

Siegel families with application to class fields

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We investigate certain families of meromorphic Siegel modular functions on which Galois groups act in a natural way. By using Shimura's reciprocity law we construct some algebraic numbers in the ray class fields of CM-fields in terms of special values of functions in these Siegel families.

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1. Introduction

For a positive integer N let \mathfrak{F}_N be the field of meromorphic modular functions of level N (defined on $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$) whose Fourier coefficients belong to the N th cyclotomic field. As is well known, \mathfrak{F}_N is a Galois extension of \mathfrak{F}_1 whose Galois group is isomorphic to $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ (see [11, § 6.1–6.2]). Now, let $N \geq 2$ and consider a set

$$V_N = \{\mathbf{v} \in \mathbb{Q}^2 \mid N \text{ is the smallest positive integer for which } N\mathbf{v} \in \mathbb{Z}^2\}$$

as the index set. We call a family $\{f_{\mathbf{v}}(\tau)\}_{\mathbf{v} \in V_N}$ of functions in \mathfrak{F}_N a *Fricke family* of level N if each $f_{\mathbf{v}}(\tau)$ depends only on $\pm\mathbf{v} \pmod{\mathbb{Z}^2}$ and satisfies

$$f_{\mathbf{v}}(\tau)^\alpha = f_{\alpha^T \mathbf{v}}(\tau) \quad (\alpha \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}),$$

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where α^T means the transpose of α . For example, Siegel functions of one variable form such a Fricke family of level N [8, ch. 2, proposition 1.3] (see also [4] or [6]).

Let K be an imaginary quadratic field with the ring of integers \mathcal{O}_K , and let \mathfrak{f} be a proper non-trivial ideal of \mathcal{O}_K . We denote by $\text{Cl}(\mathfrak{f})$ and $K_{\mathfrak{f}}$ the ray class group modulo \mathfrak{f} and its corresponding ray class field modulo \mathfrak{f} , respectively. If $\{f_{\mathfrak{v}}(\tau)\}_{\mathfrak{v}}$ is a Fricke family of level N in which every $f_{\mathfrak{v}}(\tau)$ is holomorphic on \mathbb{H} , then we can assign to each ray class $\mathcal{C} \in \text{Cl}(\mathfrak{f})$ an algebraic number $f_{\mathfrak{f}}(\mathcal{C})$ as a special value of a function in $\{f_{\mathfrak{v}}(\tau)\}_{\mathfrak{v}}$. Furthermore, we attain by Shimura’s reciprocity law that $f_{\mathfrak{f}}(\mathcal{C})$ belongs to $K_{\mathfrak{f}}$ and satisfies

$$f_{\mathfrak{f}}(\mathcal{C})^{\sigma_{\mathfrak{f}}(\mathcal{D})} = f_{\mathfrak{f}}(\mathcal{C}\mathcal{D}) \quad (\mathcal{D} \in \text{Cl}(\mathfrak{f})),$$

where $\sigma_{\mathfrak{f}}$ is the Artin reciprocity map for \mathfrak{f} (see [8, ch. 11, theorem 1.1]).

In this paper we shall define a Siegel family $\{h_M(Z)\}_M$ of level N consisting of meromorphic Siegel modular functions of (higher) genus g and level N , which is a generalization of a Fricke family of level N in the case when $g = 1$ (definition 3.1). It turns out that every Siegel family of level N is induced from a meromorphic Siegel modular function for the congruence subgroup $\Gamma^1(N)$ with rational Fourier coefficients (theorem 3.5).

Let K be a CM-field and let $\mathfrak{f} = N\mathcal{O}_K$. Given a Siegel family $\{h_M(Z)\}_M$ of level N , we shall introduce a number $h_{\mathfrak{f}}(\mathcal{C})$ as a special value of a function in $\{h_M(Z)\}_M$ for each ray class $\mathcal{C} \in \text{Cl}(\mathfrak{f})$ (definition 5.4). Under certain assumptions on K (assumption 5.1) we shall prove that if $h_{\mathfrak{f}}(\mathcal{C})$ is finite, then it lies in the ray class field $K_{\mathfrak{f}}$ whose Galois conjugates are of the same form (theorem 7.2 and corollary 7.3). To this end, we assign a principally polarized abelian variety to each non-trivial ideal of \mathcal{O}_K , and apply Shimura’s reciprocity law to $h_{\mathfrak{f}}(\mathcal{C})$.

On the other hand, we note that there is a remarkable paper by Grant [2] in which he generalized a classical formula of Eisenstein and obtained classes of S -units by evaluating abelian functions at the intersections of divisors on the Jacobian of the curve $y^2 = x^5 + \frac{1}{4}$. We hope that our invariant $h_{\mathfrak{f}}(\mathcal{C})$ obtained from a Siegel family in theorem 4.3 will contribute further towards finding a higher-dimensional analogue of an elliptic unit.

2. Actions on Siegel modular functions

First, we shall describe the Galois group between fields of meromorphic Siegel modular functions in a concrete way.

Let g be a positive integer and let

$$\eta_g = \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}.$$

For every commutative ring R with unity we define

$$\begin{aligned} \text{GSp}_{2g}(R) &= \{\alpha \in \text{GL}_{2g}(R) \mid \alpha^T \eta_g \alpha = \nu(\alpha) \eta_g \text{ with } \nu(\alpha) \in R^\times\}, \\ \text{Sp}_{2g}(R) &= \{\alpha \in \text{GSp}_{2g}(R) \mid \nu(\alpha) = 1\}. \end{aligned}$$

Let

$$G = \text{GSp}_{2g}(\mathbb{Q})$$

and let $G_{\mathbb{A}}$ be the adelization of G , let G_0 be its non-Archimedean part and let G_{∞} be its Archimedean part. One can extend the multiplier map $\nu: G \rightarrow \mathbb{Q}^{\times}$ continuously to the map $\nu: G_{\mathbb{A}} \rightarrow \mathbb{Q}_{\mathbb{A}}^{\times}$ and set

$$G_{\infty+} = \{\alpha \in G_{\infty} \mid \nu(\alpha) > 0\}, \quad G_{\mathbb{A}+} = G_0 G_{\infty+}, \quad G_+ = G \cap G_{\mathbb{A}+}.$$

Furthermore, let

$$\Delta = \left\{ \begin{bmatrix} I_g & O_g \\ O_g & sI_g \end{bmatrix} \mid s \in \prod_p \mathbb{Z}_p^{\times} \right\},$$

$$U_1 = \prod_p \text{GSp}_{2g}(\mathbb{Z}_p) \times G_{\infty+},$$

$$U_N = \{x \in U_1 \mid x_p \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z}_p)} \text{ for all rational primes } p\}$$

for every positive integer N . Then we have

$$U_N \trianglelefteq U_1 \leq G_{\mathbb{A}+} \quad \text{and} \quad G_{\mathbb{A}+} = U_N \Delta G_+$$

(see [13, lemma 8.3(1)]).

Note that the group $G_{\infty+}$ acts on the Siegel upper half-space

$$\mathbb{H}_g = \{Z \in M_g(\mathbb{C}) \mid Z^T = Z, \text{ Im}(Z) \text{ is positive definite}\}$$

by

$$\alpha(Z) = (AZ + B)(CZ + D)^{-1} \quad (\alpha \in G_{\infty+}, Z \in \mathbb{H}_g),$$

where A, B, C, D are $g \times g$ block matrices of α . Let \mathcal{F}_N be the field of meromorphic Siegel modular functions of genus g for the congruence subgroup

$$\Gamma(N) = \{\gamma \in \text{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z})}\}$$

of the symplectic group $\text{Sp}_{2g}(\mathbb{Z})$ whose Fourier coefficients belong to the N th cyclotomic field $\mathbb{Q}(\zeta_N)$ with $\zeta_N = e^{2\pi i/N}$. That is, if $f \in \mathcal{F}_N$, then

$$f(Z) = \frac{\sum_h c(h)e(\text{tr}(hZ)/N)}{\sum_h d(h)e(\text{tr}(hZ)/N)} \quad \text{for some } c(h), d(h) \in \mathbb{Q}(\zeta_N),$$

where the denominator and numerator of f are Siegel modular forms of the same weight, h runs over all $g \times g$ positive semi-definite symmetric matrices over half integers with integral diagonal entries, and $e(w) = e^{2\pi iw}$ for $w \in \mathbb{C}$ [5, § 4, theorem 1]. Let

$$\mathcal{F} = \bigcup_{N=1}^{\infty} \mathcal{F}_N.$$

PROPOSITION 2.1. *There exists a homomorphism $\tau: G_{\mathbb{A}+} \rightarrow \text{Aut}(\mathcal{F})$ satisfying the following properties. Let*

$$f(Z) = \frac{\sum_h c(h)e(\text{tr}(hZ)/N)}{\sum_h d(h)e(\text{tr}(hZ)/N)} \in \mathcal{F}_N.$$

(i) *If $\alpha \in G_+ = \{\alpha \in G \mid \nu(\alpha) > 0\}$, then*

$$f^{\tau(\alpha)} = f \circ \alpha.$$

(ii) If

$$\beta = \begin{bmatrix} I_g & O_g \\ O_g & sI_g \end{bmatrix} \in \Delta$$

and t is a positive integer such that $t \equiv s_p \pmod{N\mathbb{Z}_p}$ for all rational primes p , then

$$f^{\tau(\beta)} = \frac{\sum_h c(h)^\sigma e(\text{tr}(hZ)/N)}{\sum_h d(h)^\sigma e(\text{tr}(hZ)/N)},$$

where σ is the automorphism of $\mathbb{Q}(\zeta_N)$ given by $\zeta_N^\sigma = \zeta_N^t$.

(iii) For every positive integer N we have

$$\mathcal{F}_N = \{f \in \mathcal{F} \mid f^{\tau(x)} = f \text{ for all } x \in U_N\}.$$

(iv) We have $\ker(\tau) = \mathbb{Q}^\times G_{\infty+}$.

Proof. See [13, theorem 8.10]. □

Since

$$U_N(\mathbb{Q}^\times G_{\infty+})/\mathbb{Q}^\times G_{\infty+} \simeq U_N/(U_N \cap \mathbb{Q}^\times G_{\infty+}) \simeq \begin{cases} U_1/\pm G_{\infty+} & \text{if } N = 1, \\ U_N/G_{\infty+} & \text{if } N > 1, \end{cases}$$

we see by proposition 2.1(iii) and (iv) that \mathcal{F}_N is a Galois extension of \mathcal{F}_1 with

$$\text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq U_1/\pm U_N. \tag{2.1}$$

PROPOSITION 2.2. *We have*

$$\text{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}.$$

Proof. Let $\alpha \in U_1$. Take a matrix A in $M_{2g}(\mathbb{Z})$ for which $A \equiv \alpha_p \pmod{N \cdot M_{2g}(\mathbb{Z}_p)}$ for all rational primes p . Define a matrix $\psi(\alpha) \in M_{2g}(\mathbb{Z}/N\mathbb{Z})$ by the image of A under the natural reduction $M_{2g}(\mathbb{Z}) \rightarrow M_{2g}(\mathbb{Z}/N\mathbb{Z})$. Then, by the Chinese remainder theorem, $\psi(\alpha)$ is well defined and independent of the choice of A . Furthermore, let t be an integer relatively prime to N such that $t \equiv \nu(\alpha_p) \pmod{N\mathbb{Z}_p}$ for all rational primes p . We then derive that

$$t\eta_g \equiv \nu(\alpha_p)\eta_g \equiv \alpha_p^T \eta_g \alpha_p \equiv A^T \eta_g A \equiv \psi(\alpha)^T \eta_g \psi(\alpha) \pmod{N \cdot M_{2g}(\mathbb{Z}_p)}$$

for all rational primes p , and hence $\psi(\alpha) \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Thus, we obtain a group homomorphism

$$\psi: U_1 \rightarrow \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}).$$

Let $\beta \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and take a preimage B of β under the natural reduction $M_{2g}(\mathbb{Z}) \rightarrow M_{2g}(\mathbb{Z}/N\mathbb{Z})$. Since $\nu(\beta) \in (\mathbb{Z}/N\mathbb{Z})^\times$ and

$$B^T \eta_g B \equiv \beta^T \eta_g \beta \equiv \nu(\beta)\eta_g \pmod{N \cdot M_{2g}(\mathbb{Z})},$$

B belongs to $\text{GSp}_{2g}(\mathbb{Z}_p)$ for every rational prime p dividing N . Let $\alpha = (\alpha_p)_p$ be the element of $\prod_p \text{GSp}_{2g}(\mathbb{Z}_p)$ given by

$$\alpha_p = \begin{cases} B & \text{if } p|N, \\ I_{2g} & \text{otherwise.} \end{cases}$$

We then see that $\alpha \in U_1$ and $\psi(\alpha) = \beta$. Thus, ψ is surjective.

Clearly, U_N is contained in $\ker(\psi)$. Let $\gamma \in \ker(\psi)$. Since $\gamma_p \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z}_p)}$ for all rational primes p , we get $\gamma \in U_N$, and hence $\ker(\psi) = U_N$. Therefore, ψ induces an isomorphism $U_1/U_N \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, from which we achieve, by (2.1),

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq U_1/\pm U_N \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}.$$

□

REMARK 2.3. We have the decomposition

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq G_N \cdot \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\},$$

where

$$G_N = \left\{ \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \mid \nu \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$

By proposition 2.1 one can describe the action of $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on \mathcal{F}_N as follows.

Let

$$f(Z) = \frac{\sum_h c(h)e(\mathrm{tr}(hZ)/N)}{\sum_h d(h)e(\mathrm{tr}(hZ)/N)} \in \mathcal{F}_N.$$

(i) An element

$$\beta = \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix}$$

of G_N acts on f by

$$f^\beta = \frac{\sum_h c(h)^\sigma e(\mathrm{tr}(hZ)/N)}{\sum_h d(h)^\sigma e(\mathrm{tr}(hZ)/N)},$$

where σ is the automorphism of $\mathbb{Q}(\zeta_N)$ satisfying $\zeta_N^\sigma = \zeta_N^\nu$.

(ii) An element γ of $\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ acts on f by

$$f^\gamma = f \circ \gamma',$$

where γ' is any preimage of γ under the natural reduction

$$\mathrm{Sp}_{2g}(\mathbb{Z}) \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}.$$

3. Siegel families of level N

By making use of the description of $\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$ in § 2 we shall introduce a generalization of a Fricke family in higher dimensional cases.

Let $N \geq 2$. For $\alpha \in M_{2g}(\mathbb{Z})$ we denote by $\tilde{\alpha}$ its reduction modulo N . Define a set

$$\mathcal{V}_N = \left\{ (1/N) \begin{bmatrix} A^\mathrm{T} \\ B^\mathrm{T} \end{bmatrix} \mid \alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z}) \text{ such that } \tilde{\alpha} \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \right\}.$$

Let α, β be elements of $M_{2g}(\mathbb{Z})$ satisfying $\alpha, \beta \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. If M is an element of \mathcal{V}_N induced from α , then it is straightforward that $\beta^\mathrm{T}M$ is also an element of \mathcal{V}_N given by the product $\alpha\beta$.

DEFINITION 3.1. We call a family $\{h_M(Z)\}_{M \in \mathcal{V}_N}$ a *Siegel family* of level N if it satisfies the following:

- (S1) each $h_M(Z)$ belongs to \mathcal{F}_N ;
- (S2) $h_M(Z)$ depends only on $\pm M \pmod{M_{2g \times g}(\mathbb{Z})}$;
- (S3) $h_M(Z)^\sigma = h_{\sigma^T M}(Z)$ for all $\sigma \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq \text{Gal}(\mathcal{F}_N/\mathcal{F}_1)$.

By \mathcal{S}_N we mean the set of such Siegel families of level N .

REMARK 3.2. Let $\{h_M(Z)\}_M \in \mathcal{S}_N$.

- (i) The property (S3) yields a right action of the group $\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on $\{h_M(Z)\}_M$.
- (ii) We let

$$M = (1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix} \in \mathcal{V}_N,$$

and so there is a matrix

$$\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})$$

such that $\tilde{\alpha} \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Considering $\tilde{\alpha}$ as an element of $\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ we obtain

$$h_{(1/N) \begin{bmatrix} I_g \\ O_g \end{bmatrix}}(Z)^{\tilde{\alpha}} = h_{(1/N)\alpha^T \begin{bmatrix} I_g \\ O_g \end{bmatrix}}(Z) = h_M(Z).$$

Thus, the action of $\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on $\{h_M(Z)\}_M$ is transitive.

Let

$$\Gamma^1(N) = \left\{ \gamma \in \text{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix} \pmod{N \cdot M_{2g}(\mathbb{Z})} \right\},$$

and let $\mathcal{F}_N^1(\mathbb{Q})$ be the field of meromorphic Siegel modular functions for $\Gamma^1(N)$ with rational Fourier coefficients.

LEMMA 3.3. *If $\{h_M(Z)\}_M \in \mathcal{S}_N$, then*

$$h_{\begin{bmatrix} (1/N)I_g \\ O_g \end{bmatrix}}(Z) \in \mathcal{F}_N^1(\mathbb{Q}).$$

Proof. For any

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma^1(N)$$

we deduce by (S2) and (S3) that

$$\begin{aligned} h_{\begin{bmatrix} (1/N)I_g \\ O_g \end{bmatrix}}(\gamma(Z)) &= h_{\begin{bmatrix} (1/N)I_g \\ O_g \end{bmatrix}}(Z)^{\tilde{\gamma}} = h_{\gamma^T \begin{bmatrix} (1/N)I_g \\ O_g \end{bmatrix}}(Z) \\ &= h_{(1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix}}(Z) = h_{\begin{bmatrix} (1/N)I_g \\ O_g \end{bmatrix}}(Z) \end{aligned}$$

because

$$A \equiv I_g, \quad B \equiv O_g \pmod{N \cdot M_g(\mathbb{Z})}.$$

Thus, $h\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right](Z)$ is modular for $\Gamma^1(N)$.

For every $\nu \in (\mathbb{Z}/N\mathbb{Z})^\times$ we see by (S2) and (S3) that

$$h\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right](Z) \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} = h\left[\begin{smallmatrix} I_g & O_g \\ O_g & \nu I_g \end{smallmatrix}\right] \begin{bmatrix} (1/N)I_g \\ O_g \end{bmatrix}(Z) = h\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right](Z),$$

which implies that $h\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right](Z)$ has rational Fourier coefficients. This proves the lemma. \square

One can consider \mathcal{S}_N as a field under the binary operations

$$\begin{aligned} \{h_M(Z)\}_M + \{k_M(Z)\}_M &= \{(h_M + k_M)(Z)\}_M, \\ \{h_M(Z)\}_M \cdot \{k_M(Z)\}_M &= \{(h_M k_M)(Z)\}_M. \end{aligned}$$

By lemma 3.3 we get the ring homomorphism

$$\begin{aligned} \phi_N : \mathcal{S}_N &\rightarrow \mathcal{F}_N^1(\mathbb{Q}) \\ \{h_M(Z)\}_M &\mapsto h\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right](Z). \end{aligned}$$

LEMMA 3.4. *If $M \in \mathcal{V}_N$, then there exists*

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})$$

such that $\tilde{\gamma} \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and

$$M = (1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix}.$$

Proof. Let

$$\alpha = \begin{bmatrix} A & B \\ U & V \end{bmatrix} \in M_{2g}(\mathbb{Z})$$

such that $\tilde{\alpha} \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and

$$M = (1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix}.$$

In $M_{2g}(\mathbb{Z}/N\mathbb{Z})$, decompose $\tilde{\alpha}$ as

$$\tilde{\alpha} = \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix} \quad \text{with } \nu = \nu(\tilde{\alpha}) \in (\mathbb{Z}/N\mathbb{Z})^\times$$

so that

$$\begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix}$$

belongs to $\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Since the reduction $\mathrm{Sp}_{2g}(\mathbb{Z}) \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ is surjective (see [10]), we can take $\gamma \in M_{2g}(\mathbb{Z})$ satisfying

$$\tilde{\gamma} = \begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix}.$$

□

THEOREM 3.5. \mathcal{S}_N and $\mathcal{F}_N^1(\mathbb{Q})$ are isomorphic via ϕ_N .

Proof. Since \mathcal{S}_N and $\mathcal{F}_N^1(\mathbb{Q})$ are fields, it suffices to show that ϕ_N is surjective.

Let $h(Z) \in \mathcal{F}_N^1(\mathbb{Q})$. For each $M \in \mathcal{V}_N$, take any

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})$$

such that $\tilde{\gamma} \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and

$$M = (1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix}$$

by using lemma 3.4. We set

$$h_M(Z) = h(Z)^{\tilde{\gamma}}.$$

We claim that $h_M(Z)$ is independent of the choice of γ . Indeed, if

$$\gamma' = \begin{bmatrix} A & B \\ C' & D' \end{bmatrix} \in M_{2g}(\mathbb{Z})$$

such that $\tilde{\gamma}' \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, then we attain in $M_{2g}(\mathbb{Z}/N\mathbb{Z})$ that

$$\tilde{\gamma}'\tilde{\gamma}^{-1} = \begin{bmatrix} A & B \\ C' & D' \end{bmatrix} \begin{bmatrix} D^T & -B^T \\ -C^T & A^T \end{bmatrix} = \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix}$$

by the fact $\tilde{\gamma}, \tilde{\gamma}' \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Let δ be an element of $\mathrm{Sp}_{2g}(\mathbb{Z})$ such that $\tilde{\delta} = \tilde{\gamma}'\tilde{\gamma}^{-1}$. We then achieve

$$h(Z)^{\tilde{\gamma}'} = (h(Z)^{\tilde{\gamma}'\tilde{\gamma}^{-1}})^{\tilde{\gamma}} = h(\delta(Z))^{\tilde{\gamma}} = h(Z)^{\tilde{\gamma}}$$

because $h(Z)$ is modular for $\Gamma^1(N)$ and $\delta \in \Gamma^1(N)$.

Now, for any

$$\sigma = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$$

with $\nu = \nu(\sigma)$ we derive that

$$\begin{aligned} h_M(Z)^\sigma &= h(Z)^{\tilde{\gamma}\sigma} \\ &= h(Z) \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \\ &= h(Z) \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \begin{bmatrix} AP+BR & AQ+BS \\ \nu^{-1}(CP+DR) & \nu^{-1}(CQ+DS) \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 &= h(Z) \left[\begin{array}{cc} AP+BR & AQ+BS \\ \nu^{-1}(CP+DR) & \nu^{-1}(CQ+DS) \end{array} \right] \\
 &\quad \text{since } h(Z) \text{ has rational Fourier coefficients} \\
 &= h \left[\begin{array}{c} (AP+BR)^T \\ (AQ+BS)^T \end{array} \right] (Z) \\
 &= h \left[\begin{array}{cc} P^T & R^T \\ Q^T & S^T \end{array} \right] \left[\begin{array}{c} A^T \\ B^T \end{array} \right] (Z) \\
 &= h_{\sigma^T M}(Z).
 \end{aligned}$$

This shows that the family $\{h_M(Z)\}_M$ belongs to \mathcal{S}_N . Furthermore, since

$$\phi_N(\{h_M(Z)\}_M) = h \left[\begin{array}{c} (1/N)I_g \\ O_g \end{array} \right] (Z) = h(Z) \left[\begin{array}{cc} I_g & O_g \\ O_g & I_g \end{array} \right] = h(Z),$$

ϕ is surjective as desired. □

REMARK 3.6. (i) By proposition 2.2 and remark 2.3 we obtain

$$\text{Gal}(\mathcal{F}_N/\mathcal{F}_N^1(\mathbb{Q})) \simeq G_N \cdot \left\{ \gamma \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \mid \gamma = \pm \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix} \right\}.$$

(ii) Let $\mathcal{F}_{1,N}(\mathbb{Q})$ be the field of meromorphic Siegel modular functions for

$$\Gamma_1(N) = \left\{ \gamma \in \text{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} I_g & * \\ O_g & I_g \end{bmatrix} \pmod{N \cdot M_{2g}(\mathbb{Z})} \right\}$$

with rational Fourier coefficients. If we set

$$\omega = \begin{bmatrix} (1/\sqrt{N})I_g & O_g \\ O_g & \sqrt{N}I_g \end{bmatrix},$$

then we know that $\omega \in \text{Sp}_{2g}(\mathbb{R})$ and

$$\omega \begin{bmatrix} A & B \\ C & D \end{bmatrix} \omega^{-1} = \begin{bmatrix} A & (1/N)B \\ NC & D \end{bmatrix} \quad \text{for all } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}_{2g}(\mathbb{R}).$$

This implies

$$\omega \Gamma^1(N) \omega^{-1} = \Gamma_1(N),$$

and so $\mathcal{F}_{1,N}(\mathbb{Q})$ and $\mathcal{F}_N^1(\mathbb{Q})$ are isomorphic via

$$\begin{aligned}
 \mathcal{F}_{1,N}(\mathbb{Q}) &\rightarrow \mathcal{F}_N^1(\mathbb{Q}) \\
 h(Z) &\mapsto (h \circ \omega)(Z) = h((1/N)Z).
 \end{aligned}$$

4. An example of a Siegel family

In this section, we shall give a concrete example of a Siegel family by means of theta constants.

Let g be a positive integer. For

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_u \\ \mathbf{v}_\ell \end{bmatrix} \in \mathbb{Q}^{2g}$$

with $\mathbf{v}_u, \mathbf{v}_\ell \in \mathbb{Q}^g$, the *theta constant* $\theta_{\mathbf{v}}(Z)$ is given by

$$\theta_{\mathbf{v}}(Z) = \sum_{\mathbf{n} \in \mathbb{Z}^g} e(\frac{1}{2}(\mathbf{n} + \mathbf{v}_u)^T Z (\mathbf{n} + \mathbf{v}_u) + (\mathbf{n} + \mathbf{v}_u)^T \mathbf{v}_\ell) \quad (Z \in \mathbb{H}_g).$$

It was shown by Igusa (see [3, theorem 2]) that $\theta_{\mathbf{v}}(Z)$ is identically zero if and only if every entry of the vector \mathbf{v} is in $(1/2)\mathbb{Z}$ and $e(2\mathbf{v}_u^T \mathbf{v}_\ell) = -1$. Let

$$S_- = \left\{ \mathbf{a} = \begin{bmatrix} \mathbf{a}_u \\ \mathbf{a}_\ell \end{bmatrix} \in \{0, 1/2\}^{2g} \mid e(2\mathbf{a}_u^T \mathbf{a}_\ell) = -1 \right\} \quad \text{and} \quad S_+ = \{0, 1/2\}^{2g} \setminus S_-.$$

Now, let $\mathbf{v} \in \mathbb{Q}^{2g}$ with exact denominator $N \geq 2$. We define

$$\Theta_{\mathbf{v}}(Z) = 2^{4N} e(-2^g N(2^g - 1)(2^g + 1)\mathbf{v}_u^T \mathbf{v}_\ell) \frac{\prod_{\mathbf{a} \in S_-} \theta_{\mathbf{a}-\mathbf{v}}(Z)^{4N(2^g+1)}}{\prod_{\mathbf{b} \in S_+} \theta_{\mathbf{b}}(Z)^{4N(2^g-1)}} \quad (Z \in \mathbb{H}_g)$$

(see [7, definition 4.2]).

PROPOSITION 4.1. *The function $\Theta_{\mathbf{v}}(Z)$ depends only on $\pm \mathbf{v} \pmod{\mathbb{Z}^{2g}}$. Moreover, it belongs to \mathcal{F}_N and satisfies that*

$$\Theta_{\mathbf{v}}(Z)^\sigma = \Theta_{\sigma \tau_{\mathbf{v}}}(Z)$$

for every $\sigma \in \text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq \text{Gal}(\mathcal{F}_N/\mathcal{F}_1)$.

Proof. See [7, lemma 4.4]. □

REMARK 4.2. One can readily verify that if $g \geq 2$, then $\Theta_{\mathbf{v}}(Z)$ is identically zero if and only if $N = 2$.

THEOREM 4.3. *If $\mathbf{r} \in \mathbb{Q}^g$ with exact denominator $N \geq 3$, then $\{\Theta_{M(N\mathbf{r})}\}_{M \in \mathcal{V}_N}$ is a Siegel family of level N .*

Proof. For any $\gamma \in \Gamma^1(N)$ we derive by proposition 4.1 that

$$\Theta_{\begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix}}(\gamma(Z)) = \Theta_{\begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix}}(Z)^{\tilde{\gamma}} = \Theta_{\gamma^T \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix}}(Z) = \Theta_{\begin{bmatrix} I_g & * \\ O_g & I_g \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix}}(Z) = \Theta_{\begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix}}(Z).$$

This shows that $\Theta_{\begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix}}(Z)$ is modular for $\Gamma^1(N)$. Furthermore, for any $\nu \in (\mathbb{Z}/N\mathbb{Z})^\times$, by proposition 4.1 we see that

$$\Theta_{\begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix}}(Z) \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} = \Theta_{\begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix}}(Z) = \Theta_{\begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix}}(Z).$$

Thus, $\Theta_{\begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix}}(Z)$ has rational Fourier coefficients, and hence $\Theta_{\begin{bmatrix} \mathbf{r} \\ \mathbf{0} \end{bmatrix}}(Z)$ belongs to $\mathcal{F}_N^1(\mathbb{Q})$.

For each $M \in \mathcal{V}_N$, we can take an element

$$\gamma_M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

of $M_{2g}(\mathbb{Z})$ such that $\tilde{\gamma}_M \in \text{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and

$$M = (1/N) \begin{bmatrix} A^T \\ B^T \end{bmatrix}$$

by lemma 3.4. Then, by the proof of theorem 3.5, the family $\{\Theta_{[\mathbf{r}]}(Z)^{\tilde{\gamma}_M}\}_{M \in \mathcal{V}_N}$ turns out to be a Siegel family of level N . Lastly, we obtain by proposition 4.1 that

$$\Theta_{[\mathbf{r}]}(Z)^{\tilde{\gamma}_M} = \Theta_{\tilde{\gamma}_M[\mathbf{r}]}(Z) = \Theta_{\begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}[\mathbf{r}]}(Z) = \Theta_{\begin{bmatrix} A^T \\ B^T \end{bmatrix}\mathbf{r}}(Z) = \Theta_{M(N\mathbf{r})}(Z).$$

This completes the proof. □

5. Special values associated with a Siegel family

As an application of a Siegel family of level N we shall construct a number associated with each ray class modulo N of a CM-field.

Let n be a positive integer, K be a CM-field with $[K : \mathbb{Q}] = 2n$ and $\{\varphi_1, \dots, \varphi_n\}$ be a set of embeddings of K into \mathbb{C} such that $(K, \{\varphi_i\}_{i=1}^n)$ is a CM-type. We fix a finite Galois extension L of \mathbb{Q} containing K , and set

$$\begin{aligned} S &= \{\sigma \in \text{Gal}(L/\mathbb{Q}) \mid \sigma|_K = \varphi_i \text{ for some } i \in \{1, 2, \dots, n\}\}, \\ S^* &= \{\sigma^{-1} \mid \sigma \in S\}, \\ H^* &= \{\gamma \in \text{Gal}(L/\mathbb{Q}) \mid \gamma S^* = S^*\}. \end{aligned}$$

Let K^* be the subfield of L corresponding to the subgroup H^* of $\text{Gal}(L/\mathbb{Q})$, and let $\{\psi_1, \dots, \psi_g\}$ be the set of all embeddings of K^* into \mathbb{C} arising from the elements of S^* . Then we know that $(K^*, \{\psi_j\}_{j=1}^g)$ is a primitive CM-type and

$$K^* = \mathbb{Q}\left(\sum_{i=1}^n a^{\varphi_i} \mid a \in K\right)$$

(see [12, § 8.3, proposition 28]). We call this CM-type $(K^*, \{\psi_j\}_{j=1}^g)$ the reflex of $(K, \{\varphi_i\}_{i=1}^n)$. Using this CM-type we define an embedding

$$\begin{aligned} \Psi: K^* &\rightarrow \mathbb{C}^g \\ a &\mapsto \begin{bmatrix} a^{\psi_1} \\ \vdots \\ a^{\psi_g} \end{bmatrix}. \end{aligned}$$

For each purely imaginary element c of K^* we associate an \mathbb{R} -bilinear form

$$E_c: \mathbb{C}^g \times \mathbb{C}^g \rightarrow \mathbb{R}$$

$$(\mathbf{u}, \mathbf{v}) \mapsto \sum_{j=1}^g c^{\psi_j} (u_j \bar{v}_j - \bar{u}_j v_j) \quad \left(\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_g \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_g \end{bmatrix} \right).$$

Then, one can readily check that

$$E_c(\Psi(a), \Psi(b)) = \text{Tr}_{K^*/\mathbb{Q}}(cab) \quad \text{for all } a, b \in K^* \tag{5.1}$$

by using the fact $\overline{a^{\psi_j}} = \bar{a}^{\psi_j}$ for all $a \in K^*$ ($1 \leq j \leq g$).

ASSUMPTION 5.1. In what follows we assume the following conditions.

- (i) $(K^*)^* = K$.
- (ii) There is a purely imaginary element ξ of K^* and a \mathbb{Z} -basis $\{\mathbf{a}_1, \dots, \mathbf{a}_{2g}\}$ of the lattice $\Psi(\mathcal{O}_{K^*})$ in \mathbb{C}^g for which

$$[E_\xi(\mathbf{a}_i, \mathbf{a}_j)]_{1 \leq i, j \leq 2g} = \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}.$$

In this case, we say that the complex torus $(\mathbb{C}^g/\Psi(\mathcal{O}_{K^*}), E_\xi)$ is a principally polarized abelian variety with a symplectic basis $\{\mathbf{a}_1, \dots, \mathbf{a}_{2g}\}$. See [12, § 6.2].

- (iii) $\mathfrak{f} = N\mathcal{O}_K$ for an integer $N \geq 2$.

REMARK 5.2. The assumption 5.1(i) is equivalent to saying that $(K, \{\varphi_i\}_{i=1}^n)$ is a primitive CM-type, namely, the abelian varieties of this CM-type are simple [12, § 8.2, proposition 26].

By assumption 5.1(i) one can define a group homomorphism

$$\begin{aligned} \mathfrak{g}: K^\times &\rightarrow (K^*)^\times \\ d &\mapsto \prod_{i=1}^n d^{\varphi_i}, \end{aligned}$$

and extend it continuously to the homomorphism $\mathfrak{g}: K_{\mathbb{A}}^\times \rightarrow (K^*)_{\mathbb{A}}^\times$ of idele groups. It is also known that for each fractional ideal \mathfrak{a} of K there is a fractional ideal $\mathcal{G}(\mathfrak{a})$ of K^* such that [12, § 8.3]

$$\mathcal{G}(\mathfrak{a})\mathcal{O}_L = \prod_{i=1}^n (\mathfrak{a}\mathcal{O}_L)^{\varphi_i}.$$

Let \mathcal{C} be a given ray class in $\text{Cl}(\mathfrak{f})$. Take any integral ideal \mathfrak{c} in \mathcal{C} , and let

$$\mathcal{N}(\mathfrak{c}) = \mathcal{N}_{K/\mathbb{Q}}(\mathfrak{c}) = |\mathcal{O}_K/\mathfrak{c}|.$$

LEMMA 5.3. $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$ is also a principally polarized abelian variety.

Proof. It follows from (5.1) that

$$\begin{aligned} E_{\xi\mathcal{N}(\mathfrak{c})}(\Psi(\mathcal{G}(\mathfrak{c})^{-1}), \Psi(\mathcal{G}(\mathfrak{c})^{-1})) &= \text{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c})\mathcal{G}(\mathfrak{c})^{-1}\overline{\mathcal{G}(\mathfrak{c})^{-1}}) \\ &= \text{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{O}_{K^*}) \\ &= E_\xi(\Psi(\mathcal{O}_{K^*}), \Psi(\mathcal{O}_{K^*})) \\ &\subseteq \mathbb{Z} \end{aligned}$$

because E_ξ is a Riemann form on $\mathbb{C}^g/\Psi(\mathcal{O}_{K^*})$. Thus, $E_{\xi\mathcal{N}(\mathfrak{c})}$ defines a Riemann form on $\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1})$.

Now, let $\{\mathbf{b}_1, \dots, \mathbf{b}_{2g}\}$ be a symplectic basis of the abelian variety $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$ so that

$$\Psi(\mathcal{G}(\mathfrak{c})^{-1}) = \sum_{j=1}^{2g} \mathbb{Z}\mathbf{b}_j \quad \text{and} \quad [E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{b}_i, \mathbf{b}_j)]_{1 \leq i, j \leq 2g} = \begin{bmatrix} O_g & -\mathcal{E} \\ \mathcal{E} & O_g \end{bmatrix},$$

where

$$\mathcal{E} = \begin{bmatrix} \varepsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_g \end{bmatrix}$$

is a $g \times g$ diagonal matrix for some positive integers $\varepsilon_1, \dots, \varepsilon_g$ satisfying $\varepsilon_1 | \cdots | \varepsilon_g$. Furthermore, let b_1, \dots, b_{2g} be elements of $\mathcal{G}(\mathfrak{c})^{-1}$ such that $\mathbf{b}_j = \Psi(b_j)$ ($1 \leq j \leq 2g$). Since $\mathcal{O}_{K^*} \subseteq \mathcal{G}(\mathfrak{c})^{-1}$, we have

$$[\mathbf{a}_1 \ \cdots \ \mathbf{a}_{2g}] = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_{2g}] \alpha \quad \text{for some } \alpha \in M_{2g}(\mathbb{Z}) \cap \text{GL}_{2g}(\mathbb{Q}), \quad (5.2)$$

and hence

$$\begin{bmatrix} a_1^{\psi_1} & \cdots & a_{2g}^{\psi_1} \\ \vdots & & \vdots \\ a_1^{\psi_g} & \cdots & a_{2g}^{\psi_g} \\ a_1^{\psi_1} & \cdots & a_{2g}^{\psi_1} \\ \vdots & & \vdots \\ a_1^{\psi_g} & \cdots & a_{2g}^{\psi_g} \end{bmatrix} = \begin{bmatrix} b_1^{\psi_1} & \cdots & b_{2g}^{\psi_1} \\ \vdots & & \vdots \\ b_1^{\psi_g} & \cdots & b_{2g}^{\psi_g} \\ b_1^{\psi_1} & \cdots & b_{2g}^{\psi_1} \\ \vdots & & \vdots \\ b_1^{\psi_g} & \cdots & b_{2g}^{\psi_g} \end{bmatrix} \alpha.$$

Taking determinant and squaring gives rise to the identity

$$\Delta_{K^*/\mathbb{Q}}(a_1, \dots, a_{2g}) = \Delta_{K^*/\mathbb{Q}}(b_1, \dots, b_{2g}) \det(\alpha)^2.$$

It then follows that

$$\begin{aligned} \det(\alpha)^2 &= \frac{|\Delta_{K^*/\mathbb{Q}}(a_1, \dots, a_{2g})|}{|\Delta_{K^*/\mathbb{Q}}(b_1, \dots, b_{2g})|} = \frac{d_{K^*/\mathbb{Q}}(\mathcal{O}_{K^*})}{d_{K^*/\mathbb{Q}}(\mathcal{G}(\mathfrak{c})^{-1})} \\ &= \mathcal{N}_{K^*/\mathbb{Q}}(\mathcal{G}(\mathfrak{c}))^2 \\ &= \mathcal{N}_{K^*/\mathbb{Q}}(\mathcal{G}(\mathfrak{c})\overline{\mathcal{G}(\mathfrak{c})}) \\ &= \mathcal{N}(\mathfrak{c})^{2g}, \end{aligned} \quad (5.3)$$

where $d_{K^*/\mathbb{Q}}$ stands for the discriminant of a fractional ideal of K^* [9, ch. III, proposition 13]. Furthermore, we deduce by (5.2) that

$$\begin{aligned} \mathcal{N}(\mathfrak{c}) \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} &= [\mathcal{N}(\mathfrak{c})E_\xi(\mathbf{a}_i, \mathbf{a}_j)]_{1 \leq i, j \leq 2g} \\ &= [E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{a}_i, \mathbf{a}_j)]_{1 \leq i, j \leq 2g} \\ &= \alpha^T [E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{b}_i, \mathbf{b}_j)]_{1 \leq i, j \leq 2g} \alpha \\ &= \alpha^T \begin{bmatrix} O_g & -\mathcal{E} \\ \mathcal{E} & O_g \end{bmatrix} \alpha. \end{aligned}$$

By taking the determinant we get $\mathcal{N}(\mathfrak{c})^{2g} = \det(\alpha)^2(\varepsilon_1 \dots \varepsilon_g)^2$, which, by (5.3), yields that $\varepsilon_1 = \dots = \varepsilon_g = 1$, and so $\mathcal{E} = I_g$. Therefore, $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$ becomes a principally polarized abelian variety. \square

As in the proof of lemma 5.3 we take a symplectic basis $\{\mathbf{b}_1, \dots, \mathbf{b}_{2g}\}$ of the principally polarized abelian variety $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi_{\mathcal{N}(\mathfrak{c})}})$, and let b_1, \dots, b_{2g} be elements of $\mathcal{G}(\mathfrak{c})^{-1}$ such that $\mathbf{b}_j = \Psi(b_j)$ ($1 \leq j \leq 2g$). We then have

$$[\mathbf{a}_1 \ \cdots \ \mathbf{a}_{2g}] = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_{2g}] \alpha \quad \text{for some } \alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z}) \cap \text{GSp}_{2g}(\mathbb{Q}). \tag{5.4}$$

Since $\nu(\alpha) = \mathcal{N}(\mathfrak{c})$ is relatively prime to N , the reduction $\tilde{\alpha}$ of α modulo N belongs to $\text{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Let $Z_{\mathfrak{c}}^*$ be the CM-point associated with the symplectic basis $\{\mathbf{b}_1, \dots, \mathbf{b}_{2g}\}$, namely

$$Z_{\mathfrak{c}}^* = [\mathbf{b}_{g+1} \ \cdots \ \mathbf{b}_{2g}]^{-1} [\mathbf{b}_1 \ \cdots \ \mathbf{b}_g],$$

which belongs to \mathbb{H}_g [1, proposition 8.1.1].

DEFINITION 5.4. Let $\{h_M(Z)\}_M \in \mathcal{S}_N$. For a given ray class $\mathcal{C} \in \text{Cl}(\mathfrak{f})$ we define

$$h_{\mathfrak{f}}(\mathcal{C}) = h_{(1/N)[\frac{B}{D}]}(Z_{\mathfrak{c}}^*).$$

REMARK 5.5. Here, the index matrix

$$(1/N) \begin{bmatrix} B \\ D \end{bmatrix}$$

is obtained using the fact that

$$\left(\begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} \alpha \right)^T = \begin{bmatrix} B^T & D^T \\ -A^T & -C^T \end{bmatrix}.$$

6. Well-definedness of $h_{\mathfrak{f}}(\mathcal{C})$

In this section we shall show that the value $h_{\mathfrak{f}}(\mathcal{C})$ given in definition 5.4 depends only on the ray class \mathcal{C} , and hence it is independent of the choice of a symplectic basis and an integral ideal in \mathcal{C} .

PROPOSITION 6.1. *The value $h_{\mathfrak{f}}(\mathcal{C})$ does not depend on the choice of a symplectic basis $\{\mathbf{b}_1, \dots, \mathbf{b}_{2g}\}$ of $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi_{\mathcal{N}(\mathfrak{c})}})$.*

Proof. Let $\{\widehat{\mathbf{b}}_1, \dots, \widehat{\mathbf{b}}_{2g}\}$ be another symplectic basis of $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi_{\mathcal{N}(\mathfrak{c})}})$. Thus,

$$[\widehat{\mathbf{b}}_1 \ \cdots \ \widehat{\mathbf{b}}_{2g}] = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_{2g}] \beta \quad \text{for some } \beta = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \text{GL}_{2g}(\mathbb{Z}). \tag{6.1}$$

We then derive

$$\begin{aligned} \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} &= \begin{bmatrix} E_{\xi_{\mathcal{N}(\mathfrak{c})}}(\widehat{\mathbf{b}}_i, \widehat{\mathbf{b}}_j) \end{bmatrix}_{1 \leq i, j \leq 2g} \\ &= \beta^T \begin{bmatrix} E_{\xi_{\mathcal{N}(\mathfrak{c})}}(\mathbf{b}_i, \mathbf{b}_j) \end{bmatrix}_{1 \leq i, j \leq 2g} \beta \\ &= \beta^T \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} \beta, \end{aligned}$$

which shows that $\beta \in \text{Sp}_{2g}(\mathbb{Z})$. Since

$$[\mathbf{a}_1 \ \cdots \ \mathbf{a}_{2g}] = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_{2g}] \alpha = \begin{bmatrix} \widehat{\mathbf{b}}_1 & \cdots & \widehat{\mathbf{b}}_{2g} \end{bmatrix} \beta^{-1} \alpha$$

by (5.4) and (6.1), the special value obtained by $\{\widehat{\mathbf{b}}_1, \dots, \widehat{\mathbf{b}}_{2g}\}$ is

$$h_{(1/N)\beta^{-1}[\frac{B}{D}]}(\widehat{Z}_c^*),$$

where \widehat{Z}_c^* is the CM-point corresponding to $\{\widehat{\mathbf{b}}_1, \dots, \widehat{\mathbf{b}}_{2g}\}$.

On the other hand, we attain that

$$\begin{aligned} \widehat{Z}_c^* &= \begin{bmatrix} \widehat{\mathbf{b}}_{g+1} & \cdots & \widehat{\mathbf{b}}_{2g} \end{bmatrix}^{-1} \begin{bmatrix} \widehat{\mathbf{b}}_1 & \cdots & \widehat{\mathbf{b}}_g \end{bmatrix} \\ &= ([\mathbf{b}_1 \ \cdots \ \mathbf{b}_g] Q + [\mathbf{b}_{g+1} \ \cdots \ \mathbf{b}_{2g}] S)^{-1} \\ &\quad \times ([\mathbf{b}_1 \ \cdots \ \mathbf{b}_g] P + [\mathbf{b}_{g+1} \ \cdots \ \mathbf{b}_{2g}] R) \quad \text{by (6.1)} \\ &= (P^T [\mathbf{b}_1 \ \cdots \ \mathbf{b}_g]^T + R^T [\mathbf{b}_{g+1} \ \cdots \ \mathbf{b}_{2g}]^T) \\ &\quad \times (Q^T [\mathbf{b}_1 \ \cdots \ \mathbf{b}_g]^T + S^T [\mathbf{b}_{g+1} \ \cdots \ \mathbf{b}_{2g}]^T)^{-1}, \quad \text{since } (\widehat{Z}_c^*)^T = \widehat{Z}_c^* \\ &= (P^T ([\mathbf{b}_{g+1} \ \cdots \ \mathbf{b}_{2g}]^{-1} [\mathbf{b}_1 \ \cdots \ \mathbf{b}_g])^T + R^T) \\ &\quad \times (Q^T ([\mathbf{b}_{g+1} \ \cdots \ \mathbf{b}_{2g}]^{-1} [\mathbf{b}_1 \ \cdots \ \mathbf{b}_g])^T + S^T)^{-1} \\ &= (P^T (Z_c^*)^T + R^T)(Q^T (Z_c^*)^T + S^T)^{-1} \\ &= (P^T Z_c^* + R^T)(Q^T Z_c^* + S^T)^{-1} \quad \text{because } (Z_c^*)^T = Z_c^* \\ &= \beta^T (Z_c^*). \end{aligned} \tag{6.2}$$

Thus, we deduce that

$$\begin{aligned} h_{(1/N)\beta^{-1}[\frac{B}{D}]}(\widehat{Z}_c^*) &= h_{(1/N)\beta^{-1}[\frac{B}{D}]}(\beta^T (Z_c^*)) \quad \text{by (6.2)} \\ &= (h_{(1/N)\beta^{-1}[\frac{B}{D}]}(Z))^\beta |_{Z=Z_c^*} \\ &= h_{(1/N)(\beta^T)^\tau \beta^{-1}[\frac{B}{D}]}(Z_c^*) \quad \text{by the property (S3) of } \{h_M(Z)\}_M \\ &= h_{(1/N)[\frac{B}{D}]}(Z_c^*). \end{aligned}$$

This proves that the value $h_f(\mathcal{C})$ is independent of the choice of a symplectic basis of $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$. □

REMARK 6.2. One can analogously readily show that $h_f(\mathcal{C})$ does not depend on the choice of a symplectic basis $\{\mathbf{a}_1, \dots, \mathbf{a}_{2g}\}$ of $(\mathbb{C}^g/\Psi(\mathcal{O}_K), E_\xi)$.

PROPOSITION 6.3. $h_f(\mathcal{C})$ does not depend on the choice of an integral ideal \mathfrak{c} in \mathcal{C} .

Proof. Let \mathfrak{c}' be another integral ideal in the class \mathcal{C} , and hence

$$\mathfrak{c}'\mathfrak{c}^{-1} = (1+a)\mathcal{O}_K \quad \text{for some } a \in \mathfrak{fa}^{-1}, \tag{6.3}$$

where \mathfrak{a} is an integral ideal of K relatively prime to \mathfrak{f} . Since $1 \in \mathfrak{c}^{-1}$ and $(1 + a) \in \mathfrak{c}'\mathfrak{c}^{-1} \subseteq \mathfrak{c}^{-1}$, we get $a \in \mathfrak{c}^{-1}$. Thus, we derive that

$$\begin{aligned} \mathfrak{aac} &\subseteq \mathfrak{fc} \cap \mathfrak{a} \quad \text{by the facts that } a \in \mathfrak{fa}^{-1} \text{ and } a \in \mathfrak{c}^{-1} \\ &\subseteq \mathfrak{f} \cap \mathfrak{a} \\ &= \mathfrak{fa} \quad \text{because } \mathfrak{f} \text{ and } \mathfrak{a} \text{ are relatively prime,} \end{aligned}$$

from which it follows that $a \in \mathfrak{fc}^{-1}$. Using the fact that $\mathfrak{f} = N\mathcal{O}_K$ yields

$$\begin{aligned} \mathfrak{g}(1 + a) &= \prod_{i=1}^n (1 + a)^{\varphi_i} \in K^* \cap \prod_{i=1}^n (1 + N(\mathfrak{c}^{-1}\mathcal{O}_L)^{\varphi_i}) \subseteq K^* \cap (1 + N\mathcal{G}(\mathfrak{c})^{-1}\mathcal{O}_L) \\ &= 1 + N\mathcal{G}(\mathfrak{c})^{-1}. \end{aligned} \tag{6.4}$$

Let

$$\mathbf{b}'_j = \mathfrak{g}(1 + a)^{-1}b_j \quad \text{and} \quad \mathbf{b}'_j = \Psi(b'_j) \quad (1 \leq j \leq 2g). \tag{6.5}$$

We know that $\{\mathbf{b}'_1, \dots, \mathbf{b}'_{2g}\}$ is a \mathbb{Z} -basis of the lattice $\Psi(\mathcal{G}(\mathfrak{c}')^{-1})$ in \mathbb{C}^g and

$$\mathbf{b}'_j = T\mathbf{b}_j \quad \text{with } T = \begin{bmatrix} (\mathfrak{g}(1 + a)^{-1})^{\psi_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\mathfrak{g}(1 + a)^{-1})^{\psi_g} \end{bmatrix}. \tag{6.6}$$

Furthermore, we get that

$$\begin{aligned} &[E_{\xi\mathcal{N}(\mathfrak{c}')}(\mathbf{b}'_i, \mathbf{b}'_j)]_{1 \leq i, j \leq 2g} \\ &= [\text{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c}')\mathbf{b}'_i\overline{\mathbf{b}'_j})]_{1 \leq i, j \leq 2g} \quad \text{by (5.1)} \\ &= [\text{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c}')\mathfrak{g}(1 + a)^{-1}b_i\overline{\mathfrak{g}(1 + a)^{-1}b_j})]_{1 \leq i, j \leq 2g} \quad \text{by (6.5)} \\ &= [\text{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c}')N_{K/\mathbb{Q}}(1 + a)^{-1}b_i\overline{b_j})]_{1 \leq i, j \leq 2g} \\ &= [\text{Tr}_{K/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c})b_i\overline{b_j})]_{1 \leq i, j \leq 2g} \\ &\quad \text{by (6.3) and the fact that } N_{K/\mathbb{Q}}(1 + a) > 0 \\ &= [E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{b}_i, \mathbf{b}_j)]_{1 \leq i, j \leq 2g} \quad \text{by (5.1)} \\ &= \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}. \end{aligned}$$

Thus, $\{\mathbf{b}'_1, \dots, \mathbf{b}'_{2g}\}$ is a symplectic basis of $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c}')^{-1}), E_{\xi\mathcal{N}(\mathfrak{c}')})$, and its associated CM-point $Z_{\mathfrak{c}'}$ is given by

$$\begin{aligned} Z_{\mathfrak{c}'}^* &= [\mathbf{b}'_{g+1} \quad \cdots \quad \mathbf{b}'_{2g}]^{-1} [\mathbf{b}'_1 \quad \cdots \quad \mathbf{b}'_g] \\ &= [T\mathbf{b}_{g+1} \quad \cdots \quad T\mathbf{b}_{2g}]^{-1} [T\mathbf{b}_1 \quad \cdots \quad T\mathbf{b}_g] \quad \text{by (6.6)} \\ &= Z_{\mathfrak{c}}^*. \end{aligned} \tag{6.7}$$

Let $\alpha = [a_{ij}]$, $\alpha' = [a'_{ij}] \in M_{2g}(\mathbb{Z})$ such that

$$[\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{2g}] = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_{2g}] \alpha = [\mathbf{b}'_1 \quad \cdots \quad \mathbf{b}'_{2g}] \alpha'. \tag{6.8}$$

For each $1 \leq i \leq 2g$ we obtain that

$$\begin{aligned} \sum_{j=1}^{2g} a'_{ji} b_j &= \mathfrak{g}(1+a) \sum_{j=1}^{2g} a'_{ji} b'_j \quad \text{by (6.5)} \\ &= a_i \mathfrak{g}(1+a) \quad \text{by (6.8)} \\ &\in a_i(1 + N\mathcal{G}(\mathfrak{c})^{-1}) \quad \text{by (6.4)} \\ &\subseteq a_i + N\mathcal{G}(\mathfrak{c})^{-1} \quad \text{because } a_i \in \mathcal{O}_K \\ &= \sum_{j=1}^{2g} a_{ji} b_j + N \sum_{j=1}^{2g} \mathbb{Z} b_j \quad \text{by (6.8)}. \end{aligned}$$

This yields $\alpha \equiv \alpha' \pmod{N \cdot M_{2g}(\mathbb{Z})}$, and hence

$$(1/N)\alpha \equiv (1/N)\alpha' \pmod{M_{2g}(\mathbb{Z})}. \tag{6.9}$$

Now, the result follows from (6.7), (6.9) and the property (S2) of $\{h_M(Z)\}_M$. \square

7. Galois actions on $h_f(\mathcal{C})$

Finally, we shall show that if $h_f(\mathcal{C})$ is finite, then it lies in the ray class field K_f and satisfies the natural transformation formula under the Artin reciprocity map for f .

Let $r: K^* \rightarrow M_{2g}(\mathbb{Q})$ be the regular representation with respect to the ordered basis $\{a_1, \dots, a_{2g}\}$ of K^* over \mathbb{Q} given by

$$a \begin{bmatrix} a_1 \\ \vdots \\ a_{2g} \end{bmatrix} = r(a) \begin{bmatrix} a_1 \\ \vdots \\ a_{2g} \end{bmatrix} \quad (a \in K^*). \tag{7.1}$$

Then it can be extended to the map $r: (K^*)_{\mathbb{A}} \rightarrow M_{2g}(\mathbb{Q}_{\mathbb{A}})$ of adèle rings.

LEMMA 7.1 (Shimura’s reciprocity law). *Let f be an element of \mathcal{F} that is finite at Z_c^* .*

- (i) *The special value $f(Z_c^*)$ lies in K_{ab} .*
- (ii) *For every $s \in K_{\mathbb{A}}^{\times}$ we have $r(\mathfrak{g}(s)) \in G_{\mathbb{A}^+}$ and*

$$f(Z_c^*)^{[s, K]} = f^{\tau(r(\mathfrak{g}(s)^{-1}))}(Z_c^*).$$

Proof. See [13, lemma 9.5 and theorem 9.6]. \square

THEOREM 7.2. *If $h_f(\mathcal{C})$ is finite, then it belongs to K_f . Furthermore, it satisfies*

$$h_f(\mathcal{C})^{\sigma_f(\mathcal{D})} = h_f(\mathcal{C}\mathcal{D}) \quad \text{for every } \mathcal{D} \in \text{Cl}(f),$$

where σ_f is the Artin reciprocity map for f .

Proof. Since $h_f(\mathcal{C})$ belongs to K_{ab} by lemma 7.1(i), there is a sufficiently large positive integer M so that $N|M$ and $h_f(\mathcal{C}) \in K_{\mathfrak{m}}$ with $\mathfrak{m} = M\mathcal{O}_K$. Take an integral

ideal \mathfrak{d} in \mathcal{D} relatively prime to \mathfrak{m} by using the surjectivity of the natural map $\text{Cl}(\mathfrak{m}) \rightarrow \text{Cl}(\mathfrak{f})$. Let $\{\mathbf{d}_1, \dots, \mathbf{d}_{2g}\}$ be a symplectic basis of the principally polarized abelian variety $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c}\mathfrak{d})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c}\mathfrak{d})})$, and let d_1, \dots, d_{2g} be elements of $\mathcal{G}(\mathfrak{c}\mathfrak{d})^{-1}$ such that $\mathbf{d}_j = \Psi(d_j)$ ($1 \leq j \leq 2g$). Since $\mathcal{G}(\mathfrak{c})^{-1} \subseteq \mathcal{G}(\mathfrak{c}\mathfrak{d})^{-1}$, we get

$$[\mathbf{b}_1 \ \cdots \ \mathbf{b}_{2g}] = [\mathbf{d}_1 \ \cdots \ \mathbf{d}_{2g}] \delta \quad \text{for some } \delta \in M_{2g}(\mathbb{Z}) \cap \text{GL}_{2g}(\mathbb{Q}). \tag{7.2}$$

We then have that

$$\begin{aligned} \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} &= [E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{b}_i, \mathbf{b}_j)]_{1 \leq i, j \leq 2g} \\ &= \delta^T [E_{\xi\mathcal{N}(\mathfrak{c})}(\mathbf{d}_i, \mathbf{d}_j)]_{1 \leq i, j \leq 2g} \delta \quad \text{by (7.2)} \\ &= \delta^T [\mathcal{N}(\mathfrak{c})\mathcal{N}(\mathfrak{c}\mathfrak{d})^{-1}E_{\xi\mathcal{N}(\mathfrak{c}\mathfrak{d})}(\mathbf{d}_i, \mathbf{d}_j)]_{1 \leq i, j \leq 2g} \delta \\ &= \mathcal{N}(\mathfrak{d})^{-1}\delta^T \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} \delta. \end{aligned}$$

This claims that

$$\delta \in M_{2g}(\mathbb{Z}) \cap G_+ \text{ with } \nu(\delta) = \mathcal{N}(\mathfrak{d}). \tag{7.3}$$

Furthermore, if we let $Z_{\mathfrak{c}\mathfrak{d}}^*$ be the CM-point associated with $\{\mathbf{d}_1, \dots, \mathbf{d}_{2g}\}$, then we obtain

$$Z_{\mathfrak{c}\mathfrak{d}}^* = (\delta^{-1})^T(Z_{\mathfrak{c}}^*) \tag{7.4}$$

in a similar way to the argument in the proof of proposition 6.1.

Let $s = (s_p)_p$ be an idele of K such that

$$\left. \begin{aligned} s_p &= 1 && \text{if } p|M, \\ s_p(\mathcal{O}_K)_p &= \mathfrak{d}_p && \text{if } p \nmid M. \end{aligned} \right\} \tag{7.5}$$

If we set $\tilde{\mathcal{D}}$ to be the ray class in $\text{Cl}(\mathfrak{m})$ containing \mathfrak{d} , then by (7.5) we attain

$$[s, K]|_{K_{\mathfrak{m}}} = \sigma_{\mathfrak{m}}(\tilde{\mathcal{D}}), \tag{7.6}$$

$$\mathfrak{g}(s)_p^{-1}(\mathcal{O}_{K^*})_p = \mathcal{G}(\mathfrak{d})_p^{-1} \quad \text{for all rational primes } p. \tag{7.7}$$

It then follows from (7.1)–(7.7) that for every rational prime p , the entries of each of the vectors

$$r(\mathfrak{g}(s)^{-1})_p \begin{bmatrix} b_1 \\ \vdots \\ b_{2g} \end{bmatrix} \quad \text{and} \quad (\delta^{-1})^T \begin{bmatrix} b_1 \\ \vdots \\ b_{2g} \end{bmatrix}$$

form a basis of $\mathcal{G}(\mathfrak{c}\mathfrak{d})_p^{-1} = \mathcal{G}(\mathfrak{c})^{-1}\mathcal{G}(\mathfrak{d})_p^{-1}$. So, there exists a matrix $u = (u_p)_p \in \prod_p \text{GL}_{2g}(\mathbb{Z}_p)$ satisfying

$$r(\mathfrak{g}(s)^{-1}) = u(\delta^{-1})^T. \tag{7.8}$$

Since δ^T and

$$\begin{bmatrix} I_g & O_g \\ O_g & \mathcal{N}(\delta)I_g \end{bmatrix}$$

can be viewed as elements of $\mathrm{GSp}_{2g}(Z/M\mathbb{Z})$ by (7.3), there exists a matrix $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that

$$\delta^T \equiv \begin{bmatrix} I_g & O_g \\ O_g & \mathcal{N}(\delta)I_g \end{bmatrix} \gamma \pmod{M \cdot M_{2g}(\mathbb{Z})} \tag{7.9}$$

owing to the surjectivity of the reduction $\mathrm{Sp}_{2g}(\mathbb{Z}) \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/M\mathbb{Z})$. Since

$$r(\mathfrak{g}(s)^{-1})_p = I_{2g} \quad \text{for all } p|M$$

by (7.5), we get $u_p = \delta^T$ for all $p|M$ by (7.8). Hence, we deduce using (7.9) that

$$u_p \gamma^{-1} \equiv \begin{bmatrix} I_g & O_g \\ O_g & \mathcal{N}(\delta)I_g \end{bmatrix} \pmod{M \cdot M_{2g}(\mathbb{Z}_p)} \quad \text{for all rational primes } p. \tag{7.10}$$

On the other hand, we have by (5.4) and (7.2) that

$$\begin{aligned} [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_{2g}] &= [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_{2g}] \alpha \\ &= ([\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_{2g}] \delta^{-1})(\delta \alpha) \\ &= [\mathbf{d}_1 \quad \cdots \quad \mathbf{d}_{2g}] (\delta \alpha). \end{aligned} \tag{7.11}$$

Letting

$$\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

we induce the following:

$$\begin{aligned} h_f(\mathcal{C})^{\sigma_m(\tilde{\mathcal{D}})} &= h_f(\mathcal{C})^{[s,K]} \quad \text{by (7.6)} \\ &= h_{(1/N)[\frac{B}{D}]}(Z_c^*)^{[s,K]} \quad \text{by definition 5.4} \\ &= h_{(1/N)[\frac{B}{D}]}(Z)^{\tau(r(\mathfrak{g}(s)^{-1}))}|_{Z=Z_c^*} \quad \text{by lemma 7.1(ii)} \\ &= h_{(1/N)[\frac{B}{D}]}(Z)^{\tau(u(\delta^{-1})^T)}|_{Z=Z_c^*} \quad \text{by (7.8)} \\ &= h_{(1/N)[\frac{B}{D}]}(Z)^{\tau(u\gamma^{-1})\tau(\gamma)\tau((\delta^{-1})^T)}|_{Z=Z_c^*} \\ &= h_{(1/N)\left[\begin{smallmatrix} I_g & O_g \\ O_g & \mathcal{N}(\delta)I_g \end{smallmatrix}\right]\left[\frac{B}{D}\right]}(Z)^{\tau(\gamma)\tau((\delta^{-1})^T)}|_{Z=Z_c^*} \quad \text{by (7.10) and (S3)} \\ &= h_{(1/N)\gamma^T\left[\begin{smallmatrix} I_g & O_g \\ O_g & \mathcal{N}(\delta)I_g \end{smallmatrix}\right]\left[\frac{B}{D}\right]}(Z)^{\tau((\delta^{-1})^T)}|_{Z=Z_c^*} \quad \text{by (S3)} \\ &= h_{(1/N)\delta\left[\frac{B}{D}\right]}(Z)^{\tau((\delta^{-1})^T)}|_{Z=Z_c^*} \quad \text{by (7.9) and (S2)} \\ &= h_{(1/N)\delta\left[\frac{B}{D}\right]}((\delta^{-1})^T(Z_c^*)) \quad \text{due to the fact that } \delta \in G_+ \text{ and by (A1)} \\ &= h_f(\mathcal{CD}) \quad \text{by (7.4), (7.11) and definition 5.4.} \end{aligned}$$

In particular, suppose that $\mathfrak{d} = d\mathcal{O}_K$ for some $d \in \mathcal{O}_K$ such that $d \equiv 1 \pmod{\mathfrak{f}}$. Then \mathcal{D} is the identity class of $\mathrm{Cl}(\mathfrak{f})$, and so the above observation implies that $\sigma_m(\tilde{\mathcal{D}})$ leaves $h_f(\mathcal{C})$ fixed. Therefore, we conclude that $h_f(\mathcal{C})$ lies in K_f . \square

COROLLARY 7.3. Let H be a subgroup of $\text{Cl}(\mathfrak{f})$ defined by

$$H = \langle \mathcal{D} \in \text{Cl}(\mathfrak{f}) \mid \mathcal{D} \text{ contains an integral ideal } \mathfrak{d} \text{ of } K \text{ for which } \mathcal{G}(\mathfrak{d}) = \mathfrak{g}(d)\mathcal{O}_{K^*} \\ \text{for some } d \in \mathcal{O}_K \text{ such that } \mathfrak{g}(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}} \rangle,$$

and let $K_{\mathfrak{f}}^H$ be the fixed field of H . If $h_{\mathfrak{f}}(\mathcal{C})$ is finite, then it belongs to $K_{\mathfrak{f}}^H$.

Proof. Let \mathcal{C}_0 be the identity class of $\text{Cl}(\mathfrak{f})$. Since $h_{\mathfrak{f}}(\mathcal{C}_0) \in K_{\mathfrak{f}}$ by theorem 7.2, $K(h_{\mathfrak{f}}(\mathcal{C}_0))$ is a Galois extension of K as a subfield of $K_{\mathfrak{f}}$. Furthermore, since

$$h_{\mathfrak{f}}(\mathcal{C}_0)^{\sigma_{\mathfrak{f}}(\mathcal{C})} = h_{\mathfrak{f}}(\mathcal{C}_0\mathcal{C}) = h_{\mathfrak{f}}(\mathcal{C})$$

by theorem 7.2, $K(h_{\mathfrak{f}}(\mathcal{C}_0))$ contains $h_{\mathfrak{f}}(\mathcal{C})$. Thus, it suffices to show that $h_{\mathfrak{f}}(\mathcal{C}_0)$ belongs to $K_{\mathfrak{f}}^H$.

To this end, let \mathcal{D} be an element of $\text{Cl}(\mathfrak{f})$ containing an integral ideal \mathfrak{d} of K for which

$$\mathcal{G}(\mathfrak{d}) = \mathfrak{g}(d)\mathcal{O}_{K^*} \quad \text{for some } d \in \mathcal{O}_K \text{ such that } \mathfrak{g}(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}}.$$

Now that

$$(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{d})^{-1}), E_{\xi\mathcal{N}(\mathfrak{d})}) = (\mathbb{C}^g/\Psi(\mathfrak{g}(d)^{-1}\mathcal{O}_{K^*}), E_{\xi\mathcal{N}(d\mathcal{O}_K)}),$$

we obtain

$$h_{\mathfrak{f}}(\mathcal{C}_0)^{\sigma_{\mathfrak{f}}(\mathcal{D})} = h_{\mathfrak{f}}(\mathcal{D}) = h_{\mathfrak{f}}([d\mathcal{O}_K]),$$

where $[\mathfrak{a}]$ is the ray class containing \mathfrak{a} for a fractional ideal \mathfrak{a} of K . Moreover, since $\mathfrak{g}(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}}$, we obtain

$$h_{\mathfrak{f}}([d\mathcal{O}_K]) = h_{\mathfrak{f}}([\mathcal{O}_K]) = h_{\mathfrak{f}}(\mathcal{C}_0)$$

analogously to the proof of proposition 6.3. This proves that $h_{\mathfrak{f}}(\mathcal{C}_0)$ belongs to $K_{\mathfrak{f}}^H$. \square

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