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A Characterization of Bipartite Zero-divisor Graphs

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Abstract. In this paper we obtain a characterization for all bipartite zero-divisor graphs of commutative rings R with 1 such that R is finite or $|\operatorname{Nil}(R)| \neq 2$.

1 Introduction

For graph theory terminology in general we follow [10]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. We denote the degree of a vertex v in G by $d_G(v)$, which is the number of edges incident to v. A graph G is *complete* if there is an edge between every pair of the vertices. A subset X of the vertices of a graph G is called *independent* if there is no edge with two endpoints in X. A graph G is called *bipartite* if $V(G)$ is the union of two disjoint (possibly empty) independent sets called *partite* sets of G. A graph G is said to be *star* if G contains one vertex in which all other vertices are joined to this vertex and G has no other edges. Two graphs G_1 and G_2 are said to be *isomorphic* if there is a bijective map between the vertex set of G_1 and the vertex set of G_2 such that the adjacency relation is preserved. The *complement* $\overline{G}$ of G is the graph with vertex set $V(\overline{G}) = V(G)$, and $E(\overline{G}) = \{uv : uv \notin E(G)\}$. A *path* of length n is an ordered list of distinct vertices $v_0, v_1, \dots, v_n$ such that v_i is adjacent to v_{i+1} for $i = 1, 2, \dots, n-1$. A (u, v)-path is a path with endpoints u and v. A *cycle* is a path $v_0, v_1, \dots, v_n$ with an extra edge v_0v_n. For vertices x and y of G, let $d(x, y)$ be the length of a shortest path from x to y ($d(x, x) = 0$, and $d(x, y) = \infty$ if there is no path between x and y). The *diameter* $\operatorname{diam}(G)$ of G is maximum number $d(x, y)$, over all $x, y \in V(G)$. A graph G is *connected* if it has a (u, v)-path for each pair $u, v \in V(G)$.

The study of algebraic structures using the properties of graphs has become an exciting research topic in the last few decades, leading to many fascinating results and questions. It is interesting to study the intersection graphs $G(F)$ when the members of F have an algebraic structure. For the last few decades several mathematicians studied such graphs on various algebraic structures. These interdisciplinary studies allow us to obtain characterizations and representations of special classes of algebraic structures in terms of graphs and vice versa.

By the zero-divisor graph $\Gamma(R)$ of a ring R we mean the graph with vertices $Z(R) \setminus \{0\}$ such that there is an (undirected) edge between vertices a and b if and only if $a \neq b$ and $ab = 0$. Thus $\Gamma(R)$ is the empty graph if and only if R is an integral

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domain. The concept of zero-divisor graphs has been studied extensively by many authors. For a list of references and the history of this topic the reader is referred to [1–4, 6, 7].

Among many questions it is interesting to study bipartite zero-divisor graphs. Demeyer *et al.* [7] studied the cycle structure of $\Gamma(R)$ and determined the finite rings R for which $\Gamma(R)$ does not contain a cycle. They also gave a characterization for bipartite zero-divisor graphs of reduced and also finite rings. Akbari *et al.* [2] studied bipartite zero-divisor graphs and characterized bipartite zero-divisor graphs of reduced rings, independently. Dancheng *et al.* [6] also studied bipartite zero-divisor graphs.

In this paper we give a characterization for bipartite zero-divisor graphs of a commutative ring with identity in general.

We denote by K_n and C_n the complete graph and the cycle on n vertices. Also we denote by $K_{m,n}$ the complete bipartite graph.

Throughout, R will always be a commutative ring with $1 \neq 0$, unless we state R does not have 1. We also let $Z(R)$ denotes the set of zero-divisors of R. We make use of the following.

Theorem 1.1 ([6]) *A zero-divisor graph is bipartite if and only if it contains no triangles.*

2 Main Results

Let

$$T_4 = \frac{\mathbb{Z}_2[x]}{(x^2)}, \quad T_8 = \frac{\mathbb{Z}_2[x]}{(x^3)}, \quad T_9 = \frac{\mathbb{Z}_3[x]}{(x^2)},$$

$$T_8' = \frac{\mathbb{Z}_4[x]}{(2x, x^2 - 2)} = \{0, 1, 2, 3, \overline{x}, \overline{x} + 1, \overline{x} + 2, \overline{x} + 3\}$$

(with $2 = \overline{x}^2$ and $2\overline{x} = 0$, where $\overline{x}$ is the image of x). The main result of this paper is the following characterization.

Theorem 2.1 *Let R be a commutative ring with identity such that R is finite or $|\operatorname{Nil}(R)| \neq 2$, and R is not an integral domain. Then $\Gamma(R)$ contains no triangle if and only if R satisfies one of the following.*

(1) *$Z(R) = I \cup J$, where I, J are commutative domains as rings and $I \cap J = 0$.*
(2) *$R \cong \mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_9$, T_4, T_8, T_8', T_9, $F \times T_4$, or $F \times \mathbb{Z}_4$, where F is a field.*

As a consequence of Theorems 2.1 and 1.1, we obtain the following.

Theorem 2.2 *Let R be a commutative ring with identity such that R is finite or $|\operatorname{Nil}(R)| \neq 2$, and R is not an integral domain. Then $\Gamma(R)$ is bipartite if and only if R satisfies one of the following.*

(1) *$Z(R) = I \cup J$, where I, J are commutative domains as rings and $I \cap J = 0$.*
(2) *$R \cong \mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_9$, T_4, T_8, T_8', T_9, $F \times T_4$, or $F \times \mathbb{Z}_4$, where F is field.*

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3 Proof of Theorem 2.1

We begin with the following lemmas.

Lemma 3.1 *If x is nilpotent, then $1 - x$ and $1 + x$ are invertible.*

Lemma 3.2 *If $|R_1| = m$, $|R_2| = n$, then $\Gamma(R_1 \times R_2)$ contains $K_{m-1,n-1}$.*

Proof Notice that for any $a \in R_1$ and $b \in R_2$, $(a, 0)$ and $(0, b)$ are zero-divisors and $(a, 0)(0, b) = (0, 0)$. ■

Corollary 3.3 *Let R_1, R_2 be two rings (not necessarily with 1) of orders m, n, respectively. Then $\Gamma(R_1 \times R_2) \cong K_{m-1,n-1}$ if and only if $\Gamma(R_i)$ has no vertex for some $i \in \{1, 2\}$, and $\Gamma(R_j)$ has no edge for $j \in \{1, 2\} \setminus \{i\}$.*

Proof $(\Leftarrow)$ is obvious. For $(\Rightarrow)$, let $\Gamma(R_1 \times R_2) \cong K_{m-1,n-1}$. If $x \in V\bigl(\Gamma(R_1)\bigr)$ and $y \in V\bigl(\Gamma(R_2)\bigr)$, then $(x, y) \in \Gamma(R_1 \times R_2)$. But $|\Gamma(R_1 \times R_2)| = m + n - 2$. This is a contradiction. So $\Gamma(R_1)$ or $\Gamma(R_2)$ has no vertex. If $ab \in E\bigl(\Gamma(R_1)\bigr)$, then $(a, 0)(b, 0) \in E\bigl(\Gamma(R_1 \times R_2)\bigr)$. We deduce that $\Gamma(R_1)$ has no edge. Similarly, $\Gamma(R_2)$ has no edge. ■

Theorem 3.4 *Let $R = R_1 \times R_2$, where R_1, R_2 are two rings (not necessarily with 1) of order at least 2. Then $\Gamma(R)$ contains a C_3 if and only if either $E\bigl(\Gamma(R_1)\bigr) \cup E\bigl(\Gamma(R_2)\bigr) \neq \varnothing$ or $\min\bigl\{\bigl|V\bigl(\Gamma(R_1)\bigr)\bigr|, \bigl|V\bigl(\Gamma(R_2)\bigr)\bigr|\bigr\} \geq 1$.*

Proof $(\Rightarrow)$ follows from Corollary 3.3.

$(\Leftarrow)$ If $E\bigl(\Gamma(R_1)\bigr) \cup E\bigl(\Gamma(R_2)\bigr) \neq \varnothing$, then we let $ab \in E\bigl(\Gamma(R_1)\bigr)$. Let $d \in R_2 \setminus \{0\}$. It follows that $(a, 0)$—$(b, 0)$—$(0, d)$—$(a, 0)$ is a cycle on three vertices. So we assume that $E\bigl(\Gamma(R_1)\bigr) \cup E\bigl(\Gamma(R_2)\bigr) = \varnothing$. If $\min\bigl\{\bigl|V\bigl(\Gamma(R_1)\bigr)\bigr|, \bigl|V\bigl(\Gamma(R_2)\bigr)\bigr|\bigr\} \geq 1$, then we let $a \in V\bigl(\Gamma(R_1)\bigr)$ and $b \in V\bigl(\Gamma(R_2)\bigr)$. Then $a^2 = b^2 = 0$, and so $(a, 0)$—(a, b)—$(0, b)$—$(a, 0)$ is a cycle on three vertices. ■

Corollary 3.5 *For any three rings R_1, R_2, R_3 (not necessarily with 1), $\Gamma(R_1 \times R_2 \times R_3)$ contains a triangle.*

For a nilpotent element x in a ring R, we let $d(x)$ be the least number n such that $x^n = 0$.

Lemma 3.6 ([5]) *The nilradical of R is the intersection of all the prime ideals of R.*

Lemma 3.7 *If $\Gamma(R)$ contains no triangle, then for any nilpotent element x of R, $d(x) \leq 3$.*

Proof If $d(x) \geq 4$, then by Lemma 3.1, $x^{d(x)-1} \neq x^{d(x)-2}$. Now

$$x^{d(x)-1} — x^{d(x)-2} — (x^{d(x)-1} + x^{d(x)-2}) — x^{d(x)-1}$$

forms a triangle, a contradiction. ■

Lemma 3.8 *Let $\Gamma(R)$ contain no triangle. If $A = \{x : x \neq 0, x^2 = 0\}$, then $|A| \leq 2$.*

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Proof Suppose that $|A| \geq 3$, and let $x \in A$. We show that $A = \{0, x, -x\}$. Let y be any nonzero element of A distinct from x such that $x + y \neq 0$. If $xy = 0$, then x—$(x + y)$—y—x forms a cycle, a contradiction. If $xy \neq 0$, then by Lemma 3.1, x, xy, y are mutually distinct. Then x—xy—$(x + xy)$—x is a triangle, a contradiction. This implies that $|A| \leq 2$. ■

Lemma 3.9 *Let $\Gamma(R)$ contain no triangle. If $B = \{x : x^3 = 0, x^2 \neq 0\}$, then $|B| \leq 2$.*

Proof Without loss of generality assume that $B \neq \varnothing$. Let $x \in B$. Let A be the set introduced in Lemma 3.8. If $\operatorname{char}(A) \neq 2$ then x—x^2—$(-x^2)$—x is a triangle, a contradiction. So $\operatorname{char}(A) = 2$, and $A = \{0, x^2\}$. Let $y \in B \setminus \{x\}$. Since $(x^2)^2 = (y^2)^2 = 0$, by Lemma 3.8 $x^2 = y^2$. On the other hand $(xy)^2 = 0$, and so either $xy = 0$ or $xy = x^2$. Then $(y \pm x)^2 = 0$. We deduce that $y + x = x^2 = y - x$, and thus $2x = 0$. So $\operatorname{char}(B) = 2$. Also for any $z \in B \setminus \{x\}$, $(x + z)^2 = (x + y)^2 = 0$. Then $y = z$, and $|B| = 2$. ■

Corollary 3.10 *If $\Gamma(R)$ contains no triangle, then $\operatorname{Nil}(R)$ is in one of the following forms:*

(1) 0,
(2) $\{0, x\}$, *where* $2x = 0$,
(3) $\{0, x, -x\}$, *where* $3x = 0$,
(4) $\{0, x, x^2, x + x^2\}$, *where* $2x = 0$.

We proceed with the possiblities for $\operatorname{Nil}(R)$.

Theorem 3.11 *Let R be a ring (not necessarily with* 1*) with at least one nonzero zero-divisor, and $\operatorname{Nil}(R) = \varnothing$. Then $\Gamma(R)$ contains no triangle if and only if $Z(R) = I \cup J$, where I, J are commutative domains as rings and $I \cap J = 0$.*

Proof $(\Rightarrow)$ Since $\operatorname{Nil}(R) = \varnothing$, there are two distinct elements a, b in R such that $ab = 0$. Let $I = \operatorname{ann}(a)$ and $J = \operatorname{ann}(b)$. It follows that $a, b \notin I \cap J$. If $I \cap J \neq 0$, then we let $r \in (I \cap J) \setminus \{0\}$. Then a—r—b—a is a triangle, a contradiction. So $I \cap J = 0$ and $I + J \cong I \times J$. Since $\Gamma(I + J) \leq \Gamma(R)$, we obtain $\Gamma(I + J)$ has no triangle. But $\operatorname{Nil}(R) = \varnothing$. By Theorem 3.4, we deduce that I and J are commutative domains, as rings. It is obvious that $I \cup J \subseteq Z(R)$. Let $x \in Z(R)$. There is $r \in R \setminus \{0\}$ such that $rx = 0$. We consider two cases.

Case 1 If $x \in (I + J)$, then $x = c_1 + c_2$, where $c_1 \in I$ and $c_2 \in J$. Then $rc_1 + rc_2 = 0$, and so $rc_1 = -rc_2 \in I \cap J$. Suppose that $c_1 \neq 0, c_2 \neq 0$. Since $I \cap J = 0$, we obtain $rc_1 = 0, rc_2 = 0$. Then $(rb)c_1 = (ra)c_2 = 0$. Since I and J are domains, we have $ra = rb = 0$. This means that $r \in I \cap J = 0$, and so $r = 0$, a contradiction. We deduce that $c_1 = 0$ or $c_2 = 0$. Then $x \in I \cup J$.

Case 2 If $x \notin I + J$, then we let $K = \operatorname{ann}(x)$. If $K \cap (I + J) = 0$, then $K + (I + J) \cong K \times I \times J$ and, by Corollary 3.5, $\Gamma(K \times I \times J)$ contains a triangle, a contradiction. If $K \cap (I + J) \neq 0$, by Case 1, $K \cap (I + J) \subseteq I \cup J$. This implies that $K \cap (I + J) \subseteq I$ or $K \cap (I + J) \subseteq J$. Then $K \cap I = 0$ or $K \cap J = 0$ and so $KI = 0$ or $KJ = 0$. We deduce that $aK = 0$ or $bK = 0$. Then $K \subseteq I$ or $K \subseteq J$. Suppose that $K \subseteq I$. If $K = I$, then $xI = 0$ and so $x \in \operatorname{ann}(b) = J$, a contradiction, since $x \notin I + J$. Thus $K \subset I$. On the other hand $\langle x\rangle \cap I \neq 0$, since $xI \neq 0$. Let sx be a nontrivial element of $\langle x\rangle \cap I$. Suppose that t is a nontrivial element of K. If $(sx)x = 0$, then $(sx)^2 = 0$ which is a contradiction. So $t \neq sx$. Now t—sx—a—t is a triangle, a contradiction. We conclude that $x \in I + J$. Similarly $K \subseteq J$ produces a contradiction.

($\Leftarrow$) Since $\Gamma(R) = \Gamma(I \times J)$, the result follows from Theorem 3.4. ■

Theorem 3.12 *Let R be a finite ring and $\operatorname{Nil}(R) = \{0, x\}$. Then $\Gamma(R)$ contains no triangle if and only if $R = \mathbb{Z}_4$, T_4, $F \times T_4$, or $F \times \mathbb{Z}_4$.*

Proof Let $\operatorname{Nil}(R) = \{0, x\}$, and $r \in R$. Then rx is nilpotent. So $rx \in \{0, x\}$. Then r or $r-1$ belongs to $\operatorname{ann}(x)$. This implies that $\frac{R}{\operatorname{ann}(x)}$ has two elements. Since R is a finite commutative ring with 1, we have $R = R_1 \times R_2 \times \cdots \times R_t$, where (R_i, m_i) is a local ring for each i. Then $\operatorname{Nil}(R) = m_1 \times m_2 \times \cdots \times m_t$. By Corollary 3.5, $t \leq 2$. If $t = 1$, then $\operatorname{ann}(x) = \{0, x\} = m_1$ and so $|R| = 4$. We conclude that $R = \{0, x, 1 + x, 1\}$, where is T_4 or $\mathbb{Z}_4$. Assume that $t = 2$. Since $|m_1 \times m_2| = 2$, without loss of generality we may assume that $m_1 = 0$. Then R_1 is a field. Let $x = (0, b)$, where $b \in R_2$. We conclude that $\operatorname{ann}_{R_2}(b) = \{0, b\}$. Hence $|R_2| = 4$, and as before $R_2 \in \{T_4, \mathbb{Z}_4\}$. ■

Lemma 3.13 *If $\operatorname{char}(R) = n$, then R contains a subring isomorphic to $\mathbb{Z}_n$.*

Theorem 3.14 *Let $\operatorname{Nil}(R) = \{0, x, -x\}$, where $x \neq -x$. Then $\Gamma(R)$ contains no triangle if and only if $R = \mathbb{Z}_9$ or $\mathbb{T}_9$.*

Proof ($\Leftarrow$) Notice that $\Gamma(\mathbb{Z}_9) = \Gamma(\mathbb{T}_9) = K_2$.

($\Rightarrow$) If $r \in \operatorname{ann}(x) \setminus \{0, x, -x\}$, then r—x—$(-x)$—r is a triangle, a contradiction. So $\operatorname{ann}(x) = \{0, x, -x\}$. For any $r \in R$, rx is nilpotent. This implies that $rx \in \{0, x, -x\}$. Then r, $r - 1$ or $r + 1$ belong to $\operatorname{ann}(x)$. We deduce that $|\frac{R}{\operatorname{ann}(x)}| = 3$, and so $|R| = 9$. If $\operatorname{char}(R) = 3$, then $R = \{0, 1, -1, x, -x, 1+x, 1-x, x-1, -1-x\} \cong \mathbb{T}_9$. If $\operatorname{char}(R) = 9$, then by Lemma 3.13, $R \cong \mathbb{Z}_9$. ■

Theorem 3.15 *Let $\operatorname{Nil}(R) = \{0, x, x^2, x + x^2\}$, where $x^3 = 2x = 0$. Then $\Gamma(R)$ contains no triangle if and only if $R = \mathbb{Z}_8$, $R = \mathbb{T}_8$ or $R = \mathbb{T}'_8$.*

Proof ($\Leftarrow$) Notice that $\Gamma(\mathbb{Z}_8) = \Gamma(\mathbb{T}_8) = \Gamma(\mathbb{T}'_8) = K_{1,2}$.

($\Rightarrow$) Let $I = \operatorname{ann}(x^2)$, and $r \in R$. Since $(rx^2)^2 = 0$, either $rx^2 = 0$ or $rx^2 = x^2$. We deduce that $r \in I$ or $r - 1 \in I$. This implies that $|\frac{R}{I}| = 2$. We show that $I = \operatorname{Nil}(R)$. If $r \in I \setminus \operatorname{Nil}(R)$, then $rx^2 = 0$ and so $(rx)^2 = 0$. By assumption we obtain $rx = 0$ or $rx = x^2$. If $rx = 0$, then r—x—x^2—r is a triangle, a contradiction. So $rx = x^2$. Then $x(r + x) = 0$ and $(r + x)$—x—(x^2)—$(r + x)$ is a triangle, a contradiction. We conclude that $I = \operatorname{Nil}(R)$, and $|R| = 8$. Thus $R = \{0, x, x^2, x + x^2, 1, 1 + x, 1 + x^2, 1 + x + x^2\}$. Then $\operatorname{char}(R) \in \{2, 4, 8\}$. Now the result follows. ■

Now the result follows from Theorems 3.11, 3.12, 3.14, and 3.15.

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