On the Theory of Contours, and its Applications in Physical Science.

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PART II.

1. In the first part of this paper we have considered merely the contours of curves, that is, contour points, and the method of obtaining the various physical diagrams. In this part we shall consider chiefly the contours of surfaces; that is, contour lines.

If any curve be cut by planes parallel to that of (x, y) and if the various points of intersection be projected on any one of these planes, say $z=0$, the contour points so obtained will evidently lie on a definite line, and the line will be more accurately indicated in proportion as the number of intersecting planes is greater and their mutual distance is less. It will be given without any break in continuity by projecting every point of the curve upon the plane $z=0$. But such a line may be regarded as the intersection, by the plane $z=0$, (see fig. 48) of a cylindrical surface whose generating lines are parallel to the z-axis and are drawn from the given curve to meet that plane. We have here then the intersection of a given surface by a surface over which z is constant. But this satisfies our definition of a contour line. This case of a cylindrical surface supplies the simplest system of contour lines by giving z different values. The contours are all superposed in the diagram, but are not in general conterminous. The only case in which they would be conterminous is that in which the same values of the x and y co-ordinates of a point on the curve correspond to different values of the z-co-ordinate.

2. In the case of a non-cylindrical surface, no part of the contours will be superposable in general. The contours of a sphere, for example, are concentric circles. And, just as in the case of contour points the steepness of slope of the original curve is indicated by the closeness of the contour points on the x-axis for equal increments of y, so in the case of contour lines the steepness of slope of the surface is indicated by the closeness of the contour lines for equal increments of z (see fig. 49). The contours are closer when their radii are large.

Again, the contours of a right circular cone are also concentric circles. These circles however are all at equal distances apart for equal increments of z.

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In my paper on the Representation of the Physical Properties of substances by means of Surfaces, read before this Society two sessions ago, I have collected together the methods of deducing the general properties of a surface from its contours. In this connection Maxwell's and Cayley's investigations referred to in that paper should be consulted. The differences between Maxwell's diagram of the contours of an inland basin and that of the contours of an insular highland, as mapped in our Proceedings for 1883-84, should be noted.

3. We shall now consider special surfaces the contours of which may be used to indicate certain physical properties.

We have the equation

$$g = 4\pi^2 l / t^2$$

connecting l the length of a pendulum, t its time of oscillation, and g the value of gravity. If t^2 and l be measured along two rectangular horizontal axes, and g be measured vertically from the same origin, we obtain a surface the contours of which give the values of t^2 and l for any given value of g. This surface is obviously that which Maxwell has called a *skew screw surface* (fig. 50), and its contours are straight lines through the origin variously inclined to the axes.

The intrinsic equation of the circle is

$$s = a\phi$$

where a is the radius, and ϕ is the angle between the radius-vector and the initial line. Hence the intrinsic equation of one involute is

$$s' = \frac{a}{2}\phi^2.$$

This involute is the one which meets the circle at the position from which ϕ is reckoned, and s' is measured along it from this point. If we consider a to be the mass of a moving body, and ϕ to be its speed, s and s' are respectively its momentum and kinetic energy. Fig. 51 represents two circles with their involutes satisfying the above conditions. The curves in that figure may be regarded as the contours of a right circular cone and an associated surface so formed that its intersection by any plane parallel to that of the diagram is an involute of the circle in which the cone is cut by the same plane.

The acceleration and speed of a body falling under the action of gravity, and the space passed over by it, are given by the known equations

$$a = g.$$

Hence these various quantities can be represented also by the contours of the surfaces just considered. Only in this case s and s' must not be measured from the point for which ϕ is zero.

Again, in the case of a thermo-electric circuit composed of two dissimilar metals, the electromotive force, E, is given in terms of t, the difference of temperature of the two junctions, by means of the formula

$$E = a + bt + ct^2.$$

Also the thermo-electric power e is given by the equation

$$e = \frac{dE}{dt} = b + 2ct.$$

Hence the electromotive force and thermo-electric power are representable by means of the same surfaces.

4. As an additional example the contours of the surface showing the relation of the pressure, volume, and temperature of a substance may be taken. These are shown for water-substance in our Proceedings for 1883-84. The contours are isothermals, when pressure and volume are the co-ordinate quantities in the diagram. It is only when the temperature considered happens to be one corresponding to a contour in the diagram that the relation of pressure and volume can be found. This fact, that only a finite number of curves corresponding to different values of the quantity which is constant along each curve, but varies from one to another, can be mapped, constitutes the great defect of the method of contours. But it can be entirely got rid of by using trilinear co-ordinates, and in addition the variation of a fourth quantity can be shown. For a perfect gas we have the equation

$$pv = ct$$

where p, v, and t represent respectively the pressure, volume, and temperature, and c is a quantity which depends on the nature of the gas and varies from one to another. Fig. 52 shows the contours for different values of c the triangle being equilateral for convenience. Temperature is measured by the distance from the vertical side of the triangle of reference, and pressure and volume from the inclined sides. The equation shows that the curves are hyperbolas with vertical and horizontal axes. No part of the hyperbolas outside the triangle of reference has any physical meaning, as then either the pressure, volume, or temperature, or any two, or all of them, would be negative. This is impossible of course in a perfect gas.

It is evident from this figure that pressure, volume, and temperature for any one gas are continuously represented.

5. Figure 52 evidently represents the contours of a surface by planes parallel to the plane of the diagram which may be looked upon as that corresponding to zero value of c. When c is zero, the hyperbola becomes the straight lines coinciding with the sides AB, AC of the triangle of reference. When c is infinite, the side BC is part of the corresponding hyperbola. All points of vertical lines through B and C lie upon the surface. These lines bound the parts of one sheet of the surface which correspond to real physical states of the gas, from those which do not. Outside the triangle the surface is evidently an overhanging precipice.

If c and t have each a definite value, the point on the contour is undetermined. Let P (fig. 53) be the point giving the proper ratios of p, v, and t. Draw PM, PN, parallel to the sides of the triangle of reference. Since the asymptotes of the hyperbola are parallel to the sides AB, AC, it follows that that part of the tangent at P intercepted by the sides of the triangle is bisected at the point of contact. Therefore $AM = MQ$, and $AN = NR$. Now the compressibility k of a gas is given by the ratio dv/vdp. But $dv/dp = MQ/MP = NP/MP = v/p$ where v and p are, as formerly stated, the perpendiculars from P on the sides AB, AC respectively. Hence $k = 1/p$, that is, the compressibility of a perfect gas is the reciprocal of the pressure, which is a known result. Similarly, it can be shown that the expansibility is inversely proportional to the absolute temperature.

The work done during isothermal expansion can also be found from the diagram. The position of the point P gives the ratio of p, v, and t, but since t has a known constant value, the actual values of p and v are also known. Hence PN $(= p \operatorname{cosec} BAC)$ is a known known function of v. If P move to P′ the area $PNN'P' = \int PN dv$ $= \operatorname{cosec} BAC \int p dv$ is a given multiple of the work done.

6. The applicability of the method of contours to other physical problems is evident. Electric stream lines and equipotential lines may be regarded as contours of a surface. And the number of equipotential lines crossing unit length of a stream line may be used to indicate the strength of current. So also air-current lines and isobars, isothermals and flow-lines of heat, &c., are rectangular systems of contours.