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Poles of Siegel Eisenstein Series on $U(n,n)$

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Abstract. Let $U(n,n)$ be the rank n quasi-split unitary group over a number field. We show that the normalized Siegel Eisenstein series of $U(n,n)$ has at most simple poles at the integers or half integers in certain strip of the complex plane.

1 Introduction

The family of degenerate Eisenstein series associated to certain maximal parabolic subgroup of a classical group plays an important role in the theory of Rankin-Selberg representations of L-functions. The analytic properties of the L-functions are closely related to those of such Eisenstein series. This family of Eisenstein series, which are called Siegel Eisenstein series, is also the main object in the regularized Siegel-Weil formula [KR2], [Tan], which identifies the residue of the Eisenstein series as certain theta function. Therefore, it is essential to determine the poles of the Eisenstein series. Such results for the symplectic and orthogonal groups have been obtained by Kudla and Rallis in [KR1]. They use this result in the regularized Siegel-Weil formula for the dual pair $\mathrm{Sp}(n)$ and $O(m)$ (for any n and m) [KR2], which in turn provides information about the poles of the standard Langlands L-functions of $\mathrm{Sp}(n)$. In [Tan], the author proved the Siegel-Weil formula for the dual pair $U(2,2)$ and $U(3)$. In order to generalize the result to unitary groups of arbitrary rank, we need to know the analytic behaviour of the Eisenstein series of $U(n,n)$ for any n, which is the main goal of this paper. This information will allow us to determine the poles of the standard Langlands L-functions of $U(n,n)$ using the idea similar to that in [KR1].

We shall set up the notation and state the main theorem of the paper in this section. Then in Section 2 we introduce an intertwining operator which occurs in a functional equation of the Siegel Eisenstein series. We compute the Fourier coefficients of the Eisenstein series in Section 3. Finally, the proof of the main theorem is given in Section 4.

Let F be a totally real number field and E a quadratic extension of F. The nontrivial Galois automorphism of E/F is denoted by bar and the absolute valuation of F and E by $|\cdot|$ and $\|\cdot\|$ respectively. Let v be a place of F. When v is inert, E_v is a quadratic extension of F_v; when v splits as $\varpi_1\varpi_2$, $E_v = E_{\varpi_1} \oplus E_{\varpi_2}$ and $E_{\varpi_i} \cong F_v$. In the non-archimedean case, let $q_{F_v} = q_v$ and q_{E_v} be the order of the residue field of F_v and E_v respectively. Also, we fix uniformizing elements $\pi_{F_v} = \pi_v$ and π_{E_v} and normalize $|\cdot|_v$ and $\|\cdot\|_v$ by $|\pi_v|_v = q_v^{-1}$ and $\|\pi_{E_v}\|_v = q_{E_v}^{-1}$ respectively.

Let G be the *rank n quasi-split* unitary group $U(W)$ (or $U(n,n)$ when the underlying space is clear) where $W = W_{n,n}$ is a split Hermitian space of dimension $2n$ over E with respect to the Hermitian form $(\,,)$ given by the matrix $\left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$. Let

$$W_{n,n} = W_{n,0} \oplus W_{0,n}$$

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be a complete polarization of W and P the maximal parabolic subgroup of G that stabilizes $W_{0,n}$. We call P the Siegel parabolic subgroup of G. We have a *Levi decomposition* of P into MN where

$$M := \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t\overline{a}^{-1} \end{pmatrix} \;\Big|\; a \in \mathrm{GL}(n,E) \right\}$$

is the Levi factor of P, and

$$N := \left\{ n(b) = \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \;\Big|\; b \in SH_n(F) \right\}$$

is the unipotent radical of P where $SH_n(F)$ is the space of n by n skew-Hermitian symmetric matrices with respect to E/F, *i.e.*,

$$SH_n(F) = \{ g \in M_{n\times n}(E) \mid {}^t\bar{g} = -g \}.$$

We shall denote by $a(g)$ the $\mathrm{GL}(n,E)$ element a appearing in the Levi factor $m(a)$ of $g \in G$.

For each place υ of F, we denote $G(F_\upsilon)$ by G_υ. Then $P_\upsilon, M_\upsilon, N_\upsilon$ can be defined similarly. Note that in case υ is splits, we can identify M_υ with

$$\left\{ m(a) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \;\Big|\; a = (a_1, a_2) \in \mathrm{GL}(n, F_\upsilon) \times \mathrm{GL}(n, F_\upsilon) \right\}$$

and N_υ with

$$\left\{ n(b) = \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \;\Big|\; b \in M_{n\times n}(F_\upsilon) \right\}.$$

For each G_υ, we fix a special maximal compact subgroup K_υ. In particular, for non-archimedean υ, we choose $K_\upsilon = G(\mathcal{O}_\upsilon)$ where $\mathcal{O}_\upsilon$ is the ring of integer of F_υ. Let $K = \prod_\upsilon K_\upsilon$.

Let γ be a Hecke character of $E^\times$. We will write $\hat{\gamma}$ for the character $\hat{\gamma}(x) = \gamma(\bar{x})^{-1}$. Note that $\gamma = \hat{\gamma}$ if and only if $\gamma|_{F^\times} = 1$ or $\gamma|_{F^\times} = \omega_{E/F}$, the quadratic character of the extension E/F defined by class field theory. We denote by γ_υ the local component of γ, *i.e.*, a character of $E_\upsilon^\times$. We can view γ as a character of $M(\mathbb{A})$ by setting $\gamma\big(m(a)\big) = \gamma\big(\det(a)\big)$. Sometimes we simply write it as $\gamma(a)$.

Let $I(s,\gamma) = \mathrm{Ind}_P^G(\gamma\|\cdot\|^s)$ be the *normalized* induced representation of $G(\mathbb{A})$ inducing from the (quasi-)character $\gamma\|\cdot\|^s$ of $M(\mathbb{A})$ which extends trivially across $N(\mathbb{A})$. Its space consists of all smooth functions Φ on $G(\mathbb{A})$ such that

$$\Phi\big(nm(a)g\big) = \gamma(a)\|\det a\|^{s+\frac{n}{2}}\Phi(g)$$

and Φ is K_υ-*finite* at every archimedean place υ. We can define similarly for each υ, $I_\upsilon(s,\gamma_\upsilon) = \mathrm{Ind}_{P_\upsilon}^{G_\upsilon}(\gamma_\upsilon\|\cdot\|^s)$. In the case υ is split, $I_\upsilon(s,\gamma_\upsilon) \cong \mathrm{Ind}_{P_\upsilon}^{G_\upsilon}(\gamma_\upsilon|\cdot|^s \otimes \gamma_\upsilon|\cdot|^{-s})$. As in [KS], we can describe $\gamma_\upsilon\|\cdot\|^s$ by the pair (γ_υ, s). When υ is non-archimedean, we have to normalize s and γ by requiring $\gamma_\upsilon(\pi_{E_\upsilon}\cdot\bar{\pi}_{E_\upsilon}) = 1$ and $\mathrm{Im}(s) \in [0, \frac{\pi}{\log(q_{E_\upsilon})})$ for υ inert and $\mathrm{Im}(s) \in [0, \frac{\pi}{\log(q_{F_\upsilon})})$ for υ splits. Then there is a one-one correspondence between the quasi-character and the normalized pair (see [KS]). The points of reducibility of $I_\upsilon(s,\gamma_\upsilon)$ are then given as follows.

Lemma 1.1 *If $\gamma \neq \hat{\gamma}$, then $I_v(s, \gamma_v)$ is irreducible for all s and all v. If $\gamma = \hat{\gamma}$, then*

(i) when v is non-archimedean (character not equal 2) and inert, $I_v(s, \gamma_v)$ is irreducible except when s belongs to $\{-\frac{n-i}{2}, 1 - \frac{n-i}{2}, 2 - \frac{n-i}{2}, \dots, \frac{n-i}{2}\}$ where $\gamma_v|_{F_v^\times} = \omega^i_{E_v/F_v}$ and $i = 0$ or 1;

(ii) when v is non-archimedean and splits, $I_v(s, \gamma_v)$ is irreducible as $\mathrm{GL}(2n, F_v)$-module except when $s \in \{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots, \pm\frac{n}{2}\}$;

(iii) when v is archimedean, $I_v(s, \gamma_v)$ is irreducible except when $s \in \{m + \frac{n-i}{2} | m \in \mathbb{Z}\}$ where $\gamma_v|_{F_v^\times} = \omega^i_{E_v/F_v}$ and $i = 0$ or 1.

Proof The non-archimedean case is given in [KS]; the archimedean case can be derived easily from [Lee]. ■

Allowing s to vary over the complex plane, we obtain a *section* $\Phi(s) \in I(s, \gamma)$. A *standard* section is one whose restriction to K is independent of s. Now for v a non-archimedean place at which γ_v is *unramified*, *i.e.*, $\gamma_v(x) = 1$ for all $x \in \mathcal{O}^\times_{E_v}$, let $\Phi^0_v(s) \in I_v(s, \gamma_v)$ be the *normalized* standard local section, *i.e.*, $\Phi^0_v(k, s) = 1$ for all $k \in K_v$. Then

$$I(s, \gamma) \cong \otimes'_v I_v(s, \gamma_v)$$

where the right hand side is the restricted tensor product with respect to $\Phi^0_v(s)$.

Given $\Phi(s) \in I(s, \gamma)$, we define the Siegel Eisenstein series:

$$E^n(g, s, \Phi) = E(g, s, \Phi) = \sum_{\varepsilon \in P(F)\backslash G(F)} \Phi(\varepsilon g, s).$$

(We shall suppress the superscript n from the notation when it is understood that we are refering to the Eisenstein series of $U(n, n)$.) This series converges for $\mathrm{Re}(s) > n/2$ and has a meromorphic continuation to the whole s-plane. It satisfies a functional equation

$$E(g, s, \Phi) = E\big(g, -s, M(s)\Phi\big)$$

where $M(s)$ is an intertwining operator from $I(s, \gamma)$ to $I(-s, \hat{\gamma})$ which we will define in the next section. $E(g, s, \Phi)$ defines an element of $\mathcal{A}(G)$, *i.e.*, an automorphic form on $G(\mathbb{A})$ whenever it does not have a pole [Art].

If we normalize the Eisenstein series by setting

$$E^{n,*}(g, s, \Phi) = b_n^{*,\mathcal{S}'}(s) E^n(g, s, \Phi)$$

where $b_n^{*,\mathcal{S}'}(s)$ is a certain partial product of local Tate L-factors (to be defined in the subsequent sections), we can state the main result of this paper:

Main Theorem *The normalized Siegel Eisenstein series $E^{n,*}(g, s, \Phi)$ is entire if $\gamma \neq \hat{\gamma}$. If $\gamma = \hat{\gamma}$, then $E^{n,*}(g, s, \Phi)$ can have at most simple poles, and these can only occur at points s in the set*

$$X := \left\{-\frac{n-i}{2}, 1 - \frac{n-i}{2}, 2 - \frac{n-i}{2}, \dots, \frac{n-i}{2}\right\}$$

where $\gamma_v|_{F_v^\times} = \omega^i_{E_v/F_v}$ and $i = 0$ or 1.

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2 Intertwining Operator

We now define the *intertwining operator*

$$M(s)\colon I(s,\gamma) \longrightarrow I(-s,\hat{\gamma})$$

for $\mathrm{Re}(s) > \frac{n}{2}$ as follows:

$$M(s)\Phi(g) = \int_{N(\mathbb{A})} \Phi(wng)dn = \int_{SH_n(F)} \Phi\bigl(wn(b)g\bigr)\, db$$

where $w = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \in G(\mathbb{A})$. The Haar measure db on $SH_n(F) \cong N(\mathbb{A})$ is self-dual in the following sense. Fix once and for all a non-trivial additive character ψ of $F^\times$. For $\phi \in S\bigl(SH_n(F)\bigr)$, the space of locally compact, compactly supported functions on $SH_n(F)$, we define the Fourier transform of ϕ by $\hat{\phi}(x) = \int_{SH_n(F)} \phi(b)\psi\bigl(\mathrm{tr}(bx)\bigr)\, db$. Then db is self-dual if $\hat{\hat{\phi}}(b) = \phi(-b)$. Again we can write $M(s) = \bigotimes_v M_v(s)$ where the right hand side is the tensor product and $M_v(s)\Phi_v(g) = \int_{N_v} \Phi_v(wng)\, dn$.

Proposition 2.1 *Let v be a non-archimedean place where γ_v is unramified. Then*

$$M_v(s)\Phi_v^0(s) = \frac{a_{n,v}(s)}{b_{n,v}(s)}\Phi_v^0(-s),$$

where

$$a_{n,v}(s) = \prod_{1\le i<j\le n} L_{E_v}(2s-n+i+j-1)\prod_{i=1}^{n} L_{F_v}(2s-n+2i-1,\gamma_v)$$

and $b_{n,v}(s) = a_{n,v}(s+1/2)$.

Here $L_{F_v}(s,\chi) = \frac{1}{1-\chi(\pi_v)q_v^{-1}}$ and $L_{E_v}(s,\chi) = \frac{1}{1-\chi(\pi_{E_v})q_v^{-2}}$ (resp. $\bigl(\frac{1}{1-\chi(\pi_{E_v})q_v^{-1}}\bigr)^2$) when v is inert (resp. splits) are the local Tate L-factors and $L_{E_v}(s) = L_{E_v}(s,1)$ where 1 denotes the trivial character of E_v.

Before we go into the proof of the proposition, let us recall the notion of *root system* for an algebraic group. Let Σ be the set of (F-)roots of G with respect to the maximal F-split torus T. Let Σ^+ (resp. Σ^-) be the positive (resp. negative) roots in Σ determined by N_B, the unipotent radical of B where B is the Borel subgroup of G. Explicitly, T has elements t of the form

$$\begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & {a_1}^{-1} & & \\ & & & & \ddots & \\ & & & & & {a_n}^{-1} \end{pmatrix}$$

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where $a_i \in F^\times$. For $1 \le i, j \le n$, let $\alpha^+_{ij}(t) = a_i a_j$ and $\alpha^-_{ij}(t) = a_i a_j^{-1}$. Then

$$\Sigma = \{\alpha^{\pm}_{ij}, 1 \le i, j \le n\}$$

and

$$\Sigma^+ = \{\alpha^{\pm}_{ij}, 1 \le i < j \le n\} \cup \{\alpha^+_{ii}, 1 \le i \le n\}.$$

For an element w in the Weyl group W_G, we have

$$\Sigma^+(w) = \{\alpha \in \Sigma^+ \mid w^{-1}\alpha \in \Sigma^-\}$$

where $w^{-1}\alpha(t) = \alpha(wtw^{-1})$.

For a positive root α, we let $G_\alpha = C_G(\ker \alpha)$. This is a connected reductive quasi-split group of *semi-simple F-rank* 1. Let $N_\alpha = N_B \cap G_\alpha$, the *one parameter unipotent subgroup* of N_B associated to α.

Proof of Proposition 2.1 Since the statement of the proposition only involves one local place, we shall omit the subscript v in this proof to simplify notation unless necessary. We shall use the method of Gindikin and Karpelevich [GPS] of reduction to F-rank 1 to compute $M(s) = M_v(s)$. Since Φ^0 is K-invariant, the action of $M(s)$ is just multiplication by the following constant which we again denote by $M(s)$:

$$M(s) = \int_N \Phi^0(wn)\, dn.$$

Also note that

$$\Phi^0 \in \mathrm{ind}_P^G(\gamma\|\cdot\|^{s+\frac{n}{2}})$$

where "ind" denotes the unnormalized induced representation. This is in turn contained in the normalized induced representation from the Borel

$$\mathrm{Ind}_B^G\big(\gamma(a_1\cdots a_n)\|a_1\cdots a_n\|^{s+\frac{n}{2}}\delta_B^{-\frac{1}{2}}\big)$$

where $\delta_B := \delta_{N_B}$ is the modular character of B given by $t\colon \mapsto \prod_i \|a_i\|^{2n-2i+1}$. Hence we have

$$\Phi^0 \in \mathrm{Ind}_B^G\Big(\gamma(a_1\cdots a_n)\prod_i \|a_i\|^{\frac{2s-n+2i-1}{2}}\Big). \tag{2.2}$$

By [Lai, Proposition 4.4],

$$M(s) = \prod_{\alpha\in\Sigma^+(w)} \int_{\bar{N}_\alpha} \Phi^0_\alpha(n)\, dn$$

where Φ^0_α is the restriction of Φ^0 to G_α and $\bar{N}_\alpha$ is the opposite of N_α. We can easily check that

$$\Sigma^+(w) = \{\alpha^+_{ij},\ \ 1 \le i \le j \le n\}.$$

We shall apply the formula in [Lai, Section 3] to compute $M_\alpha = \int_{\bar{N}_\alpha} \Phi^0_\alpha(n)\,dn$ for each α in $\Sigma^+(w)$.

(i) $\underline{\alpha = \alpha^+_{ij} \text{ for } i < j}$

For this α, we check that $G_\alpha \cong \mathrm{GL}(2,E) \times (E^\times)^{n-2}$. We can further reduce our computation to the derived group of G_α since it yields the same $M_\alpha(s)$. We have $G'_\alpha \cong \mathrm{SL}(2,E)$ and T'_α is the subgroup of T_B whose element t has 1 along the diagonals except the i-th and j-th entries where they are a and $\bar{a}$ respectively for some $a \in E^\times$. The modular character $\delta_{B'_\alpha} := \delta_{N'_\alpha}$ of B'_α is given by $t \mapsto \|a\|^2$.

We have to consider the cases when υ is inert and splits in E separately. When υ is inert, we check using (2.2) that

$$\Phi^0_\alpha \in \mathrm{Ind}^{G'_\alpha}_{B'_\alpha}\big(\gamma(a\bar{a})\|a\|^{\frac{2s-n+2i-1}{2}}\|\bar{a}\|^{\frac{2s-n+2j-1}{2}}\big) = \mathrm{Ind}^{G'_\alpha}_{B'_\alpha}(\delta_{B'_\alpha}^{\frac{2s-n+i+j-1}{2}})$$

since we have normalized γ such that $\gamma(\pi_E \cdot \bar{\pi}_E) = 1$.

Hence, in this case, we have by [Lai, (3.5)]

$$M_\alpha(s) = \frac{1 - q^{-2(2s-n+i+j)}}{1 - q^{-2(2s-n+i+j-1)}}.$$

When υ splits in E as ϖ_1 and ϖ_2, we have $E_\upsilon \cong E_{\varpi_1} \oplus E_{\varpi_2}$ where $E_{\varpi_i} \cong F_\upsilon$ for $i = 1, 2$. We identify $a \in E_\upsilon$ with $(a_1, a_2) \in E_{\varpi_1} \oplus E_{\varpi_2}$. Note then that $\bar{a} = (a_2, a_1)$.

In this case, $G'_\alpha \in \mathrm{SL}(2, E_{\varpi_1}) \times \mathrm{SL}(2, E_{\varpi_2})$. Again by (2.2), we have

$$\Phi^0_\alpha \in \mathrm{Ind}^{G'_\alpha}_{B'_\alpha}\Big(\gamma\big((a_1,a_2)(a_2,a_1)\big)\|(a_1,a_2)\|^{\frac{2s-n+2i-1}{2}}\|(a_2,a_1)\|^{\frac{2s-n+2j-1}{2}}\Big).$$

Simplifying this last expression, it becomes

$$\begin{aligned}\mathrm{Ind}^{G'_\alpha}_{B'_\alpha}\big(\|(a_1,a_2)\|^{2s-n+i+j-1}\big) &\cong \mathrm{Ind}^{G'_{\alpha,\varpi_1}}_{B'_{\alpha,\varpi_1}}(|a_1|^{2s-n+i+j-1}) \times \mathrm{Ind}^{G'_{\alpha,\varpi_2}}_{B'_{\alpha,\varpi_2}}(|a_2|^{2s-n+i+j-1}) \\ &= \mathrm{Ind}^{G'_{\alpha,\varpi_1}}_{B'_{\alpha,\varpi_1}}(\delta_{B'_{\alpha,\varpi_1}}^{\frac{2s-n+i+j-1}{2}}) \times \mathrm{Ind}^{G'_{\alpha,\varpi_2}}_{B'_{\alpha,\varpi_2}}(\delta_{B'_{\alpha,\varpi_2}}^{\frac{2s-n+i+j-1}{2}})\end{aligned}$$

where $G'_{\alpha,\varpi_k} \cong \mathrm{SL}(2, E_{\varpi_k})$. We have used the fact that

$$\gamma_\upsilon\big((a,a)\big) = \gamma_{\varpi_1}(a)\gamma_{\varpi_2}(a) = 1.$$

Since $E_{\varpi_k} \cong F_\upsilon$, [Lai, (3.30)] gives

$$M_\alpha(s)_{\varpi_k} = \frac{1 - q^{-(2s-n+i+j)}}{1 - q^{-(2s-n+i+j-1)}}$$

for $k = 1, 2$ and

$$M_\alpha(s) = \left(\frac{1 - q^{-(2s-n+i+j)}}{1 - q^{-(2s-n+i+j-1)}}\right)^2.$$

(ii) $\underline{\alpha = \alpha^+_{ii}}$

For such α, $G_\alpha \cong U(1,1)(F)\times(E^\times)^{n-1}$. Hence $G'_\alpha \cong \mathrm{SL}(2,F)$ and T'_α is the subgroup of T_B whose elements t has all 1 along the diagonals except the i-th and $n+i$-th entry where it can be any $a \in F^\times$. The modular character $\delta_{B'_\alpha}$ in this case is given by $t \mapsto \|a\|$.

Since the group is isomorphic to $\mathrm{SL}(2,F)$, we do not need to concern about the splitting of υ in this case.

By (2.2) again, we have

$$\Phi^0_\alpha \in \mathrm{Ind}_{B'_\alpha}^{G'_\alpha}\big(\gamma(a)\|a\|^{\frac{2s-n+2i-1}{2}}\big) = \mathrm{Ind}_{B'_\alpha}^{G'_\alpha}(\gamma\delta_{B'_\alpha}^{\frac{2s-n+2i-1}{2}}).$$

Hence

$$M_\alpha(s) = \frac{1-\gamma(\pi)q^{-(2s-n+2i)}}{1-\gamma(\pi)q^{-(2s-n+2i-1)}}.$$

By taking the product of the $M_\alpha(s)$'s over $\Sigma^+(w)$, we have the proposition. ∎

Proposition 2.1 says that, if we multiply $M_\upsilon(s)$ by $\frac{b_{n,\upsilon}(s)}{a_{n,\upsilon}(s)}$, the resulting intertwining operator will send $\Phi^0_\upsilon(s)$ to $\Phi^0_\upsilon(-s)$ whenever γ_υ is unramified. In particular, this operator is non-vanishing on $I_\upsilon(s)$. In [KS, Section 3], it has been shown that $M_\upsilon(s)$ can be meromorphically continued to the whole s-plane with only finitely many simple poles. In other words, we can normalize $M_\upsilon(s)$ so that the resulting $M^*_\upsilon(s)$ is entire and non-vanishing at every s. The normalization is given as follows: First of all, we express the L_{E_υ}-factors appearing in $a_{n,\upsilon}(s)$ and $b_{n,\upsilon}(s)$ in terms of L_{F_υ}. Recall that

$$L_{E_\upsilon}(s) = \begin{cases} \frac{1}{1-q_\upsilon^{-2}} & \upsilon \text{ inert} \\ (\frac{1}{1-q_\upsilon^{-1}})^2 & \upsilon \text{ split.}\end{cases}$$

Then $L_{E_\upsilon}(s) = L_{F_\upsilon}(s)L_{F_\upsilon}(s,\omega_{E_\varpi/F_\upsilon})$ where $\varpi = \upsilon$ (resp. ϖ_1) if υ is inert (resp. splits). Now let $a^*_{n,\upsilon}(s)$ and $b^*_{n,\upsilon}(s)$ be given by

$$\frac{a^*_{n,\upsilon}(s)}{b^*_{n,\upsilon}(s)} = \frac{a_{n,\upsilon}(s)}{b_{n,\upsilon}(s)}$$

such that they have no common L_{F_υ}-factors. Then

$$M^*_\upsilon(s) = \frac{1}{a^*_{n,\upsilon}(s)}M_\upsilon(s).$$

Explicitly,

$$a^*_{n,\upsilon}(s) = \prod_{i=1}^n L_{F_\upsilon}(2s+i-n, \gamma_\upsilon\omega^{i-1}_{E_\varpi/F_\upsilon})$$

and

$$b^*_{n,\upsilon}(s) = \prod_{i=1}^n L_{F_\upsilon}(2s+i, \gamma_\upsilon\omega^{i+n}_{E_\varpi/F_\upsilon}) \tag{2.3}$$

Note that this coincides with the computation given in [KS]. This is true for all non-archimedean υ including those ramified ones.

We will refer to $M^*_\upsilon(s)$ as the *normalized* intertwining operator.

3 Fourier Coefficients of $E(g,s,\Phi)$

Let $\beta \in SH_n(F)$ with $\det\beta \neq 0$. With the additive character ψ, we let $\psi_\beta(x) = \psi\big(\mathrm{tr}(x\beta)\big)$ for $x \in SH_n(F)$. Then the β-th-Fourier coefficient of $E(g,s,\Phi)$ is given by

$$\begin{aligned} E_\beta(g,s,\Phi) &= \int_{N(F)\backslash N(\mathbb{A})} E\big(n(b)g,s,\Phi\big)\psi_\beta(-b)\,db \\ &= \prod_v W_{\beta,v}(g_v,s,\Phi_v) \end{aligned}$$

where $W_{\beta,v}(g_v,s,\Phi_v) = \int_{SH_n(F_v)} \Phi_v\big(w_v n(b_v)g_v,s\big)\psi_{\beta,v}(-b_v)\,db_v$ which extends to an entire function of s [Kar], [Wal]. Here $w = \left(\begin{smallmatrix} 0 & I_n \\ I_n & 0\end{smallmatrix}\right)$. Our aim is to determine the analytic properties of these non-singular Fourier coefficients.

Recall that, when γ_v is unramified, we denote by $\Phi_v^0(s)$ the normalized standard section in $I_v(s,\gamma_v)$. We let $\mathcal{S}(g,\beta) = \mathcal{S}$ be a finite set of places of F such that the complement $\mathcal{S}^{\mathbf{c}}$ satisfies

$$(3.1)\quad \begin{aligned} \mathcal{S}^{\mathbf{c}} \subseteq \{&\text{places } v \text{ of } F \text{ such that } v \text{ is non-archimedean and unramified in } E;\\ &\Phi_v = \Phi_v^0;\ \gamma_v \text{ and } \psi_v \text{ unramified; } \mathrm{ord}_v \det\beta = 0;\ g_v \in G(\mathcal{O}_v)\}\end{aligned}$$

We are going to show that, for $v \notin \mathcal{S}$, $W_{\beta,v}(e,s,\Phi_v^0)$ can be expressed as a product of local L-factors. In fact, this is given in [Shi]. Let us translate it into our notation:

Our Eisenstein series $E(g,s,\Phi)$ corresponds to $E_0(g) = E_0(g,2s,n,\gamma,\Omega)$ in the notation of Shimura [Shi, (2.17a)]. Here Ω is the integral ideal of F where

$$\Omega = \prod_{v\in\mathcal{S}} \mathcal{P}_v$$

and $\mathcal{P}_v$ is the maximal ideal in $\mathcal{O}_v$. For $g = n(b)g_\infty w_{\mathbf{f}}^{-1}$ where $n(b) \in N(\mathbb{A})$, $g_\infty \in G_\infty$, the infinite part of $G(\mathbb{A})$, and $w_{\mathbf{f}}$ the finite part of w, we have the Fourier expansion [Shi, (3.5(a)]

$$E_0(g,2s,n,\gamma,\Omega) = \sum_{\beta\in SH_n(F)} d(\beta,g_\infty,2s)\psi_\beta(b).$$

So

$$\begin{aligned} d(\beta,g_\infty,2s) &= \int_{SH_n(F)\backslash SH_n(\mathbb{A})} E_0\big(n(b)g_\infty w_{\mathbf{f}}^{-1}\big)\psi_\beta(-b)\,db \\ &= c\int_{SH_n(\mathbb{A})} \Phi\big(wn(b)w_{\mathbf{f}}^{-1}g_\infty,s\big)\psi_\beta(-b)\,db \\ &= c\prod_{v\in\infty} a_v(\beta_v,g_v,2s)\prod_{v\in\mathbf{f}} a_v(\beta_v,2s) \end{aligned}$$

with $a_v(\beta_v,2s) = \int_{SH_n(F_v)} \Phi_v\big(w_v n(b_v)w_v^{-1},s\big)\psi_{\beta,v}(-b_v)\,db_v$ for non-archimedean v.

Note that for $\Phi_v = \Phi_v^0$, $a_v(\beta_v,2s) = W_{\beta,v}(e,s,\Phi_v^0)$. Now

$$a_v(\beta_v,2s) = \alpha\big(v,\beta_v,\gamma_v(\pi_v)q_v^{-n-2s}\big)$$

where $\alpha(v, \beta_v, t)$ is a rational function in the variable t called a *Siegel series*. By [Shi, Proposition 4.6],

$$\alpha(v, \beta_v, t) = \prod_{i=0}^{n-1} (1 - \omega^i_{E_v/F_v} q^i_v t)$$

for v unramified in E. Putting all these together, we obtained

Proposition 3.2 $W_{\beta,v}(e, s, \Phi^0_v) = \prod_{i=0}^{n-1} \frac{1}{L_{F_v}(2s+n-i, \gamma_v \omega^i_{E_\varpi/F_v})}$ *for all* $v \notin \mathcal{S}$.

4 Poles of $E(g, s, \Phi)$

We shall now put together all the results we have obtained to prove our main theorem stated in Section 1. The arguments we use here is similar to those in [KR2]. Let $\mathcal{S}'$ be the set of all archimedean places of F and places where Φ_v is not K_v-invariant. Note that $\mathcal{S}'$ is a subset of $\mathcal{S}$ defined in Section 3.

Recall that the normalized Siegel Eisenstein series on $U(n, n)$ is defined by

$$E^{n,*}(g, s, \Phi) = \Big(\prod_{v \notin \mathcal{S}'} b^*_{n,v}(s) \Big) E^n(g, s, \Phi)$$

where $b^*_{n,v}(s)$ is given at the end of Section 2.

We first deal with $\mathrm{Re}(s) \geq 0$ and then extend the result to the left half plane using a functional equation. We shall consider the case when $s \in X$ and outside X separately.

Proposition 4.1 $E^{n,*}(g, s, \Phi)$ *has no pole at* $s = 0$ *and* $\mathrm{Re}(s) > 0$ *if either* $\gamma \neq \hat{\gamma}$ *or* $s \notin X$.

Proof First of all, we need a lemma.

Lemma 4.2 *The non-singular* β*-th-Fourier coefficients of* $E^{n,*}(g, s, \Phi)$ *are holomorphic in* $\mathrm{Re}(s) > 0$.

Proof We note that for non-singular β, the β-th Fourier coefficient is given by

$$E^{n,*}_\beta(g, s, \Phi) = \prod_{v \notin \mathcal{S}} b^*_{n,v}(s) W_{\beta,v}(e, s, \Phi_v) \prod_{v \in \mathcal{S}-\mathcal{S}'} b^*_{n,v}(s) \prod_{v \in \mathcal{S}} W_{\beta,v}(g, s, \Phi_v).$$

Each of the factors in the third product in the right hand side above gives rise to an entire function on s. On the other hand, we can easily check, in view of (2.3) and Proposition 3.2, that the first two products are holomorphic in $\mathrm{Re}(s) > 0$. ■

Suppose $E^{n,*}(g, s, \Phi)$ has a pole of order k at a point s' where $\mathrm{Re}(s') > 0$ but either $\gamma \neq \hat{\gamma}$ or $s' \notin X$. In view of Lemma 1.1, we can choose a non-archimedean place v' such that $I_{v'}(s', \gamma_{v'})$ is irreducible. Fix v' and consider all $\Phi(s)$ of the form

$$\Phi(s) = \Phi_{v'}(s) \otimes \prod_{v \neq v'} \Phi^0_v(s) \tag{4.3}$$

by varying only the v'-component. Let $A_{-k}(g,\Phi)$ be the leading term in the Laurent expansion of $E^{n,*}(g,s,\Phi)$ where $\Phi(s)$ is as in (4.3). Then the map defined by

$$A_{-k}\colon I_{v'}(s',\gamma_{v'}) \to \mathcal{A}(G)$$
$$\Phi_{v'}(s) \mapsto (s-s')^k E^{n,*}(g,s,\Phi)|_{s=s'}$$

is $G_{v'}$-intertwining and hence injective. By the above lemma, $E^{n,*}_{\beta}(g,s,\Phi)$ is holomorphic for all non-singular β. So $A_{-k,\beta}(g,\Phi)$, the coefficient of $(s-s')^{-k}$ in the Laurent expansion of $E^{n,*}_{\beta}(g,s,\Phi)$ is identically zero for such β's. Now $I_{v'}(s',\gamma_{v'})$ is irreducible implies it is non-singular in the sense of Howe [How]. This means that there exists a function $\phi \in S\big(SH_n(F_{v'})\big)$ such that the Fourier transform $\hat{\phi}$ of ϕ with respect to the self-dual measure db defined earlier has support in the set of non-singular β and that ϕ does not act by zero in $I_{v'}(s',\gamma_{v'})$. On the other hand, for all $\beta \in SH_n(F)$, we have

$$A_{-k,\beta}\big(g, r(\phi)\Phi\big) = \hat{\phi}(\beta) A_{-k,\beta}(g,\Phi)$$

for any $g \in G(\mathbb{A})$ with $g_{v'} = e$. The term on the right hand side vanishes since $\hat{\phi}(\beta) = 0$ for $\det\beta = 0$ and $A_{-k,\beta}(g,\Phi) = 0$ for $\det\beta \neq 0$. So $A_{-k}\big(g, r(\phi)\Phi\big) = 0$ while $r(\phi)\Phi$ is nonzero for some $\Phi(s)$. This contradicts the injectivity of A_{-k}. Hence $E^{n,*}(g,s,\Phi)$ cannot have a pole outside X in the right half plane or when $\gamma \neq \hat{\gamma}$. For the case $s = 0$, the argument is almost the same except that we may not be able to find a place v whereby $I_v(0,\gamma_v)$ is irreducible. However, this space is completely reducible with each constituent being non-singular [KS, Theorem 1.2]. So we may use one of the irreducible constituents in place of $I_v(s,\gamma_v)$ in the above argument. ■

Our next job is to look at the poles in X.

Proposition 4.4 *$E^{n,*}(g,s,\Phi)$ has at most a simple pole for $s \in X$ with $\mathrm{Re}(s) > 0$ and $\gamma = \hat{\gamma}$.*

Proof We shall prove this proposition by breaking up into a few steps. First of all, we introduce another parabolic subgroup P_1 of G and define the P_1 constant term of our Eisenstein series. P_1 is the maximal parabolic subgroup of G stabilizing the line

$$\{(0_n, a, 0_{n-1}) \mid a \in E\}$$

in W. The Levi decomposition of this parabolic subgroup is given by

$$P_1 = M_1 N_1$$

where M_1 is the Levi factor whose elements are of the form

$$m_1(a,g_0) = \begin{pmatrix} a & & & \\ & b & & c \\ & & \bar{a}^{-1} & \\ & d & & e \end{pmatrix}$$

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with $a \in E^\times$ and $g_0 = \left(\begin{smallmatrix} b & c \\ d & e \end{smallmatrix}\right) \in U(n-1, n-1)$; N_1 is the unipotent radical with elements

$$n_1(b,c,d) = \begin{pmatrix} 1 & b & c & d \\ & I_{n-1} & -{}^t\bar{d} & \\ & & 1 & \\ & & -{}^t\bar{b} & I_{n-1} \end{pmatrix}$$

where $b, d \in E^n$, $c \in E$ and $c + \bar{c} = -(b\,{}^t\bar{d} + d\,{}^t\bar{b})$.

Then the P_1-constant term

$$E^n_{P_1}(g, s, \Phi) = \int_{N_1(F)\backslash N_1(\mathbb{A})} E^n(ng, s, \Phi)\, dn.$$

Let $J(s)\colon I(s) \longrightarrow I(s-1)$ be the intertwining map

$$J(s)(\Phi)(g) = \int_{{}^{w_0}N_1(\mathbb{A})} \Phi(w_0 ng, s)\, dn,$$

where $w_0 = \left(\begin{smallmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{smallmatrix}\right)$ and ${}^{w_0}N_1$ the subgroup of N_1 with elements of the form

$$n(x,y) = \begin{pmatrix} 1 & x & y & 0 \\ & I_{n-1} & 0 & 0 \\ & & 1 & 0 \\ & & -{}^t\bar{x} & I_{n-1} \end{pmatrix}. \tag{4.5}$$

Then we have

Lemma 4.6

$$\begin{aligned} E^{n,*}_{P_1}\big(m_1(1, g_0), s, \Phi\big) &= L_F^{S'}(2s+1, \omega^n_{E/F}) E^{n-1,*}\left(g_0, s + \frac{1}{2}, i^*\Phi\right) \\ &\quad + L_F^{S'}(2s, \omega^n_{E/F}) E^{n-1,*}\left(g_0, s - \frac{1}{2}, i^*\Psi\right) \end{aligned} \tag{4.7}$$

where $i\colon U(n-1, n-1) \hookrightarrow U(n,n)$ *is the embedding* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{smallmatrix}\right)$, $i^*\colon \Phi(g_0, s + \frac{1}{2}) \mapsto \Phi\big(i(g_0), s\big)$ *and* $\Psi(s) = \Phi^{S'}(s) \prod_{v \in S'} \eta_v(s)^{-1} J_v(s)\Phi_v(s)$. *Here* $J_v(s)\Phi_v(s)$ *is the local component of* $J(s)\Phi(s)$ *and* $\eta_v(s) = \frac{a_{n,v}(s) b_{n-1,v}(s-\frac{1}{2})}{b_{n,v}(s) a_{n-1,v}(s-\frac{1}{2})}$.

Proof The expression on the right hand side of (4.7) can be derived in the standard way as in [KR1, Proposition 1.2.1] by unfolding the constant term of $E^{n,*}(g, s, \Phi)$ along P_1. ■

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Now we can prove the proposition by induction on n. We check that $\Psi(s)$ is holomorphic in $\mathrm{Re}(s) > 0$ (similar argument as in [KR1]). Therefore it can be written as a finite linear combination of standard sections with holomorphic coefficients. When $n = 1$, we check that $E^{1,*}(g, s, \Phi)$ has at most a simple pole at $s = \pm\frac{1}{2}$ if $\gamma|_{F^\times} = 1$ and no pole otherwise. Then, using Lemma 4.6, we prove by induction that in $\mathrm{Re}(s) > 0$, $E^{n,*}(g, s, \Phi)$ has at most simple poles at $s \in X$. ∎

Finally, we extend the above result to the left half plane.

Proposition 4.8 *For* $\mathrm{Re}(s) < 0$, $E^{n,*}(g, s, \Phi)$ *has no pole except at most a simple pole for* $s \in X$ *and* $\gamma = \hat{\gamma}$.

Proof We use the functional equation $E^n(g, s, \Phi) = E^n\big(g, -s, M(s)\Phi\big)$:

$$E^{n,*}(g, s, \Phi) = \prod_{v \notin \mathcal{S}'} b^*_{n,v}(s) E^n\big(g, -s, M(s)\Phi\big)$$
$$= \frac{a^*_n(s)}{b^{*,\mathcal{S}'}_n(-s)} E^{n,*}(g, -s, \hat{\Phi})$$

where $\hat{\Phi} = \prod_{v \notin \mathcal{S}'} \Phi_v \otimes \prod_{v \in \mathcal{S}'} M^*_v(s)$ is entire (see Section 2). So again $E^{n,*}(g, -s, \hat{\Phi})$ has at most a simple pole in the half plane $\mathrm{Re}(s) < 0$ and these can only occur at the points $s \in X$. On the other hand, $a^*_n(s)$ has no pole in $\mathrm{Re}(s) < 0$ while $b^{*,\mathcal{S}'}_n(-s)$ is also holomorphic and non-vanishing there. ∎

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