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DIRECT AND CONVERSE RESULTS FOR q-BERNSTEIN OPERATORS

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Abstract Direct and converse theorems are established for the q-Bernstein polynomials introduced by G. M. Phillips. The direct approximation theorems are given for the second-order Ditzian–Totik modulus of smoothness. The converse results are theorems of Berens–Lorentz type.

Keywords: q-Bernstein polynomials; K-functional; Ditzian–Totik modulus of smoothness; Berens–Lorentz-type theorem

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1. Introduction

Let $0 < q \leqslant 1$ and denote the q-integers by $[n]_q = 1 + q + \cdots + q^{n-1}$ for $n = 1, 2, \ldots$ and $[0]_q = 0$. Besides, let $[n]_q! = [1]_q[2]_q \cdots [n]_q$ for $n = 1, 2, \ldots$ and $[0]_q! = 1$. Then the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

where $0 \leqslant k \leqslant n$.

Recently, Phillips [**7**] proposed the following generalization of the Bernstein operators, based on the q-integers. For every $n = 1, 2, \ldots$ and $f \in C[0, 1]$, we define

$$B_{n,q}(f, x) \equiv (B_{n,q}f)(x) = \sum_{k=0}^{n} f_k p_{n,k,q}(x), \tag{1.1}$$

where f_k denotes $f([k]_q/[n]_q)$, $k = 0, 1, \ldots, n$, and

$$p_{n,k,q}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1 - x)(1 - xq) \cdots (1 - xq^{n-k-1}), \quad k = 0, 1, \ldots, n$$

(an empty product is taken to equal 1). $B_{n,q}(f,x)$ are called q-Bernstein polynomials. For $q=1$, $B_{n,q}(f,x)$ reduces to the well-known Bernstein polynomials:

$$B_{n,1}(f,x) \equiv B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

A useful property of (1.1) is based on the q-differences. For $f \in C[0,1]$ we define $\Delta_q^0 f_i = f_i$ for $i=0,1,\dots,n$ and, recursively,

$$\Delta_q^{k+1} f_i = \Delta_q^k f_{i+1} - q^k \Delta_q^k f_i \tag{1.2}$$

for $k=0,1,\dots,n-i-1$. It is easy to show by induction on k that q-differences satisfy the relation

$$\Delta_q^k f_i = \sum_{r=0}^{k} (-1)^r q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix}_q f_{i+k-r}. \tag{1.3}$$

Then, in view of [**7**], we may write

$$B_{n,q}(f,x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \Delta_q^k f_0 x^k. \tag{1.4}$$

The rate of convergence and a Voronovskaja-type asymptotic formula are studied in [**7**] for the new Bernstein polynomials defined in (1.1). Results concerning the convergence of derivatives of the q-Bernstein polynomials are given in [**8**]. Further properties of (1.1), such as convexity and monotonicity, are obtained in [**5**] and [**6**], respectively. For the main results of [**7**], Videnskii gave another proof in [**9**].

In the present paper we prove direct and converse global theorems for the q-Bernstein polynomials. The direct results are formulated by the second-order Ditzian–Totik modulus of smoothness, given by

$$\omega_\varphi^2(f,\delta) = \sup_{0<h\leqslant\delta} \sup_{x\pm h\varphi(x)\in[0,1]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|,$$

$\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. The corresponding K-functional is

$$K_{2,\varphi}(f,\delta^2) = \inf_{g\in W^2(\varphi)} \{\|f-g\| + \delta^2 \|\varphi^2 g''\|\},$$

where $W^2(\varphi) = \{g \in C[0,1] : g' \in \mathrm{AC_{loc}}[0,1],\ \varphi^2 g'' \in C[0,1]\}$, $\|\cdot\|$ denotes the sup-norm on $C[0,1]$ and $g' \in \mathrm{AC_{loc}}[0,1]$ means that g is differentiable such that g' is absolutely continuous on every interval $[a,b] \subset [0,1]$. It is known (see [**3**, Theorem 2.1.1, p. 11]) that $K_{2,\varphi}(f,\delta^2)$ and $\omega_\varphi^2(f,\delta)$ are equivalent, i.e. there exists an absolute constant $C>0$ such that

$$C^{-1}\omega_\varphi^2(f,\delta) \leqslant K_{2,\varphi}(f,\delta^2) \leqslant C\omega_\varphi^2(f,\delta). \tag{1.5}$$

Here we mention that $C>0$ is an absolute constant which can be different at each occurrence. The converse results are theorems of Berens–Lorentz type (see [**3**, (9.3.3), p. 117]).

2. Direct theorem

The direct results are presented in the following.

Theorem 2.1. *Let $B_{n,q}f$ be defined as in (1.1) and let $q = q(n)$ such that $0 < q(n) < 1$ and $q(n) \to 1$ as $n \to \infty$. Then there exists $C > 0$ such that*

$$\|B_{n,q(n)}f - f\| \leqslant C\omega_\varphi^2(f, [n]_{q(n)}^{-1/2})$$

for all $f \in C[0,1]$ and $n = 1, 2, \dots$.

Proof. We recall some properties of the q-Bernstein polynomials. In view of [**8**, (10), p. 264] we have

$$B_{n,q(n)}(1, x) = 1, \tag{2.1}$$

$$B_{n,q(n)}(t, x) = x, \tag{2.2}$$

$$B_{n,q(n)}(t^2, x) = x^2 + \frac{1}{[n]_{q(n)}} x(1-x). \tag{2.3}$$

Moreover, the q-Bernstein operator defined by (1.1) is a positive linear operator. Therefore, by (2.1), we obtain

$$|B_{n,q(n)}(f, x)| \leqslant \|f\| B_{n,q(n)}(1, x) = \|f\|,$$

i.e.

$$\|B_{n,q(n)}f\| \leqslant \|f\|, \tag{2.4}$$

for all $f \in C[0,1]$.

Next, for $g \in W^2(\varphi)$ we find, in view of Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t g''(u)(t-u)\,\mathrm{d}u, \quad t \in [0,1],$$

by (2.1)–(2.3) and [**3**, Lemma 9.6.1, p. 140], that

$$\begin{aligned}
|B_{n,q(n)}(g, x) - g(x)| &= \left| B_{n,q(n)}\left(\int_x^t g''(u)(t-u)\,\mathrm{d}u, x \right) \right| \\
&\leqslant B_{n,q(n)}\left(\left| \int_x^t |g''(u)|\,|t-u|\,\mathrm{d}u \right|, x \right) \\
&\leqslant \|\varphi^2 g''\| B_{n,q(n)}\left(\left| \int_x^t \frac{|t-u|}{u(1-u)}\,\mathrm{d}u \right|, x \right) \\
&\leqslant \|\varphi^2 g''\| \varphi^{-2}(x) B_{n,q(n)}((t-x)^2, x) \\
&= \frac{1}{[n]_{q(n)}} \|\varphi^2 g''\|.
\end{aligned} \tag{2.5}$$

Now for $f \in C[0,1]$, in view of (2.4) and (2.5), we find that

$$\begin{aligned}|B_{n,q(n)}(f,x)-f(x)| &\leqslant |B_{n,q(n)}(f-g,x)-(f-g)(x)|+|B_{n,q(n)}(g,x)-g(x)| \\ &\leqslant 2\|f-g\|+[n]_{q(n)}^{-1}\|\varphi^2 g''\| \\ &\leqslant 2\{\|f-g\|+[n]_{q(n)}^{-1}\|\varphi^2 g''\|\}\end{aligned}$$

Hence, $\|B_{n,q(n)}f-f\| \leqslant 2K_{2,\varphi}(f,[n]_{q(n)}^{-1})$. Using (1.5) we get the assertion of the theorem. □

3. Converse theorem

The converse results of Berens–Lorentz type are included in the following theorem.

Theorem 3.1. *Let $B_{n,q}f$ be defined as in (1.1) and let $q=q(n)$ such that $0<q(n)<1$ and satisfies the following properties:*

$$\left(3^n [n]_{q(n)}^2\left(\frac{n}{[n]_{q(n)}}\frac{n-1}{[n-1]_{q(n)}}-1\right)\right) \text{ is a bounded sequence,} \tag{3.1}$$

$$C_0 \leqslant (q(n))^n < 1 \quad \text{for all } n=1,2,\dots \text{ and for some absolute constant } C_0>0. \tag{3.2}$$

Then, for $f \in C[0,1]$ and $0<\alpha<2$, the global approximation

$$\|B_{n,q(n)}f-f\| \leqslant C[n]_{q(n)}^{-\alpha/2}, \quad n=1,2,\dots, \tag{3.3}$$

implies $\omega_\varphi^2(f,\delta) \leqslant C\delta^\alpha$, $0<\delta<1$.

First of all, we give two lemmas which are necessary to prove our theorem.

Lemma 3.2. *For $0<q<1$ and $g \in W^2(\varphi)$ we have*

$$|\Delta_q^2 g_t| \leqslant \begin{cases} \dfrac{2q(1+q)}{[n]_q}\|\varphi^2 g''\| & \text{if } t=0, \\ \dfrac{q^{2t+1}(1+q)^2}{2[n]_q^2}\varphi^{-2}\left(\dfrac{[t+1]_q}{[n]_q}\right)\|\varphi^2 g''\| & \text{if } t=1,2,\dots,n-3, \\ \dfrac{2q^{n-3}(1+q)}{[n]_q}\|\varphi^2 g''\| & \text{if } t=n-2. \end{cases}$$

Proof. Using the definition of the q-integers and the definition of the divided differences for g at points $[t]_q/[n]_q$, $[t+1]_q/[n]_q$ and $[t+2]_q/[n]_q$, we have, by (1.2),

$$\begin{aligned}\left[\frac{[t]_q}{[n]_q},\frac{[t+1]_q}{[n]_q},\frac{[t+2]_q}{[n]_q};g\right] &= \frac{[n]_q^2}{q^{2t+1}(1+q)}(g_{t+2}-(1+q)g_{t+1}+qg_t) \\ &= \frac{[n]_q^2}{q^{2t+1}(1+q)}\Delta_q^2 g_t.\end{aligned}$$

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Hence, taking into account the representation of the divided differences by the Peano kernel [**2**, (7.16), p. 123], we find that

$$\Delta_q^2 g_t = \frac{1}{2[n]_q^2} q^{2t+1}(1+q) \int_{-\infty}^{\infty} g''(u) M_{[t]_q}(u)\,\mathrm{d}u, \tag{3.4}$$

where

$$M_{[t]_q}(u) := M\biggl(u; \frac{[t]_q}{[n]_q}, \frac{[t+1]_q}{[n]_q}, \frac{[t+2]_q}{[n]_q}\biggr)$$

is the piecewise linear B-spline (cf. [**2**, pp. 137–138]). From

$$\int_{-\infty}^{\infty} M_{[t]_q}(u)\,\mathrm{d}u = 1$$

we find

$$M_{[t]_q}\biggl(\frac{[t+1]_q}{[n]_q}\biggr) = \frac{2[n]_q}{q^t(1+q)}.$$

Therefore,

$$M_{[0]_q}(u) \leqslant \frac{4[n]_q^2}{1+q} u(1-u), \qquad u \in [0,1]$$

and

$$M_{[n-2]_q}(u) \leqslant \frac{4[n]_q^2}{q^{n-2}(1+q)} u(1-u), \quad u \in [0,1].$$

We thus conclude from (3.4) that

$$\begin{aligned} |\Delta_q^2 g_0| &\leqslant \frac{1}{2[n]_q^2} q(1+q) \frac{4[n]_q^2}{1+q} \int_0^{[2]_q/[n]_q} \varphi^2(u)|g''(u)|\,\mathrm{d}u \\ &\leqslant \frac{2q(1+q)}{[n]_q} \|\varphi^2 g''\| \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} |\Delta_q^2 g_{n-2}| &\leqslant \frac{1}{2[n]_q^2} q^{2n-3}(1+q) \frac{4[n]_q^2}{q^{n-2}(1+q)} \int_{[n-2]_q/[n]_q}^{1} \varphi^2(u)|g''(u)|\,\mathrm{d}u \\ &\leqslant \frac{2q^{n-3}(1+q)}{[n]_q} \|\varphi^2 g''\|, \end{aligned} \tag{3.6}$$

respectively. If $t \in \{1, 2, \dots, n-3\}$, then, for u between $[t]_q/[n]_q$ and $[t+2]_q/[n]_q$, we have $\varphi^2([t+1]_q/[n]_q) \leqslant (1+q)\varphi^2(u)$. Hence, by (3.4),

$$\begin{aligned} |\Delta_q^2 g_t| &\leqslant \frac{1}{2[n]_q^2} q^{2t+1}(1+q) \varphi^{-2}\biggl(\frac{[t+1]_q}{[n]_q}\biggr)(1+q) \int_0^1 \varphi^2(u)|g''(u)| M_{[t]_q}(u)\,\mathrm{d}u \\ &\leqslant \frac{1}{2[n]_q^2} q^{2t+1}(1+q)^2 \varphi^{-2}\biggl(\frac{[t+1]_q}{[n]_q}\biggr) \|\varphi^2 g''\|, \end{aligned} \tag{3.7}$$

because

$$\int_{-\infty}^{\infty} M_{[t]_q}(u)\,\mathrm{d}u = 1.$$

By combining (3.5)–(3.7) we arrive at the desired estimates. □

Lemma 3.3.

(i) *We have*

$$\|\varphi^2 B''_{n,q} f\| \leqslant Cn\|f\|$$

for $0 < q < 1$, $f \in C[0,1]$ *and* $n = 1, 2, \dots$.

(ii) *Let* $q = q(n)$ *satisfy the conditions* $0 < q(n) < 1$, *(3.1) and (3.2). Then*

$$\|\varphi^2 B''_{n,q(n)} g\| \leqslant C\|\varphi^2 g''\|$$

for $g \in W^2(\varphi)$ *and* $n = 1, 2, \dots$.

Proof. (i) For $0 < q < 1$ we have

$$\begin{aligned} p_{n,k,q}(x) &= \begin{bmatrix} n \\ k \end{bmatrix}_q x^k(1-x)[(1-x)+x(1-q)]\dots[(1-x)+x(1-q^{n-k-1})] \\ &= \begin{bmatrix} n \\ k \end{bmatrix}_q \{x^k(1-x)^{n-k} + x^{k+1}(1-x)^{n-k-1}(1-q+1-q^2+\dots+1-q^{n-k-1}) \\ &\qquad + \dots + x^{n-1}(1-x)(1-q)(1-q^2)\cdots(1-q^{n-k-1})\}. \end{aligned}$$

Thus, $p_{n,k,q}(x)$ is a polynomial with positive coefficients in x and $1-x$ (see [**2**, pp. 109–113]).

Let $I_+ = \{k \in \{0, 1, \dots, n\} : f_k \geqslant 0\}$ and $I_- = \{k \in \{0, 1, \dots, n\} : f_k < 0\}$. Then

$$\sum_{k\in I_+} f_k p_{n,k,q}(x) \quad \text{and} \quad \sum_{k\in I_-} (-f_k) p_{n,k,q}(x)$$

are polynomials with positive coefficients in x and $1-x$. Using the Bernstein-type inequality for polynomials with positive coefficients in x, $1-x$ and of degree $\leqslant n$ (see [**2**, pp. 112–113] or [**4**]), we obtain

$$\begin{aligned} \varphi^2(x)|B''_{n,q}(f,x)| &= \varphi^2(x)\left|\sum_{k\in I_+} f_k p''_{n,k,q}(x) + \sum_{k\in I_-} f_k p''_{n,k,q}(x)\right| \\ &\leqslant \varphi^2(x)\left|\sum_{k\in I_+} f_k p''_{n,k,q}(x)\right| + \varphi^2(x)\left|\sum_{k\in I_-} (-f_k) p''_{n,k,q}(x)\right| \\ &= \varphi^2(x)\left|\left(\sum_{k\in I_+} f_k p_{n,k,q}(x)\right)''\right| + \varphi^2(x)\left|\left(\sum_{k\in I_-} (-f_k) p_{n,k,q}(x)\right)''\right| \end{aligned}$$

$$\begin{aligned}&\leqslant Cn\Big\|\sum_{k\in I_+} f_k p_{n,k,q}\Big\| + Cn\Big\|\sum_{k\in I_-}(-f_k)p_{n,k,q}\Big\|\\ &\leqslant Cn\|f\|\Big(\Big\|\sum_{k\in I_+} p_{n,k,q}\Big\| + \Big\|\sum_{k\in I_-} p_{n,k,q}\Big\|\Big)\\ &\leqslant Cn\|f\|,\end{aligned}$$

because $B_{n,q}(1,x)=1$ in view of (2.1). Hence, $\|\varphi^2 B''_{n,q}f\|\leqslant Cn\|f\|$, which was to be proved.

(ii) By [**8**, (21), p. 269] we have

$$B''_{n,q(n)}(f,x)=\sum_{k=0}^{n-2}\frac{(k+2)(k+1)}{[k+2]_{q(n)}[k+1]_{q(n)}}\begin{bmatrix}n-2\\k\end{bmatrix}_{q(n)}\Delta^k_{q(n)}([n]_{q(n)}[n-1]_{q(n)}\Delta^2_{q(n)}f_0)x^k. \tag{3.8}$$

Following [**8**, (12), p. 265], let us consider the next modified form of the q-Bernstein polynomials:

$$\tilde{B}_{n-2,q(n)}([n]_{q(n)}[n-1]_{q(n)}\Delta^2_{q(n)}f,x) = \sum_{k=0}^{n-2}\begin{bmatrix}n-2\\k\end{bmatrix}_{q(n)}\Delta^k_{q(n)}([n]_{q(n)}[n-1]_{q(n)}\Delta^2_{q(n)}f_0)x^k, \tag{3.9}$$

where $f\in C[0,1]$. By combining (3.8), (3.9) and (1.3), we arrive at the following inequality for $g\in W^2(\varphi)$:

$$\begin{aligned}&\Big|\varphi^2(x)B''_{n,q(n)}(g,x)-\varphi^2(x)\tilde{B}_{n-2,q(n)}([n]_{q(n)}[n-1]_{q(n)}\Delta^2_{q(n)}g,x)\Big|\\ &\quad\leqslant\sum_{k=0}^{n-2}\Big(\frac{k+2}{[k+2]_{q(n)}}\frac{k+1}{[k+1]_{q(n)}}-1\Big)\begin{bmatrix}n-2\\k\end{bmatrix}_{q(n)}[n]_{q(n)}[n-1]_{q(n)}\\ &\qquad\times\Big(\sum_{r=0}^{k}q(n)^{r(r-1)/2}\begin{bmatrix}k\\r\end{bmatrix}_{q(n)}|\Delta^2_{q(n)}g_{k-r}|\Big)x^{k+1}(1-x).\end{aligned} \tag{3.10}$$

By applying Lemma 3.2 for $t\in\{1,2,\dots,n-3\}$, we obtain

$$\begin{aligned}|\Delta^2_{q(n)}g_t|&\leqslant\frac{q(n)^{2t+1}(1+q(n))^2}{2[n]^2_{q(n)}}\frac{[n]^2_{q(n)}}{[t+1]_{q(n)}([n]_{q(n)}-[t+1]_{q(n)})}\|\varphi^2g''\|\\ &\leqslant\frac{q(n)^{2t+1}(1+q(n))^2}{2[n]^2_{q(n)}}\frac{[n]^2_{q(n)}}{q(n)^tq(n)^{t+1}}\|\varphi^2g''\|\\ &<2\|\varphi^2g''\|;\end{aligned}$$

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by applying the same Lemma 3.2 for $t \in \{0, n-2\}$, we also obtain

$$|\Delta^2_{q(n)} g_t| \leqslant \frac{4}{[n]_{q(n)}} \|\varphi^2 g''\| \leqslant 4\|\varphi^2 g''\|.$$

In conclusion, by (3.10),

$$\begin{aligned}|\varphi^2(x)B''_{n,q(n)}(g,x) - \varphi^2(x)\tilde{B}_{n-2,q(n)}([n]_{q(n)}[n-1]_{q(n)}\Delta^2_{q(n)}g,x)| \\ \leqslant \sum_{k=0}^{n-2}\left(\frac{k+2}{[k+2]_{q(n)}}\frac{k+1}{[k+1]_{q(n)}} - 1\right)\begin{bmatrix} n-2 \\ k \end{bmatrix}_{q(n)} [n]^2_{q(n)} \\ \times \left(\sum_{r=0}^{k} q(n)^{r(r-1)/2}\begin{bmatrix} k \\ r \end{bmatrix}_{q(n)}\right)4\|\varphi^2 g''\|. \quad (3.11)\end{aligned}$$

Since

$$\frac{k+2}{[k+2]_{q(n)}}\frac{k+1}{[k+1]_{q(n)}} - 1 \leqslant \frac{n}{[n]_{q(n)}}\frac{n-1}{[n-1]_{q(n)}} - 1$$

for $k \in \{0, 1, \ldots, n-2\}$ and the q-binomial coefficients are increasing functions of q, we get from (3.11) and (3.1) that

$$\begin{aligned}&\left|\varphi^2(x)B''_{n,q(n)}(g,x) - \varphi^2(x)\tilde{B}_{n-2,q(n)}([n]_{q(n)}[n-1]_{q(n)}\Delta^2_{q(n)}g,x)\right| \\ &\quad \leqslant 4[n]^2_{q(n)}\|\varphi^2 g''\|\left(\frac{n}{[n]_{q(n)}}\frac{n-1}{[n-1]_{q(n)}} - 1\right)\sum_{k=0}^{n-2}\binom{n-2}{k}\left(\sum_{r=0}^{k}\binom{r}{s}\right) \\ &\quad = 4[n]^2_{q(n)}3^{n-2}\left(\frac{n}{[n]_{q(n)}}\frac{n-1}{[n-1]_{q(n)}} - 1\right)\|\varphi^2 g''\| \\ &\quad \leqslant C\|\varphi^2 g''\|. \qquad (3.12)\end{aligned}$$

Furthermore, by (1.1),

$$\varphi^2(x)\tilde{B}_{n-2,q(n)}([n]_{q(n)}[n-1]_{q(n)}\Delta^2_{q(n)}g,x) = \sum_{k=0}^{n-2}[n]_{q(n)}[n-1]_{q(n)}\Delta^2_{q(n)}g_k\varphi^2(x)p_{n-2,k,q(n)}(x).$$

Hence, using Lemma 3.2, we find that

$$\begin{aligned}&|\varphi^2(x)\tilde{B}_{n-2,q(n)}([n]_{q(n)}[n-1]_{q(n)}\Delta^2_{q(n)}g,x)| \\ &\quad \leqslant \sum_{k=0}^{n-2}[n]^2_{q(n)}|\Delta^2_{q(n)}g_k|\varphi^2(x)p_{n-2,k,q(n)}(x) \\ &\quad \leqslant 2q(n)(1+q(n))[n]_{q(n)}\|\varphi^2 g''\|\varphi^2(x)p_{n-2,0,q(n)}(x) + \tfrac{1}{2}(1+q(n))^2 \\ &\qquad \times \|\varphi^2 g''\|\sum_{k=1}^{n-3}q(n)^{2k+1}\varphi^{-2}\left(\frac{[k+1]_{q(n)}}{[n]_{q(n)}}\right)\varphi^2(x)p_{n-2,k,q(n)}(x) \\ &\qquad + 2q(n)^{n-3}(1+q(n))[n]_{q(n)}\|\varphi^2 g''\|\varphi^2(x)p_{n-2,n-2,q(n)}(x). \qquad (3.13)\end{aligned}$$

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But

$$\varphi^2(x)p_{n-2,0,q(n)}(x) = x(1-x)^2(1-xq(n))\cdots(1-xq(n)^{n-3}) \leqslant x(1-xq(n)^{n-3})^{n-1}$$

and $x(1-xq(n)^{n-3})^{n-1}$ has its maximum at $x = 1/(nq(n)^{n-3})$ on $[0,1]$. Therefore,

$$\varphi^2(x)p_{n-2,0,q(n)}(x) \leqslant \frac{1}{nq(n)^{n-3}}\left(1-\frac{1}{n}\right)^{n-1} < \frac{1}{nq(n)^{n-3}}. \tag{3.14}$$

A similar argument gives the following estimate for $t = n-2$:

$$\varphi^2(x)p_{n-2,n-2,q(n)}(x) = x^{n-1}(1-x) \leqslant \left(1-\frac{1}{n}\right)^{n-1}\frac{1}{n} < \frac{1}{n}. \tag{3.15}$$

At the same time, for $t \in \{1,2,\dots,n-3\}$, we have

$$\begin{aligned} q(n)^{2k+1}\varphi^{-2}\left(\frac{[k+1]_{q(n)}}{[n]_{q(n)}}\right)&\varphi^2(x)p_{n-2,k,q(n)}(x) \\ &= q(n)^{2k+1}\frac{[n]^2_{q(n)}}{[k+1]_{q(n)}([n]_{q(n)}-[k+1]_{q(n)})}\begin{bmatrix} n-2 \\ k \end{bmatrix}_{q(n)} \\ &\quad\times x^{k+1}(1-x)^2(1-xq(n))\cdots(1-xq(n)^{n-3-k}) \\ &< q(n)^{2k+1}\frac{[n]^2_{q(n)}}{[k+1]_{q(n)}([n]_{q(n)}-[k+1]_{q(n)})}\begin{bmatrix} n-2 \\ k \end{bmatrix}_{q(n)} \\ &\quad\times x^{k+1}(1-x)(1-xq(n))\cdots(1-xq(n)^{n-3-k})(1-xq(n)^{n-2-k}). \end{aligned}$$

Using [**9**, (2.3), (2.4), p. 214], we get

$$\begin{aligned} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{q(n)} &= \begin{bmatrix} n-1 \\ k+1 \end{bmatrix}_{q(n)}\frac{[k+1]_{q(n)}}{[n-1]_{q(n)}} \\ &= \begin{bmatrix} n \\ k+1 \end{bmatrix}_{q(n)}\frac{[n-k-1]_{q(n)}}{[n]_{q(n)}}\frac{[k+1]_{q(n)}}{[n-1]_{q(n)}}. \end{aligned}$$

Thus,

$$\begin{aligned} q(n)^{2k+1}\varphi^{-2}\left(\frac{[k+1]_{q(n)}}{[n]_{q(n)}}\right)&\varphi^2(x)p_{n-2,k,q(n)}(x) \\ &< q(n)^{2k+1}\frac{[n]^2_{q(n)}}{[k+1]_{q(n)}([n]_{q(n)}-[k+1]_{q(n)})}\frac{[n-k-1]_{q(n)}}{[n]_{q(n)}}\frac{[k+1]_{q(n)}}{[n-1]_{q(n)}} \\ &\quad\times\begin{bmatrix} n \\ k+1 \end{bmatrix}_{q(n)} x^{k+1}(1-x)(1-xq(n))\cdots(1-xq(n)^{n-3-k})(1-xq(n)^{n-2-k}) \\ &= q(n)^{2k+1}\frac{[n]_{q(n)}}{[n-1]_{q(n)}}\frac{[n-k-1]_{q(n)}}{[n]_{q(n)}-[k+1]_{q(n)}}p_{n,k+1,q(n)}(x). \end{aligned}$$

On the other hand,

$$\frac{[n]_{q(n)}}{[n-1]_{q(n)}} = \frac{1-q(n)^n}{1-q(n)^{n-1}} < 1+q(n)$$

and

$$\frac{[n-k-1]_{q(n)}}{[n]_{q(n)}-[k+1]_{q(n)}} = \frac{1+q(n)+\cdots+q(n)^{n-k-2}}{q(n)^{k+1}+q(n)^{k+2}+\cdots+q(n)^{n-1}} = \frac{1}{q(n)^{k+1}}.$$

In conclusion,

$$\begin{aligned} q(n)^{2k+1}\varphi^{-2}\biggl(\frac{[k+1]_{q(n)}}{[n]_{q(n)}}\biggr)\varphi^2(x)p_{n-2,k,q(n)}(x) &< q(n)^k(1+q(n))p_{n,k+1,q(n)}(x) \\ &< 2p_{n,k+1,q(n)}(x). \end{aligned} \tag{3.16}$$

Combining the estimates (3.13)–(3.16), we find that

$$\begin{aligned} &|\varphi^2(x)\tilde{B}_{n-2,q(n)}([n]_{q(n)}[n-1]_{q(n)}\Delta^2_{q(n)}g,x)| \\ &\quad\leqslant 2q(n)(1+q(n))[n]_{q(n)}\frac{1}{nq(n)^{n-3}}\|\varphi^2 g''\| \\ &\qquad + \tfrac{1}{2}(1+q(n))^2\|\varphi^2 g''\|\sum_{k=1}^{n-3}2p_{n,k+1,q(n)}(x) \\ &\qquad + 2q(n)^{n-3}(1+q(n))[n]_{q(n)}\frac{1}{n}\|\varphi^2 g''\| \\ &\quad\leqslant \frac{4}{q(n)^n}\frac{[n]_{q(n)}}{n}\|\varphi^2 g''\| + 4\|\varphi^2 g''\|\sum_{k=1}^{n-3}p_{n,k+1,q(n)}(x) + 4\frac{[n]_{q(n)}}{n}\|\varphi^2 g''\| \\ &\quad\leqslant C\|\varphi^2 g''\|, \end{aligned} \tag{3.17}$$

where we have used $[n]_{q(n)} \leqslant n$, (2.1) and (3.2). Now, from (3.12) and (3.17), we arrive at the desired estimate, which completes the proof of the lemma. □

Proof of Theorem 3.1. By taking into account the definition of the K-functional $K_{2,\varphi}(f,\delta^2)$, (3.3) and Lemma 3.3, we obtain

$$\begin{aligned} K_{2,\varphi}(f,n^{-1}) &\leqslant \|f-B_{k,q(k)}f\| + n^{-1}\|\varphi^2 B''_{k,q(k)}f\| \\ &\leqslant \|f-B_{k,q(k)}f\| + n^{-1}(\|\varphi^2 B''_{k,q(k)}(f-g)\| + \|\varphi^2 B''_{k,q(k)}g\|) \\ &\leqslant C[k]^{-\alpha/2}_{q(k)} + Cn^{-1}(k\|f-g\| + \|\varphi^2 g''\|) \\ &\leqslant C\biggl([k]^{-\alpha/2}_{q(k)} + \frac{k}{n}\biggl(\|f-g\| + \frac{1}{k}\|\varphi^2 g''\|\biggr)\biggr). \end{aligned}$$

Taking infimum over all $g \in W^2(\varphi)$, we find that

$$K_{2,\varphi}(f,n^{-1}) \leqslant C\biggl([k]^{-\alpha/2}_{q(k)} + \frac{k}{n}K_{2,\varphi}(f,k^{-1})\biggr).$$

Moreover, by (3.1), we obtain that the sequence $(n/[n]_{q(n)})$ is also bounded. Then

$$K_{2,\varphi}(f,n^{-1}) \leqslant C\biggl(k^{-\alpha/2} + \frac{k}{n}K_{2,\varphi}(f,k^{-1})\biggr).$$

Hence, in view of the Berens–Lorentz lemma (see [**1**] or [**2**, p. 312]), we have

$$K_{2,\varphi}(f,n^{-1}) \leqslant Cn^{-\alpha/2}, \quad n = 1,2,\dots. \tag{3.18}$$

For every $0 < \delta < 1$ there exists n such that

$$\frac{1}{\sqrt{n+1}} < \delta < \frac{1}{\sqrt{n}}.$$

Then the definition of $K_{2,\varphi}(f,\delta^2)$ and (3.18) imply $K_{2,\varphi}(f,\delta^2) \leqslant C\delta^\alpha$. By (1.5) we obtain $\omega_\varphi^2(f,\delta) \leqslant C\delta^\alpha$, which was to be proved. □

Remark 3.4. There exist sequences $(q(n))$ which satisfy conditions (3.1) and (3.2). An example of such a sequence is $q(n) = 1 - 1/na^n$, where $a > 27$ and $n = 1,2,\dots$.

Remark 3.5. Condition (3.2) provides that $q(n) \to 1$ as $n \to \infty$. Thus, in Theorem 3.1 we have the uniform convergence of $B_{n,q(n)}f$ to f on $[0,1]$ (see [**7**, **8**]).

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