak{a}^*_{L,\mathbb{C}}.$$

By [CD84, Lemma 9] there exists $f_i \in C^\infty_c(G,K)_N$ of type (μ,μ) such that, for all $\lambda \in (\mathfrak{a}^{w_t}_{L,\mathbb{C}})^*$, $\mathrm{ind}^G_{P_L}(\delta\otimes e^\lambda)(f_i)$ equals $F'_i(\lambda_\delta+\lambda)$ on μ and equals zero on any other K type. The restriction of

$T_k^{\sigma_0}(\delta,\nu)$ to μ is a self-intertwining operator and hence a scalar $\varepsilon_k = \pm 1$. The function

$$f = \varepsilon_k \cdot \frac{\sum u_i * f_i}{\deg(\mu)}$$

can be seen to satisfy the claim of the proposition.

Suppose now that the converse holds. Since f is K-finite, one may choose a basis of $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)$, with respect to its K types, such that only finitely many matrix coefficients of $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)(f)$ are non-zero. According to [CD84, § 2.1], each non-zero matrix coefficient belongs to $\mathcal{PW}(\mathfrak{a}_L)_N$. As the operator $T_k^{\sigma_0}(\delta,\nu)$ sends the K-isotypical components of $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)$ to themselves and does not depend on ν, there are finitely many non-zero matrix coefficients of $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)(f)T_k^{\sigma_0}(\delta,\nu)$, and each of these matrix coefficients is a finite linear combination of functions in $\mathcal{PW}(\mathfrak{a}_L)_N$. Consequently, the trace of $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)(f)T_k^{\sigma_0}(\delta,\nu)$ defines a function in $\mathcal{PW}(\mathfrak{a}_L)_N$. The invariance of this function under W is obvious. □

Given $L \in \mathcal{L}(t)$, let $(\hat{L})^1_{\mathrm{lds},t}$ be the subset of representations $\delta \in (\hat{L})_{\mathrm{lds},t}$ such that w_t is a representative of the longest element in $Z_{W_\delta}(R_\delta)$, and let $(\hat{L})^2_{\mathrm{lds},t}$ be the subset of representations $\delta \in (\hat{L})_{\mathrm{lds},t}$ such that w_t is a representative of the longest element in W_δ^0. The following theorem is a σ_0-twisted version of [CD90, Théorème 1] in the case $G = \mathrm{SL}(n,\mathbb{R})$.

Theorem 1. *Suppose $N > 0$, $k = 1, 2$, and that for each $0 \leqslant t \leqslant \lfloor n/2 \rfloor$ and $1 \leqslant i \leqslant s$ such that $L_i \in \mathcal{L}(t)$ we are given a function*

$$F_{i,t}^k : (\hat{L}_i)^k_{\mathrm{lds},t} \times (\mathfrak{a}^{w_t}_{L_i,\mathbb{C}})^* \to \mathbb{C}.$$

Then the following are equivalent.

a) *There exists $f_k \in C_c^\infty(G,K)_N$ such that*

$$F_{i,t}^k(\delta,\nu) = \mathrm{tr}(\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)(f_k)T_k^{\sigma_0}(\delta,\nu)), \quad \nu \in (\mathfrak{a}^{w_t}_{L_i,\mathbb{C}})^*.$$

b) (1) *$F_{i,t}^k$ has finite support.*

(2) *$F_{i,t}^k(\delta,\cdot)$ belongs to $\mathcal{PW}(\mathfrak{a}^{w_t}_{L_i})_N$, for all $\delta \in (\hat{L}_i)^k_{\mathrm{lds},t}$.*

(3) *Suppose $\delta \in (\hat{L}_i)^k_{\mathrm{lds},t}$, $\nu \in (\mathfrak{a}^{w_t}_{L_i,\mathbb{C}})^*$, and $w \in K$ such that $w\delta \in (\hat{L}_j)^k_{\mathrm{lds},t}$ and $w\mathfrak{a}^{w_t}_{L_i} = \mathfrak{a}^{w_t}_{L_j}$. Then $F_{i,t}^k(\delta,\nu) = F_{j,t}^k(w\delta, w\nu)$.*

(4) *Suppose $1 \leqslant i,j \leqslant s$, $P_{L_i} \subset P_{L_j}$, $\delta \in (\hat{L}_i)^k_{\mathrm{lds},t}$, $\delta_1', \ldots, \delta_m' \in (\hat{L}_j)^k_{\mathrm{lds},t}$, and*

$$\mathrm{ind}_{P_{L_i}\cap L_j^1}^{L_j^1}(\delta\otimes e^0) = \delta_1' \oplus \cdots \oplus \delta_m'.$$

Then

$$F_{i,t}^k(\delta,\nu) = F_{j,t}^k(\delta_1',\nu) + \cdots + F_{j,t}^k(\delta_m',\nu),\ \nu \in (\mathfrak{a}^{w_t}_{L_j,\mathbb{C}})^*.$$

Proof. We start the proof by assuming condition a holds and showing that each condition listed under item b is satisfied. In doing this, we follow [CD84, § 2.1]. To prove item b(1) we may assume that μ_1,μ_2 are irreducible representations of K and that f_k is of type (μ_1,μ_2). It can then be shown that if $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)(f_k) \neq 0$ then $\mu_1 = \mu_2$ and μ_1 is a K type of $\mathrm{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)$. Frobenius reciprocity implies that the restriction of μ_1 to $K \cap L_i$ contains a $(K\cap L_i)$-type of δ. A result of Harish–Chandra tells us that there are only finitely many inequivalent irreducible admissible representations of L_i containing a fixed $(K\cap L_i)$-type, from which b(1) follows.

Condition b(2) holds as in the proof of Proposition 3.

Suppose the hypothesis of b(3) holds. It is then clear from the definitions of § 6 that $T_k^{\sigma_0}(\delta,\nu)$ is equal to $S(w,\delta,\nu)^{-1}T_k^{\sigma_0}(w\delta,w\nu)S(w,\delta,\nu)$. We therefore have

$$\begin{aligned}\operatorname{tr}(\operatorname{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)(f_k)T_k^{\sigma_0}(\delta,\nu)) &= \operatorname{tr}(\operatorname{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)(f_k)S(w,\delta,\nu)^{-1}T_k^{\sigma_0}(w\delta,w\nu)S(w,\delta,\nu))\\ &= \operatorname{tr}(S(w,\delta,\nu)\operatorname{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)(f_k)S(w,\delta,\nu)^{-1}T_k^{\sigma_0}(w\delta,w\nu))\\ &= \operatorname{tr}(\operatorname{ind}_{P_{L_j}}^G(w\delta\otimes e^{w\nu})(f_k)T_k^{\sigma_0}(w\delta,w\nu)). \qquad (10)\end{aligned}$$

Suppose the hypothesis of b(4) holds and that $\nu\in(\mathfrak{a}_{L_j,\mathbb{C}}^{w_t})^*$. Then $\operatorname{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)$ is equivalent to

$$\operatorname{ind}_{P_{L_j}}^G(\delta_1'\otimes e^\nu)\oplus\cdots\oplus\operatorname{ind}_{P_{L_j}}^G(\delta_m'\otimes e^\nu).$$

By definition, the restriction of $T_k^{\sigma_0}(\delta,\nu)$ to $\operatorname{ind}_{P_{L_j}}^G(\delta_b'\otimes e^\nu)$ is equal to $T_k^{\sigma_0}(\delta_b',\nu)$, for $1\leqslant b\leqslant m$. Thus the trace of $\operatorname{ind}_{P_{L_i}}^G(\delta\otimes e^\nu)(f_k)T_k^{\sigma_0}(\delta,\nu)$ is equal to the trace of

$$\operatorname{ind}_{P_{L_j}}^G(\delta_1'\otimes e^\nu)(f_k)T_k^{\sigma_0}(\delta_1',\nu)\oplus\cdots\oplus\operatorname{ind}_{P_{L_j}}^G(\delta_m'\otimes e^\nu)(f_k)T_k^{\sigma_0}(\delta_m',\nu),$$

and the conclusion of b(4) ensues.

Having proved that (a) implies (b) we prove the converse. Suppose the conditions of (b) hold and that $\delta\in(\hat{L}_i)_{\mathrm{lds},t}^k$ belongs to the discrete series for some $L_i\in\mathcal{L}(t)$. According to Proposition 3 and Proposition 1 of [CD90], there exists a function $f'\in C_c^\infty(G,K)_N$ such that

$$\operatorname{tr}(\operatorname{ind}_{P_{L_j}}^G(\delta'\otimes e^\nu)(f'))=F_{j,t}^k(\delta',\nu),\quad 1\leqslant j\leqslant s,$$

for all $\delta'\in(\hat{L}_j)_{\mathrm{lds},t}^k$ affiliated to δ and $\nu\in(\mathfrak{a}_{L_j,\mathbb{C}}^{w_t})^*$. Before we move on, some justification of the use of [CD90, Proposition 1] is in order. Clozel and Delorme make use of a space $\mathcal{PW}(E,W,\Delta^+)_r$, where $r>0$, E is a real vector space, Δ^+ is a set of positive roots of a root system Δ of a subspace of E, and W is a group of automorphisms of E containing the Weyl group W^0 of Δ as a normal subgroup such that W/W^0 is isomorphic to a product of copies of $\mathbb{Z}/2\mathbb{Z}$ [CD90, Appendice C]. We apply the machinery of $\mathcal{PW}(E,W,\Delta^+)_r$ to the present context by taking $r=N$, $E=\mathfrak{a}_{L_i}^{w_t}$ and

$$\Delta=\Delta_\delta^0\cap(\mathfrak{a}_{L_i,\mathbb{C}}^{w_t})^*.$$

We need to take

$$W=\{w\in W_\delta : w\mathfrak{a}_{L_i}^{w_t}\subset\mathfrak{a}_{L_i}^{w_t}\},$$

despite the fact that W/W^0 is not necessarily isomorphic to a product of copies of $\mathbb{Z}/2\mathbb{Z}$. This choice for W is legitimate if we replace W^0 with $W\cap W_\delta^0$ in the formalism of [CD90, Appendice C]. Indeed, $W/(W\cap W_\delta^0)$ is isomorphic to a product of copies of $\mathbb{Z}/2\mathbb{Z}$, and the remaining results of Appendice C also remain valid with this choice of W^0.

Moving on, the function f' is the sum of functions $f_\mu\in C_c^\infty(G,K)_N$, where μ is a minimal K type of $\operatorname{ind}_{P_{L_i}}^G(\delta\otimes e^0)$ and f_μ is of type (μ,μ). The restriction of the operator $T_k^{\sigma_0}(\delta,\nu)$ to μ is a self-intertwining operator and hence given by a scalar $\varepsilon_k(\delta,\mu)=\pm1$. Setting g_k to be the sum of the functions $\varepsilon_k(\delta,\mu)f_\mu$, where μ runs over the minimal K types of $\operatorname{ind}_{P_{L_i}}^G(\delta\otimes e^0)$, we see that $g_k\in C_c^\infty(G,K)_N$ and

$$\operatorname{tr}(\operatorname{ind}_{P_{L_j}}^G(\delta'\otimes e^\nu)(g_k)T_k^{\sigma_0}(\delta',\nu))=F_{j,t}^k(\delta',\nu),\quad 1\leqslant j\leqslant s,$$

for all $\delta'\in(\hat{L}_j)_{\mathrm{lds},t}$ affiliated to δ and $\nu\in(\mathfrak{a}_{L_j,\mathbb{C}}^{w_t})^*$.

The existence of the function g_k is a σ_0-twisted analogue of [CD90, Proposition 1]. It allows us to adapt the proofs of [CD90, Théorème 1] and [CD84, Théorème 1] to our context thereby completing the proof of this theorem. The cited proofs rely on an induction argument involving the support

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of $F_{i,t}^k$ and the length of minimal K types. The details of the induction argument can be found in [CD84, § 2.3]. $\square$

The next corollary is proven by combining Theorem 1 with the definitions of § 6 and equations like those of (10).

Corollary 1. *Given $\delta \in (\hat{L}_i)_{\mathrm{lds}}$, $1 \leqslant i \leqslant s$, let $w_{\delta,1}$ be the longest element of $Z_{W_\delta}(R_\delta)$ and $w_{\delta,2}$ be the longest element of W_δ^0. Suppose $N > 0$, $k = 1, 2$, and that for each $1 \leqslant i \leqslant s$ we are given a function*

$$F_i^k : (\hat{L}_i)_{\mathrm{lds}} \times (\mathfrak{a}_{L_i,\mathbb{C}}^{w_{\delta,k}})^* \to \mathbb{C}.$$

Then the following are equivalent.

a) *There exists $f_k \in C_c^\infty(G,K)_N$ such that*

$$F_i^k(\delta,\nu) = \mathrm{tr}(\mathrm{ind}_{P_{L_i}}^G(\delta \otimes e^\nu)(f_k) T_k^{\sigma_0}(\delta,\nu)), \quad \nu \in (\mathfrak{a}_{L_i,\mathbb{C}}^{w_{\delta,k}})^*.$$

b) (1) *F_i^k has finite support.*

(2) *$F_i^k(\delta,\cdot)$ belongs to $\mathcal{PW}(\mathfrak{a}_{L_i}^{w_{\delta,k}})_N$, for all $\delta \in (\hat{L}_i)_{\mathrm{lds}}$.*

(3) *Suppose $\delta \in (\hat{L}_i)_{\mathrm{lds}}$, $\nu \in (\mathfrak{a}_{L_i,\mathbb{C}}^{w_{\delta,k}})^*$, and $w \in K$ such that $w\delta \in (\hat{L}_j)_{\mathrm{lds}}$ and $w\mathfrak{a}_{L_i}^{w_{\delta,k}} = \mathfrak{a}_{L_i}^{w_{w\delta,k}}$. Then $F_i^k(\delta,\nu) = F_j^k(w\delta, w\nu)$.*

(4) *Suppose $1 \leqslant i, j \leqslant s$, $P_{L_i} \subset P_{L_j}$, $\delta \in (\hat{L}_i)_{\mathrm{lds}}$, $\delta_1', \ldots, \delta_m' \in (\hat{L}_j)_{\mathrm{lds}}$, and*

$$\mathrm{ind}_{P_{L_i} \cap L_j^1}^{L_j^1}(\delta \otimes e^0) = \delta_1' \oplus \cdots \oplus \delta_m'.$$

Then

$$F_i^k(\delta,\nu) = F_j^k(\delta_1',\nu) + \cdots + F_j^k(\delta_m',\nu), \quad \nu \in (\mathfrak{a}_{L_j,\mathbb{C}}^{w_{\delta,k}})^*.$$

8. A compatibility condition

We mentioned at the end of § 6 that if n is odd then the R groups of our discussion are all trivial. In this case the parameter $k = 1, 2$ of Corollary 1 is superfluous. That is, a *single* function in $C_c^\infty(G,K)_N$ determines any set of functions satisfying the properties of Corollary 1(b).

Let us suppose for the rest of this section that n is even and that we have functions F_j^k for $1 \leqslant j \leqslant s$ and $k = 1, 2$, which satisfy the conditions of Theorem 1(b). If $\{F_j^1\}$ and $\{F_j^2\}$ are compatible in some sense, then we should also be able to obtain a *single* function in $C_c^\infty(G,K)_N$ which satisfies an equality as in Corollary 1(a) for *both* $k = 1$ and $k = 2$.

Let us make this notion of compatibility precise. Suppose $\delta \in (\hat{L}_j)_{\mathrm{lds}}$ has trivial R group. Then the Weyl group elements $w_{\delta,1}$ and $w_{\delta,2}$ of the corollary are equal and $\mathrm{ind}_{P_{L_j}}^G(\delta \otimes e^\nu)$ is irreducible for all $\nu \in i(\mathfrak{a}_{L_j}^{w_{\delta,1}})^*$. Under these circumstances $T_1^{\sigma_0}(\delta,\nu)$ and $T_2^{\sigma_0}(\delta,\nu)$ are defined [Kna86, Theorem 14.20(d)] and $T_1^{\sigma_0}(\delta,\nu) T_2^{\sigma_0}(\delta,\nu)$ is a self-intertwining operator of $\mathrm{ind}_{P_{L_j}}^G(\delta \otimes e^\nu)$. By Schur's lemma, it is given by a scalar $c(\delta,\nu) \in \mathbb{C}$. We say that $\{F_j^1\}$ is compatible with $\{F_j^2\}$ if

$$F_j^2(\delta,\nu) = c(\delta,\nu) F_j^1(\delta,\nu), \quad \nu \in i(\mathfrak{a}_{L_j}^{w_{\delta,1}})^*$$

for any δ as above. Such a compatibility condition is necessary, for if δ and ν are as above and $f \in C_c^\infty(G,K)_N$ then

$$\begin{aligned}\mathrm{tr}(\mathrm{ind}_{P_{L_j}}^G(\delta \otimes e^\nu)(f) T_2^{\sigma_0}(\delta,\nu)) &= \mathrm{tr}(\mathrm{ind}_{P_{L_j}}^G(\delta \otimes e^\nu)(f)(T_1^{\sigma_0}(\delta,\nu))^2 T_2^{\sigma_0}(\delta,\nu)) \\ &= c(\delta,\nu)\,\mathrm{tr}(\mathrm{ind}_{P_{L_j}}^G(\delta \otimes e^\nu)(f) T_1^{\sigma_0}(\delta,\nu)).\end{aligned}$$

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To prove the existence of the desired function in $C_c^\infty(G,K)_N$ from the compatibility condition, we wish to define a function on $\mathcal{PW}(\mathfrak{a}_{L_j})$ which restricts to $F_j^k(\delta,\cdot)$ on $i(\mathfrak{a}_{L_j}^{w_{\delta,k}})^*$ for *any* $k=1,2$ and $\delta \in (\hat{L}_j)_{\mathrm{lds}}$. This can be accomplished using the following.

CONJECTURE[1] 1. Suppose that n is even, $N>0$, $1 \leqslant i \leqslant s$, $\delta \in (\hat{L}_i)_{\mathrm{lds}}$ lies in the discrete series, $w_{\delta,1}$ is the longest element of $Z_{W_\delta}(R_\delta)$, and $w_{\delta,2}$ is the longest element of W_δ^0. Suppose further that $\phi \in \mathcal{PW}(\mathfrak{a}_{L_i}^{w_{\delta,2}})_N$ is invariant under

$$\{w \in W_\delta : w\mathfrak{a}_{L_i}^{w_{\delta,2}} \subset \mathfrak{a}_{L_i}^{w_{\delta,2}}\}$$

and vanishes on $(\mathfrak{a}_{L_i,\mathbb{C}}^{w_{\delta,2}})^* \cap (\mathfrak{a}_{L_i,\mathbb{C}}^{w_{\delta,1}})^*$. Then ϕ extends to a W_δ-invariant function in $\mathcal{PW}(\mathfrak{a}_{L_i})_N$ which vanishes on $(\mathfrak{a}_{L_i,\mathbb{C}}^{w_{\delta,1}})^*$.

PROPOSITION 4. *Suppose n is even, Conjecture 1 holds, and $\{F_i^1\}$ is compatible with $\{F_i^2\}$. Then there exists a function $f \in C_c^\infty(G,K)_N$ such that*

$$\mathrm{tr}(\mathrm{ind}_{P_{L_i}}^G(\delta \otimes e^\nu)(f)T_k^{\sigma_0}(\delta,\nu)) = F_i^k(\delta,\nu)$$

for all $\delta \in (\hat{L}_i)_{\mathrm{lds}}$, $\nu \in (\mathfrak{a}_{L_i,\mathbb{C}}^{w_{\delta,k}})^*$, $1 \leqslant i \leqslant s$, and $k=1,2$.

Sketch of proof. The proof proceeds inductively as in [CD84, § 2.3]. The only obstacle is to find, for a given $\delta \in (\hat{L}_j)_{\mathrm{lds}}$, a function $h \in C_c^\infty(G,K)_N$ such that

$$\mathrm{tr}(\mathrm{ind}_{P_{L_j}}^G(\delta_0 \otimes e^\nu)(h)T_k^{\sigma_0}(\delta,\nu)) = F_j^k(\delta,\nu), \quad \nu \in (\mathfrak{a}_{L_j,\mathbb{C}}^{w_{\delta,k}})^*, \ k=1,2. \tag{11}$$

We can obtain h in the following manner. According to [CD90, Proposition 1], for each minimal K type μ of $\mathrm{ind}_{P_{L_j}}^G(\delta \otimes e^0)$ there exists a function $f_{k,\mu} \in C_c^\infty(G,K)_N$ such that

$$\mathrm{tr}\left(\mathrm{ind}_{P_{L_j}}^G(\delta \otimes e^\nu)\left(\sum_\mu f_{k,\mu}\right)T_k^{\sigma_0}(\delta,\nu)\right) = F_j^k(\delta,\nu), \quad \nu \in (\mathfrak{a}_{L_j,\mathbb{C}}^{w_{\delta,k}})^*.$$

The compatibility of $\{F_i^1\}$ with $\{F_i^2\}$ then implies that

$$\phi_\mu(\nu) = \mathrm{tr}(\mathrm{ind}_{P_{L_j}}^G(\delta \otimes e^\nu)(f_{2,\mu})) - \mathrm{tr}(\mathrm{ind}_{P_{L_j}}^G(\delta \otimes e^\nu)(f_{1,\mu})), \quad \nu \in (\mathfrak{a}_{L_j,\mathbb{C}}^{w_{\delta_0,2}})^*$$

vanishes on $(\mathfrak{a}_{L_i,\mathbb{C}}^{w_{\delta_0,1}})^* \cap (\mathfrak{a}_{L_i,\mathbb{C}}^{w_{\delta_0,2}})^*$. We may therefore apply Conjecture 1 to extend ϕ_μ to a W_δ-invariant function in $\mathcal{PW}(\mathfrak{a}_{L_j})_N$ which vanishes on $(\mathfrak{a}_{L_i,\mathbb{C}}^{w_{\delta_0,1}})^*$. By [CD84, Proposition 1] and [Cow86], there exists a function $h_\mu \in C_c^\infty(G,K)_N$ of type (μ,μ) such that

$$\mathrm{tr}(\mathrm{ind}_{P_{L_j}}^G(\delta \otimes e^\nu)(h_\mu)) = \phi_\mu(\nu), \quad \nu \in (\mathfrak{a}_{L_i,\mathbb{C}}^{w_{\delta_0,2}})^*.$$

Equation (11) is satisfied for $h = \sum_\mu h_\mu + f_{1,\mu}$. ☐

9. Applications

Suppose $y \in G$. Obviously, $\sigma_y\sigma_0\sigma_y^{-1}$ is an involution of G so we should have a $\sigma_y\sigma\sigma_y^{-1}$-twisted trace Paley–Wiener theorem. It is easily verified that for $\delta \in (\hat{L}_i)_{\mathrm{lds}}$ and $\nu \in \mathfrak{a}_{L_i,\mathbb{C}}^*$ the representation $\mathrm{ind}_{P_{L_i}}^G(\delta \otimes e^\nu)$ is $\sigma_y\sigma_0\sigma_y^{-1}$-stable if and only if it is σ_0-stable. If this is the case then

$$T^{\sigma_y\sigma_0\sigma_y^{-1}}(\delta,\nu) = \mathrm{ind}_{P_{L_i}}^G(\delta \otimes e^\nu)(y^{-1}) \cdot T^{\sigma_0}(\delta,\nu) \cdot \mathrm{ind}_{P_{L_i}}^G(\delta \otimes e^\nu)(y)$$

[1]We have proven a version of this conjecture in which the support of the extension is weakened to $N+\epsilon$, $\epsilon>0$.

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intertwines $\mathrm{ind}_{P_{L_i}}^{G}(\delta \otimes e^{\nu})$ with its $\sigma_y\sigma_0\sigma_y^{-1}$-conjugate. Furthermore, since taking the trace is invariant under conjugation, we have

$$\mathrm{tr}(\mathrm{ind}_{P_{L_i}}^{G}(\delta \otimes e^{\nu})(f)T^{\sigma}(\delta,\nu)) = \mathrm{tr}(\mathrm{ind}_{P_{L_i}}^{G}(\delta \otimes e^{\nu})(f^{y^{-1}})T^{\sigma_y\sigma\sigma_y^{-1}}(\delta,\nu)),$$

where

$$f^{y}(x) = f(y^{-1}xy), \quad x \in G,$$

and $f \in C_c^{\infty}(G,K)$. This equation shows that a $\sigma_y\sigma\sigma_y^{-1}$-twisted trace Paley–Wiener theorem can be derived from a σ_0-twisted trace Paley–Wiener theorem of §§ 7 and 8 simply by replacing $C_c^{\infty}(G,K)_N$ with $C_c^{\infty}(G,yKy^{-1})_N$.

Let us now consider σ_0 as an involution of $\mathrm{GL}(n,\mathbb{R})$. As the differential of σ_0 sends λ to $-\lambda$ and $g\lambda = \lambda$ for all $g \in \mathrm{GL}(n,\mathbb{R})$ and $\lambda \in \mathfrak{a}_G^*$, Langlands' classification tells us that the σ_0-stable representations of $\mathrm{GL}(n,\mathbb{R})$ are the σ_0-stable representations of $\mathrm{SL}^{\pm}(n,\mathbb{R})$ twisted by the trivial character of $\mathfrak{a}_G^*$. We therefore obtain a σ_0-twisted trace Paley–Wiener theorem for $\mathrm{GL}(n,\mathbb{R})$ if we have one for $\mathrm{SL}^{\pm}(n,\mathbb{R})$. With regard to the latter case, it is well-known that the R group of any discrete series representation of a Levi subgroup of $\mathrm{SL}^{\pm}(n,\mathbb{R})$ is trivial. Consequently, if we adjust the arguments of §§ 4–7 by merely ignoring any reference to R groups, we obtain the unique σ_0-twisted trace Paley–Wiener theorem for $\mathrm{SL}^{\pm}(n,\mathbb{R})$.

To summarize, we now have $\sigma_y\sigma_0\sigma_y^{-1}$-twisted trace Paley–Wiener theorems on $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}^{\pm}(n,\mathbb{R})$, and $\mathrm{GL}(n,\mathbb{R})$. Our remaining applications are twisted versions of the application given in [CD90, § 5.1] in the cases that the real reductive group $\mathbf{G}(\mathbb{R})$ given there is equal to one of $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}^{\pm}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{R})$. These applications are important for the Arthur–Selberg trace formula [Art88, Proposition 1.1]. We shall give a thorough treatment of these applications only for the σ_0-twisted case of $\mathrm{SL}(n,\mathbb{R})$. The cases in which the underlying group is $\mathrm{SL}^{\pm}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{R})$ are simpler than the σ_0-twisted case of $\mathrm{SL}(n,\mathbb{R})$ and do not depend on Conjecture 1. Conjecture 1 is also unnecessary in the σ_0-twisted case of $\mathrm{SL}(n,\mathbb{R})$ if n is odd.

Suppose that $P = MU_M$ is a standard parabolic subgroup of G. Its normalizer $\tilde{P}$ in $G \rtimes \langle\sigma_0\rangle$ is readily computed to be P if the block sizes $n_1,\dots,n_\ell$ of M do not satisfy

$$n_j = n_{\ell+1-j}, \quad 1 \leqslant j \leqslant \ell. \tag{12}$$

If the block sizes do satisfy (12) then $\tilde{P}$ is the disjoint union of P and $w_0P \rtimes \sigma_0$. Similarly the normalizer $\tilde{M}$ of M in $G \rtimes \langle\sigma_0\rangle$ is M, if (12) does not hold, and the disjoint union of M with $w_0M \rtimes \sigma_0$ otherwise. In [Art89, § 1], Arthur defines a *Levi subset* of $G \rtimes \sigma_0$ to be the intersection of $G \rtimes \sigma_0$ with $\tilde{M} \cap \tilde{P}$. In consequence, the Levi subsets of $G \rtimes \sigma_0$ have the form $w_0M \rtimes \sigma_0$, where $M \supset M_0$ is a Levi subgroup of G whose block sizes satisfy (12).

Suppose M satisfies (12). Then there is an obvious group isomorphism,

$$\tilde{M} = M \cup (w_0M \rtimes \sigma_0) \cong M \rtimes \langle\sigma_{w_0}\sigma_0\rangle. \tag{13}$$

We shall identify $\tilde{M}$ with the semidirect product on the right. Arthur defines $\mathfrak{a}_{M\rtimes\sigma_{w_0}\sigma_0}$ as $\mathrm{Hom}(X(M),\mathbb{R})$, where $X(M)$ is the group of rational characters of $\tilde{M}$. We compute $\mathfrak{a}_{M\rtimes\sigma_{w_0}\sigma_0}$ to be isomorphic to be the subspace

$$\{X \in \mathfrak{a}_M : \mathrm{Ad}(w_0)(X) = -X\} = \mathfrak{a}_M^{w_0}.$$

Given a representation τ of $\tilde{M}$ and $\lambda \in (\mathfrak{a}_{M,\mathbb{C}}^{w_0})^*$ define τ_λ by

$$\tau_\lambda(x) = \tau(x)e^{\lambda(\log(a))}, \quad x \in \tilde{M},$$

where a is the projection of x onto A_M.

Let $\Pi_{\mathrm{temp}}(G \rtimes \sigma_0)$ be the set of (equivalence classes of) irreducible tempered representations π of $G \rtimes \langle\sigma_0\rangle$ such that the restriction of π to G is irreducible. Given a complex-valued function ϕ on $\Pi_{\mathrm{temp}}(G \rtimes \sigma_0)$, we denote its extension to the free $\mathbb{Z}$-module generated by $\Pi_{\mathrm{temp}}(G \rtimes \sigma_0)$ as $\tilde{\phi}$.

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Suppose Γ is a finite set of (equivalence classes of) irreducible representations of K and N is a positive real number. In keeping with [Art89, § 11], we define $\mathcal{I}_N(G \rtimes \sigma_0)_\Gamma$ to be the space of complex-valued functions ϕ on $\Pi_{\mathrm{temp}}(G \rtimes \sigma_0)$ which satisfy the following three properties.

1) Suppose sgn is the non-trivial character of $\langle \sigma_0 \rangle$. Then
$$\phi(\pi \otimes \mathrm{sgn}) = -\phi(\pi), \quad \pi \in \Pi_{\mathrm{temp}}(G \rtimes \sigma_0).$$
2) Suppose that the restriction of $\pi \in \Pi_{\mathrm{temp}}(G \rtimes \sigma_0)$ to K does not contain any representation of Γ. Then $\phi(\pi) = 0$.
3) Suppose that $w_0 M \rtimes \sigma_0$ is a Levi subset $G \rtimes \sigma_0$ and that τ is an irreducible tempered representation of $\tilde{M}$ which remains irreducible when restricted to M. Then the integral
$$\phi(\tau, X) = \int_{i(\mathfrak{a}_M^{w_0})^*} \tilde{\phi}(\mathrm{ind}_{\tilde{P}}^{G \rtimes \langle \sigma_0 \rangle}(\tau_\lambda)) e^{-\lambda(X)}\, d\lambda, \quad X \in \mathfrak{a}_M^{w_0}$$
converges to a smooth function of X which has support in the closed ball of radius N centered about the origin in $\mathfrak{a}_M^{w_0}$.

Define $C_c^\infty(G,K)_{N,\Gamma}$ to be the subspace of functions of $C_c^\infty(G,K)_N$ which transform according to representations occurring in Γ under the bilateral action of K.

Theorem 2. *Suppose Conjecture 1 is true if n is even. Suppose $f \in C_c^\infty(G,K)_{N,\Gamma}$ and*
$$\phi(f)(\pi) = \mathrm{tr}\left(\int_G f(x)\pi(x,\sigma_0)\, dx\right), \quad \pi \in \Pi_{\mathrm{temp}}(G \rtimes \sigma_0).$$
Then the map given by $f \mapsto \phi(f)$ is surjective onto $\mathcal{I}_N(G \rtimes \sigma_0)_\Gamma$.

Proof. Using the notation of § 7, suppose that $1 \leqslant i \leqslant s$, $0 \leqslant t \leqslant \lfloor n/2 \rfloor$, $L_i \in \mathcal{L}(t)$, $\delta \in (\hat{L}_i)^1_{\mathrm{lds},t}$, w_t is the longest element in $Z_{W_\delta}(R_\delta)$, and $\nu \in i(\mathfrak{a}_{L_i}^{w_t})^*$. Then $\mathrm{ind}_{P_{L_i}}^G(\delta \otimes e^v)$ is σ_0-stable and decomposes as a finite sum,
$$\mathrm{ind}_{P_{L_i}}^G(\delta \otimes e^v) = \pi_1^0 \oplus \cdots \oplus \pi_m^0,$$
of irreducible tempered representations of G. The corresponding intertwining operator, $T_1^{\sigma_0}(\delta,\nu)$, can be used to define representations $\pi_1, \ldots, \pi_m \in \Pi_{\mathrm{temp}}(G \rtimes \sigma_0)$ by setting T_j equal to the restriction of $T_1^{\sigma_0}(\delta,\nu)$ to the space of π_j^0 and
$$\pi_j(x,\sigma_0) = \pi_j^0(x)T_j, \quad x \in G.$$
We define
$$F_{i,t}^1(\delta,\nu) = \phi(\pi_1) + \cdots + \phi(\pi_m).$$
This procedure can be repeated with W_δ^0 in place of $Z_{W_\delta}(R_\delta)$, and $T_2(\delta,\nu)$ in place of $T_1(\delta,\nu)$ to define
$$F_{i,t}^2(\delta,\nu) = \phi(\pi_1) + \cdots + \phi(\pi_m), \quad \delta \in (\hat{L}_i)^2_{\mathrm{lds},t},\ \nu \in i(\mathfrak{a}_{L_i}^{w_t})^*$$
(we apologize for the use of i as an index as well as the customary imaginary number). Recall the Levi subgroup $M_{L_i,t}$ defined in § 4. By construction, the block sizes of $M_{L_i,t}$ satisfy (12) and $\mathfrak{a}_{M_{L_i,t}}^{w_0} = \mathfrak{a}_{L_i}^{w_t}$. In addition, the representations
$$\mathrm{ind}_{P_{L_i} \cap M_{L_i,t}}^{M_{L_i,t}}(\delta \otimes e^\nu) = (\mathrm{ind}_{P_{L_i} \cap M_{L_i,t}^1}^{M_{L_i,t}^1} \delta) \otimes e^\nu, \quad \delta \in (\hat{L}_i)^k_{\mathrm{lds},t}, \quad \nu \in i(\mathfrak{a}_{L_i}^{w_t})^*, \quad k = 1,2,$$
are irreducible, as their R groups are trivial. Using the ideas of § 4, it is simple to show that these operators are $\sigma_{w_0}\sigma_0$-stable. Therefore, for every $\nu \in i(\mathfrak{a}_{L_i}^{w_t})^*$ we may define a representation τ_ν of

the Levi subset $\tilde{M} = \tilde{M}_{L_i,t}$ such that its restriction to $M_{L_i,t}$ is equal to $\mathrm{ind}_{P_{L_i} \cap M^1_{L_i,t}}^{M^1_{L_i}}(\delta \otimes e^{\nu})$ and

$$\tilde{\phi}(\mathrm{ind}_{\tilde{P}}^{G \rtimes \sigma_0}(\tau_{\nu})) = F^k_{i,t}(\delta, \nu), \quad k = 1, 2.$$

Property 3 of ϕ implies that $F^1_{i,t}(\delta, \cdot)$ and $F^2_{i,t}(\delta, \cdot)$ extend to functions of complex variables in $\mathcal{PW}(\mathfrak{a}^{w_t}_{L_i})_N$.

Retaining the same notation for these extensions, we now have functions

$$F^k_{i,t} : (\hat{L}_i)^k_{\mathrm{lds},t} \times (\mathfrak{a}^{w_t}_{L_i,\mathbb{C}})^* \to \mathbb{C}, \quad k = 1, 2$$

for all $1 \leqslant i \leqslant s$ such that $L_i \in \mathcal{L}(t)$, which satisfy condition b(2) of Theorem 1. We wish to show that these functions satisfy the other conditions of Theorem 1(b). In the proof of Theorem 1 we appeal to the result of Harish–Chandra which tells us that there are only finitely many inequivalent irreducible admissible representations of L_i containing a fixed $(K \cap L_i)$-type. This result and property 2 of ϕ imply that our functions satisfy the finite support condition of Theorem 1b(1). The remaining two conditions, Theorem 1b(3) and (4), are easily seen to be satisfied from the definitions of $T^{\sigma_0}_k(\delta, \nu)$, as these families of operators have been defined to be compatible under conjugation by permutation matrices (cf. Theorem 1b(3)), and compatible under affiliation (cf. Theorem 1b(4)).

As we know from § 5, for any $k = 1, 2$ and $1 \leqslant j \leqslant s$, every $\delta \in (\hat{L}_j)_{\mathrm{lds}}$ is conjugate to a representation in $(\hat{L}_i)^k_{\mathrm{lds},t}$, for some $L_i \in \mathcal{L}(t)$. We can therefore define functions

$$F^k_j : (\hat{L}_j)_{\mathrm{lds}} \times (\mathfrak{a}^{w_{\delta,k}}_{L_j,\mathbb{C}})^* \to \mathbb{C}, \quad 1 \leqslant j \leqslant s, \ k = 1, 2,$$

satisfying the conditions of Corollary 1(b), from the functions of $\{F^k_{i,t}\}$ in an obvious manner. By Proposition 4, the theorem is proven if we show that $\{F^1_j\}$ is compatible with $\{F^2_j\}$. Suppose, therefore, that the R group of $\delta \in (\hat{L}_j)_{\mathrm{lds}}$ is trivial. This implies that for any $\nu \in i(\mathfrak{a}^{w_{\delta,1}}_{L_j})^*$, the induced representation

$$\pi^0 = \mathrm{ind}^G_{P_{L_j}}(\delta \otimes e^{\nu})$$

is an irreducible tempered representation. Combined with each of the intertwining operators, $T^{\sigma_0}_1(\delta, \nu)$ and $T^{\sigma_0}_2(\delta, \nu)$, the representation π^0 determines respective representations π_1 and π_2 in $\Pi_{\mathrm{temp}}(G \rtimes \sigma_0)$. By definition, $F^k_j(\delta, \nu) = \phi(\pi_k)$ for $k = 1, 2$. If $T^{\sigma_0}_1(\delta, \nu) = T^{\sigma_0}_2(\delta, \nu)$ then, by definition (cf. § 7), $c_{\delta} = 1$ and

$$F^2_j(\delta, \nu) = \phi(\pi_2) = \phi(\pi_1) = c_{\delta} F^1_j(\delta, \nu).$$

Otherwise, $\pi_2 = \pi_1 \otimes \mathrm{sgn}$, $c_{\delta} = -1$, and, by property 1 of ϕ, we have

$$F^2_j(\delta, \nu) = \phi(\pi_2) = \phi(\pi_1 \otimes \mathrm{sgn}) = -\phi(\pi_1) = c_{\delta} F^1(\delta, \nu).$$

Hence, $\{F^1_j\}$ is compatible with $\{F^2_j\}$ and the theorem is complete. □

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