BULL. AUSTRAL. MATH. SOC. MOS 22A15, 20M20

VOL. 4 (1971), 137-139. (54H99)

A remark on compact semigroups having certain decomposition spaces embeddable in the plane

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Let S be a compact connected semigroup. If the decomposition space of S, under the action of a compact connected subgroup at the identity, is a plane continuum then this decomposition is a congruence.

Let S be a compact semigroup with identity 1 and H_1 the maximal subgroup at 1. In general, the orbits xH_1 do not form a congruence on S as various examples of semi-direct products show.

In [2] it is shown that if G is a compact connected subgroup of H_1 such that S/G is one dimensional then the orbits of G form a congruence. Such an S must then contain a local thread lying in the centralizer of G.

We remark here that the same result holds if S/G is a plane continuum. It should be noted that there exist one dimensional compact connected semigroups with identity and zero which are not plane continua [4].

PROPOSITION. *Let S be a compact connected semigroup with identity and let G be a compact connected subgroup of H_1 such that S/G, the hyperspace of orbits xG is embeddable in the plane. Then these orbits form a congruence. Moreover S contains a local thread at 1 which lies in the centralizer of G.*

Received 18 September 1970.

https://doi.org/10.1017/S0004972700046347 Published online by Cambridge University Press

Indication of proof. If we do not have $Gx \subseteq xG$ for all $x \in S$, which is the condition that the orbits form a congruence, then G can be considered as a transformation group of the plane continuum S/G, the action being, of course, $g\{xG\} = \{gxG\}$. If the orbit $G\bar{x}$ (where $\bar{x} = \{xG\}$) is non-degenerate it must be a simple closed curve. Now by arguments quite similar to those employed in [4] or [1] it follows that given two non-degenerate orbits $G\bar{x}_0$ and $G\bar{x}$, then one of these simple closed curves must enclose the other. Next we note that a non-degenerate orbit say, $G\bar{t}$ unioned with that portion of S/G lying in the unbounded complementary domain of $G\bar{t}$, is a subcontinuum $S_{\bar{t}}$ of S/G filled up in a continuous manner by the orbits. In particular, the union of the non-degenerate orbits of points of $S_{\bar{t}}$ is a closed subset of $S_{\bar{t}}$. The degenerate orbits clearly form a closed set. Since $S_{\bar{t}}$ is a continuum all this is impossible were there to exist a degenerate orbit defined by a point of $S_{\bar{t}}$. Thus a degenerate orbit must be enclosed by every non-degenerate one, and, in particular $G\bar{1} = \{\bar{1}\}$ is enclosed by every non-degenerate orbit. Since the collection of orbits is a continuous collection it follows that every orbit save $G\bar{1}$ is non-degenerate. (Every non-degenerate orbit must enclose yet another.) In fact, under the supposition that the orbits did not form a congruence, we see that S/G is now a disc with $\bar{1}$ as an interior point.

It now follows that K/G separates the plane. However, S/K, the the maximal ideal of S. However, S/K, the Rees quotient, acts on $S/G/T$, where T is the decomposition formed by collapsing K/G to a point. All of this is impossible since the latter space is a sphere or a sphere tangent to a disc and cannot be acted upon by a compact connected semigroup with identity and zero.

Thus the natural map $\rho : S \to S/G$ is a continuous (open) homomorphism of S onto a plane semigroup. By the proposition of [5] we know that S/G contains a local thread at the identity, say I. Considering the closure of $\rho^{-1}(I)$ the desired local thread exists, arguing as in [2], [3].

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https://doi.org/10.1017/S0004972700046347 Published online by Cambridge University Press