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Abstract

Let f be a modular form of weight $k \geqslant 2$ and level N, let K be a quadratic imaginary field and assume that there is a prime p exactly dividing N. Under certain arithmetic conditions on the level N and the field K, one can attach to this data a p-adic L-function $L_p(f, K, s)$, as done by Bertolini–Darmon–Iovita–Spieß in [*Teitelbaum's exceptional zero conjecture in the anticyclotomic setting*, Amer. J. Math. **124** (2002), 411–449]. In the case of p being inert in K, this analytic function of a p-adic variable s vanishes in the critical range $s = 1, \dots, k-1$, and one may be interested in the values of its derivative in this range. We construct, for $k \geqslant 4$, a Chow motive endowed with a distinguished collection of algebraic cycles which encode these values, via the p-adic Abel–Jacobi map. Our main result generalizes the result obtained by Iovita and Spieß in [*Derivatives of p-adic L-functions, Heegner cycles and monodromy modules attached to modular forms*, Invent. Math. **154** (2003), 333–384], which gives a similar formula for the central value $s = k/2$. Even in this case our construction is different from the one found by Iovita and Spieß.

Introduction

Fix a quadratic imaginary field K and let f be a modular form defined over $\mathbb{Q}$. The goal of the different theories of p-adic L-functions is to produce rigid-analytic functions attached to f that interpolate the Rankin–Selberg L-function $L(f/K, s)$ in different ways. The theory has so far developed in two directions, which correspond to the two independent $\mathbb{Z}_p$-extensions of the field K: the cyclotomic and anti-cyclotomic extensions.

The first approach to such p-adic analogues was constructed by Mazur and Swinnerton-Dyer in [MS74], where they introduced a p-adic L-function associated to a modular form f of arbitrary even weight $n+2$ using the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Mazur, Tate and Teitelbaum formulated in [MTT86] a conjectural formula that related the order of vanishing of this p-adic L-function to that $L(f, s)$ and, in [GS93], Greenberg and Stevens proved that formula in the case of weight 2. Also, Perrin-Riou [Rio92] obtained a Gross–Zagier-type formula for the central value using p-adic heights.

Nekovář [Nek95] extended the result of Perrin-Riou to higher weights, by using the definition of p-adic height that he had already introduced in his earlier paper [Nek93]. Combining this result with his previous work on Euler systems [Nek92], he obtained a result of Kolyvagin type for the cyclotomic p-adic L-function.

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Bertolini and Darmon, in a series of papers [BD96, BD98, BD99], constructed another p-adic L-function which depends instead on the *anti-cyclotomic* $\mathbb{Z}_p$-extension of a fixed quadratic imaginary field K. One important feature of this construction is that it is purely p-adic, unlike its cyclotomic counterpart. Bertolini and Darmon formulated the analogous conjectures to those of Teitelbaum [Tei90], and proved them in the case of weight 2.

Assume in the introduction that the level N of f is the product of an even number of distinct primes, and let K be a quadratic imaginary number field on which all prime divisors of N are inert. Choose a prime p dividing N.

In [BDIS02], taking ideas from the work of Schneider in [Sch84], the four authors constructed under these restrictions the anti-cyclotomic p-adic L-function attached to the rigid modular form f and the quadratic imaginary field K, and obtained a formula which computes the derivative of this p-adic L-function at the central point in terms of an integral on the p-adic upper half-plane, using the integration theory introduced by Coleman in [Col85]. Assume for simplicity that the ideal class number of K is 1. Using their techniques, one can easily show that, when p is inert in K, the anti-cyclotomic p-adic L-function vanishes at all the critical values. Moreover, one computes a formula for the derivative at all the values in the critical range: if f is a modular form of even weight $n+2$, and we denote by $L_p(f,K,s)$ the anti-cyclotomic p-adic L-function attached to f and K, then, for all $0 \leqslant j \leqslant n$, one has

$$L_p'(f,K,j+1) = \int_{\overline{z}_0}^{z_0} f(z)(z-z_0)^j (z-\overline{z}_0)^{n-j}\, dz, \tag{1}$$

where $z_0, \overline{z}_0 \in \mathcal{H}_p(K)$ are certain conjugate Heegner points on the p-adic upper half-plane. In 2003, Iovita and Spieß [IS03] interpreted the quantity appearing in the right-hand side of formula (1), in the case of $j = n/2$, as the image of a Heegner cycle under a p-adic analogue to the Abel–Jacobi map. This paper gives a similar geometric interpretation of the quantity appearing in the right-hand side of the previous formula, for all values of j.

Let $M_{n+2}(X)$ denote the space of modular forms on a Shimura curve X, of weight $n+2 \geqslant 4$. The case of weight 2 is excluded for technical reasons, and because it has already been studied by other authors. Let K be a quadratic imaginary field in which p is inert, and fix an elliptic curve E with complex multiplication. In this setting, we construct a Chow motive $\mathcal{D}_n$ over X, and a collection of algebraic cycles Δ_φ supported in the fibers over CM-points of X, indexed by isogenies $\varphi : E \to E'$, of elliptic curves with complex multiplication. The motive $\mathcal{D}_n$ is obtained from a self-product of a certain number of abelian surfaces, together with a self-product of the elliptic curve E. The cycles Δ_φ are essentially the graph of φ, and are expected to carry more information than the classical Heegner cycles.

The étale p-adic Abel–Jacobi map, denoted $\mathrm{AJ}_{K,p}$, assigns to a null-homologous algebraic cycle an element in the dual of the de Rham realization of the motive $\mathcal{D}_n$. This motive has precisely been constructed so that this realization is

$$M_{n+2}(X) \otimes_{\mathbb{Q}_p} \mathrm{Sym}^n H^1_{\mathrm{dR}}(E/K).$$

One can choose generators ω and η for the group $H^1_{\mathrm{dR}}(E/K)$, and it thus makes sense to evaluate $\mathrm{AJ}_{K,p}(\Delta_\varphi)$ on an element of the form $f \wedge \omega^j \eta^{n-j}$. By explicitly computing this map, and combining the result with the formula in (1), we obtain the following result (see Corollary 7.7 for a more precise and general statement).

Theorem. *There exist explicit isogenies φ and $\overline{\varphi}$ as above and a constant $\Omega \in K^\times$ such that for all $0 \leqslant j \leqslant n$,*

$$\mathrm{AJ}_{K,p}(\Delta_\varphi - \Delta_{\overline{\varphi}})(f \wedge \omega^j \eta^{n-j}) = \Omega^{j-n} L_p'(f,K,j+1).$$

This result is to be regarded as a p-adic Gross–Zagier-type formula for the anti-cyclotomic p-adic L-function. Note however that instead of heights it involves the p-adic Abel–Jacobi map. It can also be seen as a generalization of the main result of Iovita and Spieß in [IS03] to all values in the critical range.

The paper is structured as follows: in Section 1, we recall concepts from rigid-analytic geometry and p-adic integration. Section 2 deals with the theory of Shimura curves, their p-adic uniformization and modular forms defined on them. Section 3 recalls Fontaine's theory of filtered Frobenius monodromy modules and their relationship with semistable Galois representations. In Section 4, we describe the extra structure of certain de Rham cohomology groups as done by Coleman and Iovita in [CI10], and describe pairings in certain cases. In Section 5, we recall the definition of the anti-cyclotomic p-adic L-function $L_p(f, K, s)$, and prove a formula for its derivative in terms of integration on the p-adic upper half-plane. Section 6 contains the construction of the Chow motive $\mathcal{D}_n$ and the computation of its realizations. In Section 7, we state and prove the main result of this paper. We also give some concluding remarks and future directions of research.

An expanded, mostly self-contained version of this paper can be found in the ArXiv [Mas11].

1. Integration on quotients of $\mathcal{H}_p$

In this section, we describe the integration theory constructed initially by Coleman in [Col82, Col85, Col89], and further developed by Coleman–Iovita in [CI10] and by de Shalit in [dS89], among others. In fact, we will specialize Coleman's integration theory to those rigid spaces which allow a covering by a certain type of open subsets of $\mathbb{P}^1(\mathbb{C}_p)$, called basic wide opens. The p-adic upper half-plane $\mathcal{H}_p$ as described below admits such a covering, and hence we will obtain a theory of integration on $\mathcal{H}_p$ and on Mumford–Schottky curves.

We are in fact interested in the integration of general vector bundles over the spaces mentioned in the previous paragraph. It turns out, however, that the bundles that we will encounter in this work have a basis of horizontal sections, and therefore one can integrate component-wise, thus reducing to integration with trivial coefficients. For more details of rigid-analytic geometry, the reader is invited to refer to [BGR84] or [FV04]. Here we will use the notation of this latter reference.

Fix a rational prime p, and denote by $\mathbb{C}_p$ the topological completion of the algebraic closure of $\mathbb{Q}_p$. On the projective line over $\mathbb{C}_p$, denoted $\mathbb{P}^1(\mathbb{C}_p)$, we consider the strong G-topology [FV04, Definition 2.6.7]. We proceed to define a certain analytic subspace of $\mathbb{P}^1(\mathbb{C}_p)$, the p-adic upper half-plane $\mathcal{H}_p$ defined over $\mathbb{Q}_p$. It can be defined as a formal scheme over $\mathbb{Z}_p$, but we are only interested in the rigid-analytic space associated to its generic fiber, which is a subset of $\mathbb{P}^1(\mathbb{C}_p)$, together with a collection of affinoids that define its rigid-analytic structure. One can find more details of its construction in [DT08, § 3]. We also need to fix a *branch of the p-adic logarithm*, which is a locally analytic homomorphism $\log_p : \mathbb{C}_p^\times \to \mathbb{C}_p^+$ such that $(d/dz)\log_p(1) = 1$. It can be easily shown that $\log_p(z)$ is analytic on the ball with center x and radius $|x|$, for all $x \in \mathbb{C}_p^\times$.

Denote by $\mathcal{T}$ the Bruhat–Tits tree of $\mathrm{GL}_2(\mathbb{Q}_p)$, as explained in [Dar04]. It is an unoriented $(p+1)$-regular tree $\mathcal{T}$, with a natural action of $\mathrm{PGL}_2(\mathbb{Q}_p)$ by continuous graph automorphisms. Fix an ordering of the edges of $\mathcal{T}$, and denote by $\vec{\mathfrak{E}}(\mathcal{T})$ the set of ordered edges. If the ordered edge e connects the vertices v_1 and v_2, we write $v_1 = o(e)$ and $v_2 = t(e)$. We also write $\bar{e}$ for the opposite edge, which has $o(\bar{e}) = v_2$ and $t(\bar{e}) = v_1$. The Bruhat–Tits tree $\mathcal{T}$ has a distinguished

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vertex, written v_0, which corresponds to the homothety class of the standard lattice $\mathbb{Z}_p^2$ inside $\mathbb{Q}_p^2$. The edges e with $o(e)=v_0$ correspond to the $p+1$ sublattices of index p in $\mathbb{Z}_p^2$, which in turn are in bijection with the points t of $\mathbb{P}^1(\mathbb{F}_p)=\{0,1,\dots,p-1,\infty\}$. For such a point t, we denote by $e_t\in\vec{\mathfrak{E}}(\mathcal{T})$ the corresponding edge. Since $\mathcal{T}$ is locally finite, it can be endowed with a natural topology, and it thus becomes a contractible topological space. Given an edge $e\in\mathfrak{E}(\mathcal{T})$, we write $[e]\subset\mathcal{T}$ for the closed edge, which contains the two vertices that e connects, and $]e[$ for the corresponding open edge.

As a set, the *p-adic upper half-plane* is defined as $\mathcal{H}_p(\mathbb{C}_p):=\mathbb{P}^1(\mathbb{C}_p)\backslash\mathbb{P}^1(\mathbb{Q}_p)$. The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on $\mathcal{H}_p(\mathbb{C}_p)$ by fractional linear transformations. We describe a covering by basic affinoids and annuli, using the Bruhat–Tits tree. Let

$$\mathrm{red}:\mathbb{P}^1(\mathbb{C}_p)\to\mathbb{P}^1(\overline{\mathbb{F}}_p)$$

denote the natural map given by reduction modulo $\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}$, the maximal ideal of the ring of integers of $\mathbb{C}_p$. Given a point $\tilde{x}$ in $\mathbb{P}^1(\overline{\mathbb{F}}_p)$, the *residue class* of $\tilde{x}$ is the subset $\mathrm{red}^{-1}(\{\tilde{x}\})$ of $\mathbb{P}^1(\mathbb{C}_p)$.

Define the set A_0 to be $\mathrm{red}^{-1}(\mathbb{P}^1(\overline{\mathbb{F}}_p)\backslash\mathbb{P}^1(\mathbb{F}_p))$. This is the prototypical example of a *standard affinoid*. Define also a collection of annuli W_t, for $t\in\mathbb{P}^1(\mathbb{F}_p)$, as

$$W_t:=\left\{\tau\in\mathbb{P}^1(\mathbb{C}_p)\ \middle|\ \frac{1}{p}<|\tau-t|<1\right\},\quad 0\leqslant t\leqslant p-1,\quad W_\infty:=\{\tau\ \mid\ 1<|\tau|<p\}.$$

Note that A_0 and the annuli W_t are mutually disjoint. It is easy to see [Dar04, Proposition 5.1] that there is a unique $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant 'reduction map' $r:\mathcal{H}_p(\mathbb{C}_p)\to\mathcal{T}$ which maps A_0 to v_0 and W_t to e_t, where we recall that we labeled the edges with origin v_0 as e_t, with $t\in\mathbb{P}^1(\mathbb{F}_p)$. For each vertex $v\in\mathfrak{V}(\mathcal{T})$, let $\mathcal{A}_v:=r^{-1}(\{v\})$. For each edge $e\in\mathfrak{E}(\mathcal{T})$, write $\mathcal{A}_{[e]}:=r^{-1}([e])$ and $\mathcal{A}_{]e[}:=r^{-1}(]e[)$. Then the collection $\{\mathcal{A}_{[e]}\}_{e\in\mathfrak{E}(\mathcal{T})}$ gives a covering of $\mathcal{H}_p(\mathbb{C}_p)$ by standard affinoids, and their intersections are

$$\mathcal{A}_{[e]}\cap\mathcal{A}_{[e']}=\begin{cases}\emptyset & \text{if } [e]\cap[e']=\emptyset,\\ \mathcal{A}_v & \text{if } [e]\cap[e']=\{v\}.\end{cases}$$

This covering endows $\mathcal{H}_p(\mathbb{C}_p)$ with the structure of a rigid-analytic space, and its nerve is precisely the Bruhat–Tits tree $\mathcal{T}$.

The *boundary* of $\mathcal{H}_p$ is the set $\mathbb{P}^1(\mathbb{Q}_p)$, which has been removed from $\mathbb{P}^1(\mathbb{C}_p)$ in order to obtain $\mathcal{H}_p(\mathbb{C}_p)$. An *end* of $\mathcal{T}$ is an equivalence class of sequences $\{e_i\}_{i\geqslant 1}$ of edges $e_i\in\vec{\mathfrak{E}}(\mathcal{T})$, such that $t(e_i)=o(e_{i+1})$, and such that $t(e_{i+1})\neq o(e_i)$. Two such sequences are identified if a shift of one is the same as the other, for large enough i. Write $\mathfrak{E}_\infty(\mathcal{T})$ for the space of ends, which can be identified with $\mathbb{P}^1(\mathbb{Q}_p)$, thus endowing $\mathfrak{E}_\infty(\mathcal{T})$ with a topology. There is a basis of $\mathbb{P}^1(\mathbb{Q}_p)$ of compact open sets indexed by $\vec{\mathfrak{E}}(\mathcal{T})$: given an oriented edge e, the corresponding compact open is the set $U(e)$ of ends passing through e. Moreover, one can compactify $\mathcal{T}$ by adding to it its ends: writing $\mathcal{T}^*$ for this compactification, one can extend the reduction map to $r:\mathbb{P}^1(\mathbb{C}_p)\to\mathcal{T}^*$.

Let Γ be a discrete cocompact subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Suppose for simplicity that Γ contains no elliptic points, and consider the topological quotient $\pi:\mathcal{H}_p\to X_\Gamma:=\Gamma\backslash\mathcal{H}_p$. The space X_Γ can be given a structure of a rigid-analytic space in a way so that π is a morphism of rigid-analytic spaces. An admissible covering is indexed by the quotient graph $\Gamma\backslash\mathcal{T}$, in the same way that was done for $\mathcal{H}_p$. One thus obtains a complete curve called a *Mumford–Shottky curve*. In § 2, we will explain how these curves are related to Shimura curves.

A *wide open* is a set of the form

$$U:=\{z\in\mathbb{P}^1(\mathbb{C}_p)\ \mid\ |f(z)|<e_f,\ f\in S\},$$

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where S a finite set of rational functions over $\mathbb{C}_p$ containing at least one non-constant function, and $e_f \in \{1, \infty\}$ for all f. For example, the open balls $B(a, r)$ and the open annuli $\mathcal{A}_{]e[}$ are all wide open sets. In [Col82, p. 177], Coleman defined a *basic wide open*. The main example is

$$X_1 := A_0 \cup \bigg(\bigcup_{t \in \mathbb{P}^1(\mathbb{F}_p)} W_t \bigg), \tag{2}$$

as well as its translates by the action of $\mathrm{GL}_2(\mathbb{Q}_p)$. These basic wide opens can always be written as the disjoint union of a connected affinoid (in our example A_1) and a finite number of wide open annuli (the W_t in our example).

Given any open subset X of $\mathbb{P}^1(\mathbb{C}_p)$, we will denote by $\mathcal{O}(X)$ (respectively $\mathcal{L}(X)$) the ring of rigid-analytic functions (respectively of locally analytic functions) on X. Coleman defined in [Col82] the notion of a logarithmic F-crystal on C for any basic wide open C. This is a certain type of $\mathcal{O}(C)$-submodule of $\mathcal{L}(C)$ satisfying several technical conditions (see [Col82, p. 184, conditions A–F]). One of these conditions is the uniqueness property: if M is a logarithmic F-crystal on C, then any element $f \in M$ that vanishes in a non-empty open subset of C must be zero. The ring $\mathcal{O}(C)$ is the simplest example of a logarithmic F-crystal on C. Coleman defined also in [Col82, pp. 177–179] the modules $\mathcal{O}_{\mathrm{Log}}(X)$ and $\Omega_M(X)$. Write ϕ for the Frobenius morphism.

Lemma 1.1 [Col82, Lemma 4.4]. *Let M be a logarithmic F-crystal on a basic wide open space C, and let ω be an element of $\Omega_M(C)$. There exists a locally analytic function $F_\omega \in \mathcal{L}(C)$, unique up to an additive constant, which satisfies:*

(i) *$dF_\omega = \omega$;*

(ii) *there is a wide open neighborhood V of C such that $\phi^* F_\omega - bF_\omega \in M(V)$, for some $b \in \mathbb{C}_p$ which is not a root of unity; and*

(iii) *the restriction of F_ω to the underlying affinoid X of C is analytic in each residue class of X, and the restriction to each of the open annuli V is in $\mathcal{O}_{\mathrm{Log}}(V)$.*

Define $A^1(C)$ as the unique minimal logarithmic F-crystal on C which contains M and such that $dA^1(C) \supseteq \Omega_M(C)$. It can be described as

$$A^1(C) = \mathcal{O}(C) + \sum_{\omega \in \Omega_M(C)} F_\omega \mathcal{O}(C).$$

In [Col82, Theorem 5.1], Coleman showed that, given $\omega \in \Omega^1(C)$, there exists a unique (up to a constant) function $F_\omega \in A^1(C)$ such that $dF_\omega = \omega$.

Let Y be a rigid-analytic space which can be covered by a family $\mathcal{C}$ of basic wide opens, which intersect at basic wide opens, and such that the nerve of the covering is simply connected. Let $\mathcal{A}^1$ be the sheaf of $\mathcal{O}_Y$-modules defined by $\mathcal{A}^1(U) := A^1(U)$ for each $U \in \mathcal{C}$. The following corollary is an easy consequence of the results stated so far.

Corollary 1.2. *There is a short exact sequence*

$$0 \to \mathbb{C}_p \to H^0(Y, \mathcal{A}^1) \xrightarrow{d} H^0(Y, \mathcal{A}^1) \otimes_{\mathcal{O}_Y(Y)} \Omega^1(Y) \to 0.$$

2. Shimura curves and modular forms

In this section, we introduce the different ways in which Shimura curves appear in this work. A good exposition of the theory of Shimura curves and their p-adic uniformization can be found in [BC91]. Here we just recall the basic facts.

Fix for the rest of the paper an integer N which can be factored as $N = pN^-N^+$, where p is a prime which will remain fixed, N^- is a positive square-free integer with an odd number of prime divisors none of which equals p and N^+ is a positive integer relatively prime to pN^-. Let $\mathcal{B}$ be the indefinite rational quaternion algebra of discriminant pN^-. Fix a maximal order $\mathcal{R}^{\max}$ in $\mathcal{B}$, and an Eichler order $\mathcal{R}$ of level N^+ contained in $\mathcal{R}^{\max}$.

Definition 2.1. Let S be a $\mathbb{Q}$-scheme. An *abelian surface with quaternionic multiplication* (by $\mathcal{R}^{\max}$) and level-N^+ structure over S is a triple (A, i, G), where:

(i) A is a (principally polarized) abelian scheme over S of relative dimension 2;

(ii) $i : \mathcal{R}^{\max} \hookrightarrow \mathrm{End}_S(A)$ is a ring homomorphism;

(iii) G is a subgroup scheme of A which is étale-locally isomorphic to $(\mathbb{Z}/N^+\mathbb{Z})^2$ and is stable and locally cyclic under the action of $\mathcal{R}$.

When no confusion may arise, such a triple will be called an *abelian surface with QM*.

The *Shimura curve* $X := X_{N^+,pN^-}/\mathbb{Q}$ is the *coarse* moduli scheme representing the moduli problem over $\mathbb{Q}$:

$$S \mapsto \{\text{isomorphism classes of abelian surfaces with QM over } S\}.$$

Proposition 2.2 (Drinfel'd [BC91, § III]). *The Shimura curve X_{N^+,pN^-} is a smooth, projective and geometrically connected curve over $\mathbb{Q}$.*

We will need to work with a Shimura curve which is actually a fine moduli space. For that, we need to rigidify the moduli problem, as follows.

Definition 2.3. Let $M \geqslant 3$ be an integer relatively prime to N. Let S be a $\mathbb{Q}$-scheme. An *abelian surface with QM and full level-M structure* (QM by $\mathcal{R}^{\max}$ and level-N^+ structure is understood) is a quadruple $(A, i, G, \overline{\nu})$, where (A, i, G) is as before and $\overline{\nu} : (\mathcal{R}^{\max}/M\mathcal{R}^{\max})_S \to A[M]$ is a $\mathcal{R}^{\max}$-equivariant isomorphism from the constant group scheme $(\mathcal{R}^{\max}/M\mathcal{R}^{\max})_S$ to the group scheme of M-division points of A.

The Shimura curve $X_M = X_{N^+,pN^-,M}$ is defined to be the fine moduli scheme classifying the abelian surfaces with QM and full level-M structure. It is still a smooth and projective curve over $\mathbb{Q}$. However, it is not geometrically connected. In fact, as we will see below, it is the disjoint union of $\#(\mathbb{Z}/M\mathbb{Z})^\times$ components. Note that forgetting the level-M structure yields a Galois covering $X_M \to X$ with Galois group

$$(\mathcal{R}^{\max}/M\mathcal{R}^{\max})^\times/\{\pm 1\} \cong \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})/\{\pm 1\}.$$

We proceed to define Heegner points on the Shimura curve X_M. Let F be a field of characteristic 0. An abelian surface A defined over F (with $i : \mathcal{R}^{\max} \hookrightarrow \mathrm{End}_F(A)$ and level-N structure) is said to have *complex multiplication* (CM) if $\mathrm{End}_{\mathcal{R}^{\max}}(A) \neq \mathbb{Z}$. In that case, $\mathcal{O} := \mathrm{End}_{\mathcal{R}^{\max}}(A)$ is an order in an imaginary quadratic number field K, and one says that A has CM by $\mathcal{O}$.

Definition 2.4. A point on the Shimura curve X_M is called a *CM point* if it can be represented by a quadruple $(A, i, G, \overline{\nu})$ such that A has complex multiplication by $\mathcal{O}$. A *Heegner point* is a CM point for which the subgroup G is $\mathcal{O}$-stable.

Remark 2.5. Suppose that A has QM by $\mathcal{R}^{\max}$ and CM by $\mathcal{O}_K$. Suppose that $\mathcal{O}_K$ splits $\mathcal{R}^{\max}$. Then

$$\mathrm{End}_F(A) \cong \mathcal{O}_K \otimes \mathcal{R}^{\max} \cong M_2(\mathcal{O}_K).$$

By $\mathrm{End}_F(A)$, we mean the endomorphisms of A as an algebraic variety over F. Fixing an isomorphism $\mathrm{End}(A) \cong M_2(\mathcal{O}_K)$ yields an isomorphism $A \cong E \times E$, where E is an elliptic curve defined over H, the Hilbert class field of F, with $\mathrm{End}_H(E) \cong \mathcal{O}_K$. In particular, E is an elliptic curve with complex multiplication.

We use crucially a uniformization result due to Čerednik and Drinfel'd, which gives an explicit realization of the Shimura curves X and X_M as quotients of the p-adic upper half-plane. Let B be the *definite* rational quaternion algebra of discriminant N^-, and let R be an Eichler $\mathbb{Z}[1/p]$-order of level N^+ in B. Write $R_1^\times$ for the group of units of reduced norm one in R, and fix an isomorphism $\iota_p : B \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} M_2(\mathbb{Q}_p)$.

In [Shi94, Proposition 9.3], it was shown that ι_p identifies the group $R_1^\times$ with a discrete cocompact subgroup Γ of $\mathrm{SL}_2(\mathbb{Q}_p)$, and therefore one may consider the quotient $X_\Gamma := \Gamma\backslash\mathcal{H}_p$. The celebrated result of Čerednik–Drinfel'd gives a deep relationship of the Shimura curves X and X_M defined above, with X_Γ. Although the result is originally by Čerednik and Drinfel'd, a more detailed exposition of the proof can be found in [BC91, ch. III, 5.3.1].

Theorem 2.6 (Čerednik–Drinfel'd). *There is an isomorphism of rigid-analytic varieties:*

$$(X_{\mathbb{Q}_p^{\mathrm{ur}}})^{\mathrm{an}} \cong X_\Gamma := \Gamma\backslash\mathcal{H}_p.$$

Moreover, for any integer $M \geqslant 3$, let Γ_M be the subgroup of units of reduced norm congruent to 1 modulo M. There is an isomorphism of rigid-analytic varieties:

$$(X_M)^{\mathrm{an}}_{\mathbb{Q}_p^{\mathrm{ur}}} \cong \Gamma\backslash(\mathcal{H}_p \times (R/MR)^\times) \cong \coprod_{(\mathbb{Z}/M\mathbb{Z})^\times} \Gamma_M\backslash\mathcal{H}_p,$$

which exhibits X_M as a disjoint union of Mumford curves, and hence it is semistable.

Let $n \geqslant 0$ be an even integer. We first define modular forms as global sections of certain sheaves associated to the Shimura curve $X := X_{N^+,pN^-}$. Let $\mathcal{B}, \mathcal{R}^{\max}, \mathcal{R}$ be as in the definition of X, and choose an auxiliary Eichler order $\tilde{\mathcal{R}} \subseteq \mathcal{R}$ with the property that the group of units of norm one $\tilde{\mathcal{R}}_1^\times$ is a finite-index free subgroup of $\mathcal{R}_1^\times$. Let $\tilde{X}$ be the Shimura curve associated to the order $\tilde{\mathcal{R}}$, and let G be the finite group $\mathcal{R}_1^\times/\tilde{\mathcal{R}}_1^\times$ corresponding to the cover $\mathrm{pr} : \tilde{X} \to X$.

Definition 2.7. Let K be a field of characteristic 0. A *modular form of weight $n+2$ on X defined over K* is a global section of the sheaf $\mathrm{pr}_* \Omega^{\otimes(n+2)/2}_{\tilde{X}_K/K}$ on X_K which is invariant under the action of G. We denote by $M_{n+2}(X, K)$ the space of such modular forms.

A simple argument shows that this definition does not depend on the choice of the auxiliary Eichler order $\tilde{\mathcal{R}}$. Let K be either $\mathbb{Q}_p^{\mathrm{ur}}$ or any complete field contained in $\mathbb{C}_p$ which contains $\mathbb{Q}_{p^2}$. Using the result of Čerednik–Drinfel'd stated in Theorem 2.6, we can give a more concrete description of $M_{n+2}(X, K)$.

Definition 2.8. A p-adic modular form of weight $n+2$ for Γ is a rigid-analytic function $f : \mathcal{H}_p(\mathbb{C}_p) \to \mathbb{C}_p$, defined over K, such that

$$f(\gamma z) = (cz+d)^{n+2} f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Denote the space of such p-adic modular forms by $M_{n+2}(\Gamma) = M_{n+2}(\Gamma, K)$.

The map $f \mapsto \omega_f := f(z)\, dz^{\otimes(n+2)/2}$ induces a canonical isomorphism $M_{n+2}(\Gamma, K) \cong M_{n+2}(X, K)$. In order to justify our interest in modular forms over Shimura curves, we would like to relate them to more familiar objects. Let $\mathbb{T}$ be the abstract Hecke algebra generated by the Hecke operators T_ℓ for $\ell \nmid N$ and U_ℓ for $\ell \mid N$. The Hecke algebra $\mathbb{T}$ acts naturally on the space $M_{n+2}(X, K)$, on which also act the Atkin–Lehner involutions.

Theorem 2.9 (Jacquet–Langlands). *Let K be a field. There is an isomorphism*

$$M_{n+2}(X, K) \xrightarrow{\sim} S_{n+2}(\Gamma_0(N), K)^{pN^{-}\text{-new}},$$

which is compatible with the action of $\mathbb{T}$ and the Atkin–Lehner involutions on each of the spaces.

Therefore, to a classical modular pN^--new eigenform f_∞ on the modular curve $X_0(N)$, there is associated an eigenform f on the Shimura curve X, which is unique up to scaling. In § 5, we will consider a p-adic L-function attached to f which interpolates special values of the classical L-function associated to f_∞.

3. Filtered (ϕ, N)-modules

Let K be a field of characteristic 0, which is complete with respect to a discrete valuation and has perfect residue field κ, of characteristic $p > 0$. Let $K_0 \subseteq K$ be the maximal unramified subfield of K. Concretely, K_0 is the fraction field of the ring of Witt vectors of κ. Let $\sigma : K_0 \to K_0$ be the absolute Frobenius automorphism.

Consider the category of filtered Frobenius monodromy modules (or filtered (ϕ, N)-modules for short) over K, denoted by $MF_K^{(\phi,N)}$. Its objects are quadruples $(D, \mathrm{Fil}^\bullet, \phi, N)$, where D is a finite-dimensional K_0-vector space, $\mathrm{Fil}^\bullet = \mathrm{Fil}_D^\bullet$ is an exhaustive and separated decreasing filtration on the vector space $D_K := D \otimes_{K_0} K$ over K (called the *Hodge filtration*), $\phi = \phi_D : D \to D$ is a σ-linear automorphism, called the *Frobenius on* D, and $N = N_D : D \to D$ is a K-linear endomorphism, called the *monodromy operator*, which satisfies $N\phi = p\phi N$. Sometimes we write D to refer to the tuple $(D, \mathrm{Fil}_D^\bullet, \phi_D, N_D)$. For a precise definition of this category, please refer to [Fon94] or to the lecture notes [BC09].

Forgetting the monodromy action or, equivalently, setting $N = 0$ gives a full subcategory of $\mathrm{MF}_K^{(\phi,N)}$, called the category of *filtered F-isocrystals over K*. The full subcategory obtained by additionally forgetting the filtration is the category of *isocrystals over* K_0, which were studied and classified by Dieudonné and Manin.

The category $MF_K^{(\phi,N)}$ is an additive tensor category which admits kernels and cokernels. Also, if $D = (D, F_D^\bullet, \phi_D, N_D)$ is a filtered (ϕ, N)-module, and j is an integer, we define another filtered (ϕ, N)-module $D[j]$, the jth *Tate twist* of D, as $D[j] = (D, F_D^{\bullet - j}, p^j \phi_D, N_D)$, where, by $F_D^{\bullet - i}$, we mean

$$F^i(D[j]_K) = F^{i-j}(D_K) \quad \text{for all } i \in \mathbb{Z}.$$

Consider the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of p-adic representations of G_K, whose objects are finite-dimensional $\mathbb{Q}_p$-vector spaces with a continuous linear G_K-action. It is an abelian tensor category, with twists given by tensoring with powers of the Tate representation $\mathbb{Q}_p(1) := (\varprojlim_n \boldsymbol{\mu}_{p^n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

The functors D_{st} and V_{st} of Fontaine, constructed originally in [Fon94], are functors relating the category of p-adic representations of G_K with that of filtered Frobenius monodromy modules:

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_K) \underset{\mathrm{V_{st}}}{\overset{\mathrm{D_{st}}}{\rightleftarrows}} \mathrm{MF}_K^{(\phi,N)}.$$

The functors D_{st} and V_{st} induce an equivalence of categories between certain full subcategories of those on which they are defined, and we wish to recall here this result. Let X/K be a proper variety with a semistable model. Consider the étale cohomology groups

$$H^i_{\mathrm{et}}(\overline{X}, \mathbb{Q}_p) := \left(\varprojlim_n H^i_{\mathrm{et}}(\overline{X}, \mathbb{Z}/p^n\mathbb{Z})\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

These vector spaces are naturally finite-dimensional continuous G_K-representations. In the category of $\mathrm{Rep}(G_K)$, one has the notion of semistability, and results of Fontaine–Messing, Hyodo–Kato, Faltings and Tsuji imply that these representations are semistable. They constitute in fact the main source of semistable representations.

In general, write $\mathrm{Rep}_{\mathrm{st}}(G_K)$ for the full subcategory of $\mathrm{Rep}(G_K)$ of semistable objects. Also, call a filtered (ϕ, N)-module D *admissible* if it is in the essential image of the restriction of the functor $\mathrm{D_{st}}$ to $\mathrm{Rep}_{\mathrm{st}}(G_K)$. One also has an intrinsic notion of admissibility (called weak admissibility), but we will not use this notion here. Write $\mathrm{MF}_K^{\mathrm{ad},(\phi,N)}$ for the full subcategory of admissible filtered (ϕ, N)-modules.

THEOREM 3.1 [CF00, Theorem A]. *The functors* $\mathrm{D_{st}}$ *and* $\mathrm{V_{st}}$ *give an equivalence of categories between* $\mathrm{Rep}_{\mathrm{st}}(G_K)$ *and* $\mathrm{MF}_K^{\mathrm{ad},(\phi,N)}$*, which is compatible with exact sequences, tensor products and duality.*

The main use that we have for this fact is the following corollary.

COROLLARY 3.2. *Let V, W be two objects in* $\mathrm{Rep}_{\mathrm{st}}(G_K)$*. The functors* $\mathrm{D_{st}}$ *and* $\mathrm{V_{st}}$ *induce a canonical group isomorphism*

$$\mathrm{Ext}^1_{\mathrm{Rep}_{\mathrm{st}}(G_K)}(V, W) \cong \mathrm{Ext}^1_{\mathrm{MF}_K^{\mathrm{ad},(\phi,N)}}(\mathrm{D_{st}}(V), \mathrm{D_{st}}(W)),$$

where $\mathrm{Ext}^1_{\mathcal{C}}$ *denotes the extension-group bi-functor in the category* $\mathcal{C}$.

Next we study the extensions of filtered (ϕ, N)-modules. Let D be an object in this category. Given a rational number $\lambda = r/s$, where $r, s \in \mathbb{Z}$ are such that $(r, s) = 1$ and $s > 0$, define D_λ to be the largest subspace of D which has an $\mathcal{O}_{K_0}$-stable lattice M satisfying $\phi^s(M) = p^r M$. The subspace D_λ is called the *isotypical component* of D of slope λ. The *slopes* of D are the rational numbers λ such that $D_\lambda \neq 0$, and D is called *isotypical* of slope λ_0 if $D = D_{\lambda_0}$. The Dieudonné–Manin classification gives a slope decomposition of isocrystals:

$$D = \bigoplus_{\lambda \in \mathbb{Q}} D_\lambda.$$

Note also that $N(D_\lambda) \subseteq D_{\lambda-1}$ for all $\lambda \in \mathbb{Q}$. The following result appears in [IS03, Lemma 2.1], although its proof is mostly omitted. For completeness, we present here a fully detailed proof.

LEMMA 3.3. *Let D be a filtered (ϕ, N)-module, n an integer and assume that N induces an isomorphism between the isotypical components D_{n+1} and D_n. There is a canonical isomorphism*

$$\mathrm{Ext}^1_{\mathrm{MF}_K^{(\phi,N)}}(K[n+1], D) \cong D/\operatorname{Fil}^{n+1} D$$

mapping the class of an extension

$$0 \to D \xrightarrow{\iota} E \xrightarrow{\pi} K[n+1] \to 0$$

to $(s_1(1) - s_2(1)) + \mathrm{Fil}^{n+1} D_K$*, where:*

(i) $s_1 : K[n+1] \to E$ *is a splitting of* π *which is compatible with the Frobenius and monodromy operators, but not necessarily with filtrations; and*

(ii) $s_2 : K[n+1] \to E$ *is a splitting of* π *compatible with the filtrations, but not necessarily with the Frobenius and monodromy operators.*

Proof. First, note that by applying the snake lemma to the following diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Fil}^{n+1} D_K \arrow[r] \arrow[d, hook] & \mathrm{Fil}^{n+1} E_K \arrow[r] \arrow[d, hook] & \mathrm{Fil}^{n+1} K \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & D_K \arrow[r] & E_K \arrow[r] & K \arrow[r] & 0 \end{tikzcd}$$

we get an isomorphism $D_K/\,\mathrm{Fil}^{n+1} D_K \cong E_K/\,\mathrm{Fil}^{n+1} E_K$, and hence we just need to find an element in $E_K/\,\mathrm{Fil}^{n+1} E_K$. Explicitly, once we get $s_1(1) \in E_K$, we can consider $s_1(1) - s_2(1)$, where s_2 is a splitting of the extension which is compatible with the filtrations. Such a splitting s_2 exists because the category of K-vector spaces is semisimple. Since $\pi(s_1(1) - s_2(1)) = 0$, we can view $s_1(1) - s_2(1)$ as an element of D_K (via ι), thus making the isomorphism explicit.

The filtered (ϕ, N)-module $K[n+1]$ is pure of slope $n+1$, and the hypothesis on the monodromy action N on D gives a commutative diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & D_{n+1} \arrow[r] \arrow[d, "N", "\cong"'] & E_{n+1} \arrow[r] \arrow[d, "N"] & K_0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D_n \arrow[r] & E_n \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

An application of the snake lemma and the fact that the left vertical arrow is an isomorphism yields another isomorphism

$$\pi_| : \ker(E_{n+1} \xrightarrow{N} E_n) \xrightarrow{\sim} K_0,$$

and we define $s_1 : K \to E_K$ as its inverse. Then s_1 is compatible with the action of ϕ and N, by construction.

We check that the assignment of $s_1(1) + \mathrm{Fil}^{n+1} E_K$ to an extension $0 \to D \to E \to K[n+1] \to 0$ is well defined: if the extension is trivial, then s_1 can be chosen to be compatible with Fil, and we then get

$$s_1(1) \in s_1(\mathrm{Fil}^n K[n+1]) \subseteq \mathrm{Fil}^{n+1} E_K.$$

Conversely, given $d + \mathrm{Fil}^{n+1} D_K \in D_K/\,\mathrm{Fil}^{n+1} D_K$, we construct a filtered (ϕ, N)-module $E^{(d)}$ as an extension of $K[n+1]$ by D. We define $E_0^{(d)} = D_0 \oplus (K_0[n+1])$, as (ϕ, N)-modules. The filtration on $E_K^{(d)} = E_0^{(d)} \otimes_{K_0} K$ is defined as follows:

$$\mathrm{Fil}^j E_K^{(d)} := \{(x, t) \in D_K \oplus K \mid t \in \mathrm{Fil}^{j-n-1} K,\ x + td \in \mathrm{Fil}^j D\}.$$

Consider the isomorphism class of the extension

$$\Xi : \quad 0 \to D \xrightarrow{\iota} E^{(d)} \xrightarrow{\pi} K[n+1] \to 0,$$

where the map ι is the canonical inclusion, and the map π is the canonical projection. Note that this sequence is exact and well defined, since

$$\pi(\mathrm{Fil}^j E_K^{(d)}) = \mathrm{Fil}^{j-n-1} K = \mathrm{Fil}^j K[n+1].$$

Moreover, if $d \in \mathrm{Fil}^{n+1} D_K$, then the map $1 \mapsto (0,1)$ splits the extension Ξ in the category of filtered (ϕ, N)-modules. Hence, the map

$$D_K / \mathrm{Fil}^{n+1} D_K \to \mathrm{Ext}^1(K[n+1], D),$$

which assigns the extension Ξ to $d \in D/\mathrm{Fil}^{n+1} D$, is well defined.

To end the proof, we need to check that the two assignments are mutually inverse. Starting with $d + \mathrm{Fil}^{n+1} D_K$, the vector space splitting $1 \mapsto (0,1)$ is compatible with the Frobenius and monodromy actions. Also, the vector space splitting $1 \mapsto (-d, 1)$ is compatible with the filtrations. We obtain the class of d in $D_K / \mathrm{Fil}^{n+1} D_K$, as wanted.

Conversely, start with an arbitrary extension

$$0 \to D \xrightarrow{\iota} E \xrightarrow{\pi} K[n+1] \to 0.$$

Choose s_1 and s_2 two splittings of π as before, and define $d \in D_K$ such that $\iota_K(d) = s_1(1) - s_2(1)$. Consider now the map $E^{(d)} \to E$ sending

$$(x, t) \mapsto \iota(x) + s_1(t) = \iota(x + td) + s_2(t).$$

The first expression shows that this is a map of (ϕ, N)-modules. The second expression shows that it respects the filtrations. Its inverse is the map

$$y \mapsto (\iota^{-1}(y - s_1(\pi(y))), \pi(y)) = (\iota^{-1}(y - s_2(\pi(y))) - \pi(y)d, \pi(y)).$$

Again, the first expression shows that it is respects the Frobenius and monodromy actions, while the second shows that it respects the filtrations. This concludes the proof. $\square$

4. The cohomology of X_Γ, and pairings

Let $\mathfrak{X} \to \mathrm{Spec}(\mathcal{O}_K)$ be a proper semistable curve with connected fibers. Suppose that its generic fiber X is smooth and projective, that the irreducible components $C_1, \ldots, C_r$ of the special fiber C are smooth and geometrically connected and that there are at least two of them. Assume also that the singular points of C are κ-rational ordinary double points.

Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a smooth proper morphism and let Y be the generic fiber of $\mathfrak{Y}$. The relative de Rham cohomology $\mathcal{H}^q_{\mathrm{dR}}(Y/X) := R^q f_* \mathcal{O}_{\widehat{\mathfrak{Y}}/K}$ can be given the structure of a filtered convergent F-isocrystal as explained in [CI10, Example 3.4.c]. In turn, the de Rham cohomology of X with coefficients in $\mathcal{H}^q_{\mathrm{dR}}(Y/X)$ can be given the structure of a filtered (ϕ, N)-module. Moreover, if S is a finite set of points of X which are smooth (when considered as points on $\mathfrak{X}$), and we set $U := X \backslash S$, one can also give this structure to $H^1_{\mathrm{dR}}(U, \mathcal{H}^q_{\mathrm{dR}}(Y/X))$. A detailed construction can be found in [CI10, § 2].

The following result relates the $G_{\mathbb{Q}}$-representation $H^1_{\mathrm{et}}(\overline{X}, R^q f_* \mathbb{Q}_p)$ with the filtered (ϕ, N)-module $H^1_{\mathrm{dR}}(X, \mathcal{H}^q_{\mathrm{dR}}(Y/X))$.

THEOREM 4.1 (Faltings, Coleman–Iovita [CI10, Theorem 7.5]). (i) *The representation $H^1_{\mathrm{et}}(\overline{X}, R^q f_* \mathbb{Q}_p)$ is semistable, and there is a canonical isomorphism of filtered (ϕ, N)-modules*

$$\mathrm{D}_{\mathrm{st}}(H^1_{\mathrm{et}}(\overline{X}, R^q f_* \mathbb{Q}_p)) \cong H^1_{\mathrm{dR}}(X, \mathcal{H}^q_{\mathrm{dR}}(Y/X)).$$

(ii) *Let S be a finite set of smooth sections of $f:\mathfrak{X}\to \mathrm{Spec}(\mathcal{O}_K)$, which specialize to pairwise-different points on C. Write $U=X\backslash S$, $\overline{U}=U\otimes_K\overline{K}$ and let $Y_{\overline{x}}$ be the geometric fiber of $f:Y\to X$ over $x\in S$. Then there is an exact sequence of semistable Galois representations*

$$0\to H^1_{\mathrm{et}}(\overline{X},R^qf_*\mathbb{Q}_p)\to H^1_{\mathrm{et}}(\overline{U},R^qf_*\mathbb{Q}_p)\to\bigoplus_{s\in S}H^q_{\mathrm{et}}(Y_{\overline{x}},\mathbb{Q}_p(-1)),$$

which, after applying the functor D_{st}, becomes isomorphic to the sequence

$$0\to H^1_{\mathrm{dR}}(X,\mathcal{H}^1_{\mathrm{dR}}(Y/X))\to H^1_{\mathrm{dR}}(U,\mathcal{H}^1_{\mathrm{dR}}(Y/X))\to\bigoplus_{x\in S}\mathcal{H}^1_{\mathrm{dR}}(Y/X)_x[1].$$

The constructions above can be particularized to our setting. For that, let Γ be a discrete cocompact subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$, and denote by X_Γ the Mumford curve over $\mathbb{Q}_p^{\mathrm{ur}}$ whose associated rigid-analytic space is $\Gamma\backslash\mathcal{H}_p$. Let V be an object of $\mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{GL}_2\times\mathrm{GL}_2)$. In [IS03, § 4], the authors associated to V a filtered convergent F-isocrystal on the canonical formal $\mathbb{Z}_p^{\mathrm{ur}}$-model of the upper half-plane $\widehat{\mathcal{H}}$, which is denoted $\mathcal{E}(V)$. Also, for every $\mathbb{Q}_{p^2}$-rational point $\Psi\in\mathrm{Hom}(\mathbb{Q}_{p^2},M_2(\mathbb{Q}_p))$ of $\widehat{\mathcal{H}}$, they computed the stalk $\mathcal{E}(V)_\Psi$ as a filtered (ϕ,N)-module $V_\Psi\in\mathrm{MF}^{(\phi,N)}_{\mathbb{Q}_p^{\mathrm{ur}}}$. The assignment $V\mapsto\mathcal{E}(V)$ is an exact tensor functor. This construction can be descended to give isocrystals on X_Γ. Denote the new filtered isocrystal on X_Γ by the symbol $\mathcal{E}(V)$ as well. Let $E(V)$ be the coherent locally free $\mathcal{O}_{X_\Gamma}$-module with connection and filtration corresponding to $\mathcal{E}(V)$, so that $\mathcal{E}(V)=E(V)^{\mathrm{an}}$.

In [IS03], the authors gave a concrete description of the filtered (ϕ,N)-module $H^1_{\mathrm{dR}}(X_\Gamma,E(V))$ and, if $U\subseteq X_\Gamma$ is an open subscheme as before, also of the filtered (ϕ,N)-module $H^1_{\mathrm{dR}}(U,E(V))$. We may reduce to the case of Γ being torsion free as follows: choose $\Gamma'\subset\Gamma$ a free normal subgroup of finite index. The group Γ/Γ' acts on the filtered (ϕ,N)-modules $H^1_{\mathrm{dR}}(X_\Gamma,E(V))$ and $H^1_{\mathrm{dR}}(U,E(V))$ as automorphisms preserving the operators and the filtration. Hence, it induces a structure of filtered (ϕ,N)-modules on

$$H^1_{\mathrm{dR}}(X_\Gamma,E(V))=H^1_{\mathrm{dR}}(X_{\Gamma'},E(V))^{\Gamma/\Gamma'},$$

and similarly for $H^1_{\mathrm{dR}}(U,E(V))$.

The fact that $\mathcal{H}_p$ is a Stein space in the rigid-analytic sense allows for the computation of $H^1_{\mathrm{dR}}(X_\Gamma,E(V))$ as group hypercohomology, via the Leray spectral sequence. Therefore, the elements in $H^1_{\mathrm{dR}}(X,E(V))$ are represented by pairs (ω,f_γ), where ω belongs to $\Omega^1(\mathcal{H}_p)\otimes V$ and f_γ is a $\mathcal{O}_{\mathcal{H}_p}(\mathcal{H}_p)\otimes V$-valued 1-cocycle for Γ. They are required to satisfy the relation

$$\gamma\omega-\omega=df_\gamma\quad\text{for all }\gamma\in\Gamma.$$

Let M be a $\mathbb{C}_p[\Gamma]$-module. An *M-valued cocycle* on $\mathcal{T}$ is an M-valued function c on $\vec{\mathfrak{E}}(\mathcal{T})$, such that $c(\overline{e})=-c(e)$. The $\mathbb{C}_p$-vector space of M-valued cocycles is written $C^1(M)$. An M-valued cocycle c is called *harmonic* if it satisfies

$$\sum_{o(e)=v}c(e)=0\quad\text{for all }v\in\mathfrak{V}(\mathcal{T}).$$

The $\mathbb{C}_p$-vector space of M-valued harmonic cocycles is written $C^1_{\mathrm{har}}(M)$. The actions of Γ on $\mathcal{T}$ and M induce a natural left action of Γ on the space $C^1_{\mathrm{har}}(M)$.

Now we specialize the above discussion to a particular $\mathbb{C}_p[\Gamma]$ module. Let $\mathcal{P}_n$ be the $(n+1)$-dimensional $\mathbb{Q}_p$-vector space of polynomials of degree at most n with coefficients in $\mathbb{Q}_p$.

The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on the right on $\mathcal{P}_n$ by

$$P(x)\cdot\beta := (cx+d)^n P\left(\frac{ax+b}{cx+d}\right),$$

if $\beta = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. In this way, its $\mathbb{Q}_p$-linear dual $V_n := \mathcal{P}_n^\vee$ is endowed with a left action of $\mathrm{GL}_2(\mathbb{Q}_p)$. A *harmonic cocycle of weight* $n+2$ *on* $\mathcal{T}$ is a V_n-valued harmonic cocycle.

Define now $\mathcal{U}$ as the subspace of $M_2(\mathbb{Q}_p)$ given by matrices of trace 0. They have a right action of $\mathrm{GL}_2(\mathbb{Q}_p)$ given by

$$u\cdot\beta := \overline{\beta}u\beta,$$

where $\overline{\beta}$ is the matrix such that $\overline{\beta}\beta = \det(\beta)$. There is a map $\Phi : \mathcal{U}\to\mathcal{P}_2$ intertwining the action $\mathrm{GL}_2(\mathbb{Q}_p)$, given by

$$u\mapsto P_u(x) := \mathrm{tr}\left(u\begin{pmatrix} x & -x^2 \\ 1 & -x\end{pmatrix}\right) = \mathrm{tr}\left(u\begin{pmatrix} x \\ 1\end{pmatrix}(1\ \ -x)\right) = (1-x)u\begin{pmatrix} x \\ 1\end{pmatrix}. \tag{3}$$

One easily checks that the map Φ induces an isomorphism of right $\mathrm{GL}_2(\mathbb{Q}_p)$-modules. On $\mathcal{U}$, there is a pairing defined by $\langle u, v\rangle := -\mathrm{tr}(u\overline{v})$, which induces a pairing on $\mathcal{P}_2$ by transport of structure, and on the dual V_2 of $\mathcal{P}_2$ by canonically identifying $\mathcal{P}_2$ with V_2 using the pairing itself. Furthermore, the pairing $\langle\cdot,\cdot\rangle$ on V_2 induces a perfect Γ-invariant symmetric pairing on $\mathrm{Sym}^n V_2 = V$ given by the formula

$$\langle v_1\cdots v_n, v_1'\cdots v_n'\rangle_V := \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\langle v_1, v'_{\sigma(1)}\rangle\cdots\langle v_n, v'_{\sigma(n)}\rangle.$$

There is a natural injection

$$\iota : H^1(\Gamma, V_{\mathbb{Q}_p^{\mathrm{ur}}}) \to H^1_{\mathrm{dR}}(X_\Gamma, E(V)),$$

and Schneider constructed a map $I : H^1_{\mathrm{dR}}(X_\Gamma, E(V)) \to C^1_{\mathrm{har}}(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma$. It is called 'Schneider integration' in [dS89, dS06, IS03]. Denote by $\Omega^{\mathrm{II}}_{X_\Gamma^{\mathrm{an}}}$ the space of meromorphic differentials of the second kind on X_Γ^{an}. There is a map $\Omega^{\mathrm{II}}_{X_\Gamma^{\mathrm{an}}}\otimes_{\mathcal{O}_{X_\Gamma}} V \to C^1(V_{\mathbb{Q}_p^{\mathrm{ur}}})$:

$$\omega\mapsto (e\mapsto \mathrm{res}_e(\omega)),$$

and this map induces I. The residue theorem implies that the image of I lies in $C^1_{\mathrm{har}}(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma$.

LEMMA 4.2 (Iovita–Spieß [IS03, Lemma 4.3]). *Suppose that* Γ *is arithmetic. Then the following sequence is exact:*

$$0 \longrightarrow H^1(\Gamma, V_{\mathbb{Q}_p^{\mathrm{ur}}}) \xrightarrow{\ \iota\ } H^1_{\mathrm{dR}}(X, E(V)) \xrightarrow{\ I\ } C^1_{\mathrm{har}}(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma \longrightarrow 0. \tag{4}$$

There is a retraction P of ι, given by Coleman integration. This assigns to (ω, f_γ) the 1-cocycle

$$\gamma\mapsto f_\gamma + (\gamma F_\omega - F_\omega),$$

where F_ω is a Coleman primitive for ω as at the end of § 1. Note that the $V_{\mathbb{Q}_p^{\mathrm{ur}}}$-valued function $\gamma F_\omega - F_\omega$ is constant, so that we can think of it as a well-defined element of $V_{\mathbb{Q}_p^{\mathrm{ur}}}$.

The splitting P thus defines actions of Frobenius on the left and right terms of the exact sequence (4), as follows: there is a natural action ϕ_1 of Frobenius on $H^1(\Gamma, V_{\mathbb{Q}_p^{\mathrm{ur}}})$. Define an action ϕ_2 on $C^1_{\mathrm{har}}(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma$ as $\phi_2 := p(\varepsilon^{-1}\circ\phi_1\circ\varepsilon)$, so that the following equality holds:

$$\varepsilon\phi_2 = p\phi_1\varepsilon.$$

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We have now all the maps needed in the definition of the Frobenius and monodromy operators. Define first N to be the composition $\iota \circ (-\varepsilon) \circ I$. Since $I \circ \iota = 0$, it follows that $N^2 = 0$. Actually, Lemma 4.2 implies that $\ker N = \operatorname{img} N$. Let T be the right inverse to I corresponding to P:

$$\begin{tikzcd} 0 \arrow[r] & H^1(\Gamma, V_{\mathbb{Q}_p^{\mathrm{ur}}}) \arrow[loop above, "\phi_1"] \arrow[r, "\iota"] & H^1_{\mathrm{dR}}(X_\Gamma, E(V)) \arrow[l, bend right, dashed, "P"'] \arrow[r, "I"] & C^1(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma / C^0(V_{\mathbb{Q}_p^{\mathrm{ur}}})^\Gamma \arrow[loop above, "\phi_2"] \arrow[l, bend right, dashed, "T"'] \arrow[r] & 0. \end{tikzcd}$$

Define the Frobenius operator Φ on $H^1_{\mathrm{dR}}(X_\Gamma, E(V))$ as

$$\Phi(\omega) := \iota\phi_1(P\omega) + T(\phi_2(I\omega)).$$

It can easily be checked that this definition satisfies $N\Phi = p\Phi N$, and that Φ is the unique such action which is compatible with the maps P and ι.

Let S be a finite set of points of X_Γ, and let $U = X_\Gamma \backslash S$ be the open subscheme obtained by removing the points in S. The space $H^1_{\mathrm{dR}}(U, E(V))$ is identified with the space of V-valued differential forms on X_Γ^{an} which are of the second kind when restricted to U. The monodromy is defined in the same way as before. The Frobenius is defined so that the Gysin sequence

$$0 \longrightarrow H^1_{\mathrm{dR}}(X_\Gamma, E(V)) \longrightarrow H^1_{\mathrm{dR}}(U, E(V)) \xrightarrow{\bigoplus_{x\in S} \mathrm{res}_x} \bigoplus_{x \in S} V_{\Psi_x}[1] \tag{5}$$

is a sequence of (ϕ, N)-modules, and such that P is compatible with the Frobenii.

In [IS03], the authors applied the previous constructions to a filtered isocrystal on $\mathcal{H}_p$ denoted by $\mathcal{E}(M_2)$. It was shown in [CI10, Lemma 5.10] that $\mathcal{E}(M_2)$ is regular, and therefore one can define a structure of a filtered (ϕ, N)-module on its cohomology groups.

Next we make more explicit some of the constructions carried out in [IS03, Appendix]. Let $U = X \backslash \{x\}$, where x is a closed point of X defined over the base field K. Write $j : U \to X = X_\Gamma$ for the canonical inclusion. Let z be a lift of x to $\mathcal{H}_p(K)$, taken inside a good fundamental domain $\mathcal{F}$. We assume that the stabilizer of z under the action of Γ is trivial. Let $\mathrm{Ind}^\Gamma(V)$ be the Γ-representation given by $\mathrm{Maps}(\Gamma, V)$, with Γ-action

$$(\gamma \cdot f)(\tau) := \gamma f(\gamma^{-1}\tau).$$

Let $\mathrm{ad} : V \to \mathrm{Ind}^\Gamma(V)$ be defined as the constant map: $\mathrm{ad}(v)(\tau) := v$. Consider the complex $\mathcal{K}^\bullet(V)$, concentrated on degrees zero and one, defined as

$$\mathcal{K}^\bullet(V) : \quad V \xrightarrow{\mathrm{ad}} \mathrm{Ind}^\Gamma(V).$$

Consider also the complex $C^\bullet(V)$ defined as follows:

$$C^\bullet(V) : \quad \mathcal{O}_{\mathcal{H}_p}(\mathcal{H}_p) \otimes V \xrightarrow{(d, \mathrm{ev}_z)} \Omega^1(\mathcal{H}_p) \otimes V \oplus \mathrm{Ind}^\Gamma(V).$$

The *cohomology with compact support* on U with coefficients in $E(V)$ is the hypercohomology group:

$$H^1_{\mathrm{dR,c}}(U, E(V)) \cong \mathbb{H}^1(\Gamma, C^\bullet(V)).$$

The inclusion j induces natural maps

$$H^1_{\mathrm{dR,c}}(U, E(V)) \xrightarrow{j_*} H^1_{\mathrm{dR}}(X, E(V)), \quad H^1_{\mathrm{dR}}(X, E(V)) \xrightarrow{j^*} H^1_{\mathrm{dR}}(U, E(V)).$$

One also has a short exact sequence

$$0 \to \mathbb{H}^1(\Gamma, \mathcal{K}^\bullet(V)) \xrightarrow{\iota_{U,c}} H^1_{\mathrm{dR,c}}(U, E(V)) \xrightarrow{I_{U,c}} C^1_{\mathrm{har}}(V)^\Gamma \to 0,$$

and $\iota_{U,c}$ has a retraction $P_{U,c}$ defined as follows. Let $\mathcal{F}(V)$ be the subspace of those V-valued locally analytic functions on $\mathcal{H}_p$ which are primitives of elements of $\Omega^1(\mathcal{H}_p) \otimes V$. One checks that the complex

$$\mathcal{F}(V) \to \Omega^1(\mathcal{H}_p) \otimes V \oplus \mathrm{Ind}^\Gamma(V)$$

is quasi-isomorphic to $\mathcal{K}^\bullet$. The map $P_{U,c}$ is the natural map:

$$P_{U,c} : H^1_{\mathrm{dR,c}}(U, E(V)) = \mathbb{H}^1(\Gamma, C^\bullet) \to \mathbb{H}^1(\Gamma, \mathcal{K}^\bullet).$$

There is a surjective map $\delta : C^1(V) \to C^0(V)$ defined by

$$\delta(f)(v) := \sum_{o(e)=v} f(e).$$

Let $v_0 := \mathrm{red}(z)$. Define $C_U(V)$ to be the Γ-module

$$C_U(\Gamma) = \{(f, g) \in C^1(V) \oplus \mathrm{Ind}^\Gamma(V) \mid \mathrm{supp}(f) = \Gamma v_0, f(\gamma v_0) = g(\gamma)\}.$$

The map $I_U : H^1_{\mathrm{dR}}(U, E(V)) \to C_U(V)^\Gamma$ is naturally induced from the map $\tilde{I}_U : \Omega^1(\mathcal{H}_p)(\log(|z|)) \otimes V \to C_U(V)$, defined by

$$\tilde{I}_U(\omega)(e, \gamma) := (\mathrm{res}_e(\omega), \mathrm{res}_{\gamma(z)}(\omega)).$$

Also, the map ι_U is induced from the natural inclusion

$$V \to \mathcal{O}_{\mathcal{H}_p}(\mathcal{H}_p)(\log(|z|)) \otimes V.$$

Finally, the splitting P_U is defined by the same formula as the one defining P.

To sum up, we have described a commutative diagram with exact split rows.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{H}^1(\Gamma, \mathcal{K}^\bullet(V)) \arrow[r, "\iota_{U,c}"] \arrow[d, "q_*"] & H^1_{\mathrm{dR,c}}(U, E(V)) \arrow[r, "I_{U,c}"] \arrow[d, "j_*"] \arrow[l, dashed, bend right, "P_{U,c}"'] & C^1_{\mathrm{har}}(V)^\Gamma \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^1(\Gamma, V) \arrow[r, "\iota"] \arrow[d, equal] & H^1_{\mathrm{dR}}(X, E(V)) \arrow[r, "I"] \arrow[d, "j^*"] \arrow[l, dashed, bend right, "P"'] & C^1_{\mathrm{har}}(V)^\Gamma \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & H^1(\Gamma, V) \arrow[r, "\iota_U"] & H^1_{\mathrm{dR}}(U, E(V)) \arrow[r, "I_U"] \arrow[l, dashed, bend right, "P_U"'] & C_U(V)^\Gamma \arrow[r] & 0
\end{tikzcd} \tag{6}$$

Here the bent arrows mean splittings of the corresponding maps, and the vertical dotted arrow means the natural induced map on the quotient.

We end this section by describing certain pairings among the above spaces. Recall the pairing $\langle \cdot, \cdot \rangle_V$ defined on $V = V_n$. It induces a pairing $\langle \cdot, \cdot \rangle_\Gamma$:

$$\langle \cdot, \cdot \rangle_\Gamma : C^1_{\mathrm{har}}(V)^\Gamma \otimes H^1(\Gamma, V) \to K,$$

given as follows: choose a free subgroup $\Gamma' \subset \Gamma$ of finite index, and let $\mathfrak{F}$ be a good fundamental domain for Γ' as in [dS89, § 2.5]. Let $b_1, \ldots, b_g, c_1, \ldots, c_g$ be the free edges for $\mathfrak{F}$. For $f \in C^1_{\mathrm{har}}(V)^\Gamma$ and $[z] \in H^1(\Gamma, V)$, the pairing is given by the formula

$$\langle [z], f \rangle_\Gamma = \frac{1}{[\Gamma : \Gamma']} \sum_{i=1}^{g} \langle z(\gamma_i), f(c_i) \rangle_{V_n}.$$

The cup product induces a pairing $\langle \cdot, \cdot \rangle_{X_\Gamma}$ on $H^1_{\mathrm{dR}}(X_\Gamma, E(V))$, which is related to the above by the following formula.

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Theorem 4.3 (de Shalit [dS88, dS89], Iovita–Spieß [IS03]). *For any two classes* x, y *in* $H^1_{\mathrm{dR}}(X_\Gamma, E(V))$ *the cup product* $\langle x, y\rangle_{X_\Gamma}$ *satisfies*

$$\langle x, y\rangle_{X_\Gamma} = \langle P(x), I(y)\rangle_\Gamma - \langle I(x), P(y)\rangle_\Gamma. \tag{7}$$

The cup product also induces a pairing:

$$\langle\cdot,\cdot\rangle_U : H^1_{\mathrm{dR,c}}(U, E(V)) \times H^1_{\mathrm{dR}}(U, E(V)) \to K,$$

satisfying

$$\langle j_* y_1, y_2\rangle_X = \langle y_1, j^* y_2\rangle_U.$$

Finally, one can obtain an explicit description of a pairing

$$\langle\cdot,\cdot\rangle_{\Gamma,U} : H^1(\Gamma, \mathcal{K}^\bullet(V)) \otimes C_U(V)^\Gamma \to K.$$

Proposition 4.4 (Iovita–Spieß [IS03, Appendix]). *Let* $x \in H^1(\Gamma, \mathcal{K}^\bullet(V))$ *be represented by* (ζ, f)*, such that*

$$\mathrm{ad} \circ \zeta = \partial(f),$$

with $\zeta \in Z^1(\Gamma, V)$ *a* 1*-cocycle and* $f \in \mathrm{Ind}^\Gamma(V)$ *satisfying*

$$(\partial f)(\gamma) = \gamma f - f.$$

Let $(g, g') \in C^1(V) \oplus \mathrm{Ind}^\Gamma(V)$ *be an element in* $C_U(V)^\Gamma$*. Choose a free subgroup* $\Gamma' \subset \Gamma$ *of finite index, and let* $\mathfrak{F}$ *be a good fundamental domain for* Γ' *as in [dS89, § 2.5]. Let* $b_1, \ldots, b_g, c_1, \ldots, c_g$ *be the free edges for* $\mathfrak{F}$*. Then*

$$\langle[(\zeta, f)], (g, g')\rangle_{\Gamma,U} = \frac{1}{[\Gamma : \Gamma']}\sum_{i=1}^{g}\langle\zeta(\gamma_i), g(c_i)\rangle + \langle f(1), g'(1)\rangle.$$

5. The anti-cyclotomic p-adic L-function

In [BDIS02], the authors defined the anti-cyclotomic p-adic L-function which interpolates special values of a classical L-function (see [BDIS02, § 2.5]). Assume from now on that p is inert in K, and fix an isomorphism $\iota : B_p \to M_2(\mathbb{Q}_p)$.

Let f be a rigid-analytic modular form of weight $n + 2$ on Γ, and denote by $\mathcal{A}_n$ the set of $\mathbb{C}_p$-valued functions on $\mathbb{P}^1(\mathbb{Q}_p)$ which are locally analytic except for a pole of order at most n at ∞. Note that the subspace $\mathcal{P}_n$ of polynomials of degree at most n is dense in $\mathcal{A}_n$. In [Tei90], Teitelbaum associated to f a distribution μ_f on $\mathcal{A}_n$, in such a way that it vanishes on $\mathcal{P}_n$. This is done by extending to $\mathcal{A}_n$ the measure defined by

$$\mu_f(P \cdot \chi_{U(e)}) := \int_{U(e)} P(x)\, d\mu_f(x) := \mathrm{res}_e(f(z)P(z)\, dz),$$

where the polynomial P belongs to $\mathcal{P}_n$, and e is an edge in $\vec{\mathfrak{E}}(\mathcal{T})$ such that $\infty \notin U(e)$. The p-adic residue formula gives the following lemma.

Lemma 5.1. *If* $P \in \mathcal{P}_n$*, then*

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} P(x)\, d\mu_f(x) = 0.$$

Proof. Decompose $\mathbb{P}^1(\mathbb{Q}_p) = \coprod_{i=0}^{p} U(e_i)$ as a disjoint union of compact open subsets, where $e_0, \ldots, e_p$ are the edges leaving the distinguished vertex v_0. Then, since c_f is a harmonic cocycle,

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} P(x)\,d\mu_f(x) = \sum_{i=0}^{p} c_f(e_i)(P(X)) = \left(\sum_{i=0}^{p} c_f(e_i)\right)(P(X)) = 0. \qquad \square$$

The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts also on $\mathcal{A}_n$ with weight n, by the rule

$$(\varphi * \beta)(x) := (cx+d)^n \varphi(\beta \cdot x), \quad \varphi \in \mathcal{A}_n, \quad \text{and} \quad \beta \in \mathrm{PGL}_2(\mathbb{Q}_p).$$

One can also recover a modular form f from its associated distribution.

PROPOSITION 5.2 (Teitelbaum [Tei90, Theorem 3]). *Let f be a rigid-analytic modular form of weight $n+2$ on Γ, and let μ_f be the associated distribution on $\mathbb{P}^1(\mathbb{Q}_p)$. Then*

$$f(z) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{1}{z-t}\,d\mu_f(t).$$

For the rest of the paper, assume that K is a number field satisfying:

(i) all primes dividing pN^- are inert in K; and

(ii) all primes dividing N^+ are split in K.

In particular, note that we require the discriminant of K to be coprime to $N = pN^-N^+$. An embedding $\Psi : K \to B$ is called *optimal* if $\Psi(K) \cap R = \Psi(\mathcal{O})$, so that Ψ induces an embedding of $\mathcal{O}$ into R. The *partial p-adic L-function* depends on a pair $(\Psi, \star)$, of an optimal embedding $\Psi : K \to B$ and a base point $\star \in \mathbb{P}^1(\mathbb{Q}_p)$, and it is constructed in [BDIS02, § 2] from μ_f. One defines a measure on $G = K_{p,1}^\times$ denoted $\mu_{f,\Psi,\star}$, and the *partial p-adic L-function* is the p-adic Mellin transform

$$L_p(f, \Psi, \star, s) := \int_G x^{s-(n+2)/2}\,d\mu_{f,\Psi,\star}(x).$$

An abelian extension L/K is called *anti-cyclotomic* if it is Galois over $\mathbb{Q}$ and if the involution in $\mathrm{Gal}(K/\mathbb{Q})$ acts (by conjugation) as -1 on $\mathrm{Gal}(L/K)$. Let K_∞ denote the maximal anti-cyclotomic extension of K unramified outside p, and let H be the Hilbert class field of K. There exists a tower of extensions

$$\mathbb{Q} \subset K \subset H \subset K_\infty.$$

Assume for simplicity that $\mathcal{O}_K^\times = \{\pm 1\}$. By class field theory, the p-adic group G is isomorphic to $\mathrm{Gal}(K_\infty/H)$. Let $\Delta := \mathrm{Gal}(H/K)$, and write also $\tilde{G} := \mathrm{Gal}(K_\infty/K)$. These fit into an exact sequence

$$1 \to G \to \tilde{G} \to \Delta \to 1,$$

and in [BDIS02, Lemma 2.13] it was shown how the natural action of $\Delta := \mathrm{Pic}(\mathcal{O})$ on the set of (oriented) optimal embeddings $\mathrm{emb}(\mathcal{O}, R)$ lifts to an action of $\tilde{G}$ on the same set. The logarithm $\log_p$ extends uniquely to $\tilde{G}$, and thus one can define x^s for $s \in \mathbb{Z}_p$ and $x \in \tilde{G}$. One also defines what it means for a function $\varphi : \tilde{G} \to \mathbb{C}_p$ to be locally analytic and, by averaging over the finite set Δ, one defines another distribution $\mu_{f,K}$ on the space of analytic functions on $\tilde{G}$. Finally, $L_p(f, K, s)$ is defined as

$$L_p(f, K, s) := \int_{\tilde{G}} \alpha^{s-(n+2)/2}\,d\mu_{f,K}(\alpha), \quad s \in \mathbb{Z}_p.$$

Let as before p be an inert prime. By the very definition of $\mu_{f,K}$ as above, the anti-cyclotomic p-adic L-function $L_p(f, K, s)$ vanishes at the values $s = 1, \ldots, n+1$ (see Lemma 5.1). One is then interested in the first derivative. Write first

$$L_p'(f, K, j+1) = \int_{\widetilde{G}} \log(\alpha)\alpha^{j-n/2}\, d\mu_{f,K}(\alpha) = \sum_{i=1}^{h} L_p'(f, \Psi_i, j+1),$$

where

$$L_p'(f, \Psi_i, \infty, j+1) := \int_G \log(\alpha)\alpha^{j-n/2}\, d\mu_{f,\Psi_i,\infty}(\alpha).$$

The following formula is a generalization of [BDIS02, Theorem 3.5], which, although immediate, is not currently present in the literature.

Theorem 5.3. *Let Ψ be an optimal oriented embedding. For all j with $0 \leqslant j \leqslant n$, the following equality holds:*

$$L_p'(f, \Psi, \infty, j+1) = \int_{\overline{z}_0}^{z_0} f(z)(z-z_0)^j(z-\overline{z}_0)^{n-j}\, dz,$$

where the right-hand side is to be understood as a Coleman integral on $\mathcal{H}_p$.

Proof. Start by manipulating the expression for $L_p'(f, \Psi, j+1)$:

$$\begin{aligned} L_p'(f, \Psi, \infty, j+1) &= \int_G \log(\alpha)\alpha^{j-n/2}\, d\mu_{f,\Psi}(\alpha) \\ &= \int_{\mathbb{P}_1(\mathbb{Q}_p)} \log\left(\frac{x-z_0}{x-\overline{z}_0}\right)\left(\frac{x-z_0}{x-\overline{z}_0}\right)^{j-n/2} P_\Psi^{n/2}(x)\, d\mu_f(x) \\ &= \int_{\mathbb{P}_1(\mathbb{Q}_p)} \left(\int_{\overline{z}_0}^{z_0} \frac{dz}{z-x}\right)\left(\frac{x-z_0}{x-\overline{z}_0}\right)^{j-n/2} P_\Psi^{n/2}(x)\, d\mu_f(x), \end{aligned}$$

where the second equality follows from the change of variables $x = \eta_\Psi(\alpha)$ and the third from the definition of the logarithm. Note that from the defining property of μ_f, it follows that

$$\begin{aligned} &\int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{((x-z_0)/(x-\overline{z}_0))^{j-n/2} P_\Psi^{n/2}(x)}{z-x}\, d\mu_f(x) \\ &\quad = \int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{((z-z_0)/(z-\overline{z}_0))^{j-n/2} P_\Psi^{n/2}(z)}{z-x}\, d\mu_f(x), \end{aligned} \tag{8}$$

since the difference of the integrands is a polynomial in x of degree at most n. Using (8), a change of order of integration and Proposition 5.2, we obtain

$$\begin{aligned} L_p'(f, \Psi, \infty, j+1) &= \int_{\mathbb{P}_1(\mathbb{Q}_p)} \int_{\overline{z}_0}^{z_0} \frac{dz}{z-x}\left(\frac{x-z_0}{x-\overline{z}_0}\right)^{j-n/2} P_\Psi^{n/2}(x)\, d\mu_f(x) \\ &= \int_{\overline{z}_0}^{z_0} \left(\int_{\mathbb{P}_1(\mathbb{Q}_p)} \frac{d\mu_f(x)}{z-x}\right)\left(\frac{z-z_0}{z-\overline{z}_0}\right)^{j-n/2} P_\Psi^{n/2}(z)\, dz \\ &= \int_{\overline{z}_0}^{z_0} f(z)\left(\frac{z-z_0}{z-\overline{z}_0}\right)^{j-n/2} P_\Psi^{n/2}(z)\, dz. \end{aligned}$$

A justification for the validity of the change of the order of integration can be found in the proof given in [Tei90, Theorem 4]. □

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6. A motive

In this section, we define a certain Chow motive and calculate its realizations. We will recall the motive described in [IS03], and modify it in the spirit of [BDP09] to define the motive $\mathcal{D}_n$. Finally, we will compute the realizations of this newly constructed motive. For the definitions concerning the category of motives, we follow the construction given in [Kün94, § 2].

Let K be a field of characteristic 0. Let S be a smooth quasi-projective connected scheme over K. For simplicity, assume that S is of dimension 1, as this is the only situation that we will need in the following. Denote by $\mathbf{Sch}(S)$ the category of smooth projective schemes $X \to S$. We denote by $\mathrm{CH}^i(X)$ the ith Chow group of X, of algebraic cycles on X of codimension i, modulo rational equivalence. We denote by $\mathrm{CH}(X)$ the Chow ring of X, the product given by intersection of cycles. Given X, Y two smooth projective S-schemes, the *ring of S-correspondences* is defined as

$$\mathrm{Corr}_S(X, Y) := \mathrm{CH}(X \times_S Y).$$

Denote by $\mathbf{Mot}^0(S)$ the category of relative Chow S-motives as explained in [Kün94, § 2]. It is an additive, pseudo-abelian $\mathbb{Q}$-category with a canonical tensor product and coproducts. There is also a duality theory, and a form of Poincaré duality. Its objects are triples (X, p, i), where X/S is in $\mathbf{Sch}(S)$, p is an idempotent in $\mathrm{CH}(X \times_S X)$ and i is an integer.

The importance of Chow motives lies in their universality for the *realization functors*. For us, this means that given a motive (X, p, i), the correspondence p induces a projector on any Weil cohomology $H^*(X)$, and therefore we obtain functors H^* from the category $\mathbf{Mot}^0(S)$ to the same category where $H^*(X)$ would live, by sending (X, p, i) to $pH^*(X)$. These functors are called *realization functors*, and we will concentrate on the l-adic étale and de Rham realizations.

Fix $M \geqslant 3$, and let $X_M/\mathbb{Q}$ be the Shimura curve parameterizing abelian surfaces with quaternionic multiplication by $\mathcal{R}^{\max} \subseteq \mathcal{B}$ and level-M structure, as described in § 2. Let $\pi : \mathcal{A} \to X_M$ be the universal abelian surface with quaternionic multiplication. Consider the relative motive $h(\mathcal{A})$ as an object of $\mathbf{Mot}(X_M)$, where h is the contravariant functor

$$h : \mathbf{Sch}(X_M) \to \mathbf{Mot}^0_+(X_M)$$

from the category of smooth and proper schemes over X_M to the category of Chow motives, as explained in [Kün94, § 2]. In general, the realization functors of a motive give the corresponding cohomology groups as graded vector spaces with extra structures, and one cannot isolate the ith cohomology groups at the motivic level, without assuming the so-called 'standard conjectures'. If the underlying scheme has extra endomorphisms, then one can hope to annihilate some of these groups and thus obtain only the desired degree. The following result establishes this for abelian schemes.

THEOREM 6.1 (Deninger–Murre [DM91, Theorem 3.1, Proposition 3.3], Künnemann [Kün94]). *The motive $h(\mathcal{A})$ admits a canonical decomposition $h(\mathcal{A}) = \bigoplus_{i=0}^4 h^i(\mathcal{A})$, with $h^i(\mathcal{A}) \cong \wedge^i h^1(\mathcal{A})$ and $h^i(\mathcal{A})^\vee \cong h^{4-i}(\mathcal{A})(2)$.*

Fix an integer $M \geqslant 3$. In [IS03, Appendix], the authors defined a motive $\mathcal{M}_n^{(M)}$ for *even* $n \geqslant 2$. We now recall their construction. Let e_2 be the unique non-zero idempotent in $\mathrm{End}(\wedge^2 h^1(\mathcal{A})) = \mathrm{End}(h^2(\mathcal{A}))$ such that

$$x \cdot e_2 = \mathrm{nrd}(x) e_2 \quad \text{for all } x \in \mathcal{B}.$$

Define ε_2 to be the projector in the ring $\mathrm{Corr}_{X_M}(\mathcal{A},\mathcal{A})$ such that

$$(\mathcal{A},\varepsilon_2)=\widetilde{\mathcal{M}}_2^{(M)}:=\ker(e_2).$$

Set m as $n/2$ and define $\widetilde{\mathcal{M}}_n^{(M)}:=\mathrm{Sym}^m\,\widetilde{\mathcal{M}}_2^{(M)}$. There is a symmetric pairing, given by the cup product,

$$h^2(\mathcal{A})\otimes h^2(\mathcal{A})\to\wedge^4 h^1(\mathcal{A})\cong\mathbb{Q}(-2).$$

Let $\langle\cdot,\cdot\rangle$ be its restriction to $\widetilde{\mathcal{M}}_2^{(M)}\otimes\widetilde{\mathcal{M}}_2^{(M)}$. It induces a *Laplace operator*

$$\Delta_m:\widetilde{\mathcal{M}}_n^{(M)}\to\widetilde{\mathcal{M}}_{n-2}^{(M)}(-2),$$

given symbolically by

$$\Delta_m(x_1x_2\cdots x_m)=\sum_{1\leqslant i<j\leqslant m}\langle x_i,x_j\rangle x_1\cdots\widehat{x}_i\cdots\widehat{x}_j\cdots x_m.$$

In [IS03, § 10.1], it was shown that $\ker(\Delta_m)$ exists as a motive. Therefore, there is a correspondence ε_n in $\mathrm{Corr}_{X_M}(\mathcal{A}^m,\mathcal{A}^m)$ such that

$$(\mathcal{A}^m,\varepsilon_n)=(\mathcal{M}_n)^{(M)}:=\ker(\Delta_m).$$

Fix A an abelian surface with quaternionic multiplication. Assume also that A has CM. By Remark 2.5, A is isomorphic to $E\times E$. Fix such an isomorphism.

Let $\mathfrak{S}_n$ be the symmetric group on n letters, and consider the semidirect product

$$\Xi_n:=(\mu_2)^n\rtimes\mathfrak{S}_n,$$

with $\sigma\in\mathfrak{S}_n$ acting on $(\mu_2)^n$ by $(x_1,\dots,x_n)^\sigma=(x_{\sigma(1)},\dots,x_{\sigma(n)})$. This is isomorphic to the group of signed permutation matrices of degree n.

The group Ξ_n acts on E^n as follows: each of the copies of μ_2 acts by multiplication by -1 on the corresponding copy of E, and $\mathfrak{S}_n$ permutes the n copies.

Let $j:\Xi_n\to\{\pm1\}$ be the homomorphism which sends $-1\in\mu_2$ to -1, and which is the sign character on $\mathfrak{S}_n$, and let

$$\varepsilon_E:=\frac{1}{2^n(n)!}\sum_{\sigma\in\Xi_n}j(\sigma)\sigma\in\mathbb{Q}[\mathrm{Aut}(E^n)],$$

which is an idempotent in the rational group ring of $\mathrm{Aut}(E^n)$.

By functoriality, ε_E induces a projector in $\mathrm{Corr}_{X_M}(E^n,E^n)$.

Lemma 6.2 [BDP09, Lemma 1.8]. *The image of ε_E action on $H^*(E^n)$ is $\mathrm{Sym}^n H^1(E)$, where $H(-)$ means either $H_{\mathrm{et}}(-,\mathbb{Q}_l)$ or $H_{\mathrm{dR}}(-)$.*

We want to generalize the construction of [IS03] in the spirit of [BDP09]. Let n be a positive even integer, and set $m:=n/2$.

Definition 6.3. The motive $\mathcal{D}_n^{(M)}$ over X_M is defined as

$$\mathcal{D}_n^{(M)}:=(\mathcal{A}^m\times E^n,\varepsilon_n^{(M)}):=\mathcal{M}_n^{(M)}\otimes(E^n,\varepsilon_E),$$

where $E^n\to X_M$ is seen as a constant family $E^n\times X_M$, with fibers E^n.

We descend this construction to the Shimura curve X. For that, consider the group $G=(\mathcal{R}^{\max}/M\mathcal{R}^{\max})\cong\mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$, which acts canonically (through X-automorphisms) on X_M, on $\mathcal{A}^m$ and on E^n. Hence, we can consider the projector

$$p_G := \frac{1}{|G|}\sum_{g\in G} g \in \mathrm{Corr}_X(\mathcal{A}^m \times E^n, \mathcal{A}^m \times E^n).$$

The projector p_G commutes with both ε_n and ε_E. In fact, p_G acts trivially on E^n. So, the composition of these projectors is also a projector, which will be denoted ε.

Definition 6.4. The *generalized Kuga–Sato motive* $\mathcal{D}_n$ is defined to be

$$\mathcal{D}_n := (\mathcal{A}^m \times E^n, \varepsilon) := p_G(\mathcal{D}_n^{(M)}) = p_G(\mathcal{M}_n^{(M)}) \otimes (E^n, \varepsilon_E).$$

We now proceed to calculate the p-adic étale and de Rham realizations of the motive $\mathcal{D}_n$. Consider the p-adic étale sheaf $R^2\pi_*\mathbb{Q}_p$, which has fibers at each geometric point $\tau \to X_M$ given by $H^2_{\mathrm{et}}(\mathcal{A}_\tau, \mathbb{Q}_p)$. We want to work with a subsheaf of $R^2\pi_*\mathbb{Q}_p$. For this, note that the action of $\mathcal{R}^{\max}$ on $\mathcal{A}$ induces an action of $\mathcal{B}^\times$ on $R^2\pi_*\mathbb{Q}_p$.

Consider the p-adic étale sheaf

$$\mathbb{L}_2 := \bigcap_{b\in\mathcal{B}^\times} \ker(b - \mathrm{nrd}(b) : R^2\pi_*\mathbb{Q}_p \to R^2\pi_*\mathbb{Q}_p) \subseteq R^2\pi_*\mathbb{Q}_p,$$

which is the subsheaf on which $\mathcal{B}^\times$ acts as the reduced norm nrd of $\mathcal{B}$. It is a three-dimensional locally free sheaf on X_M. Set m to be $n/2$, and consider the map $\Delta_m : \mathrm{Sym}^m \mathbb{L}_2 \to (\mathrm{Sym}^{m-2}\mathbb{L}_2)(-2)$ given by the Laplace operator. That is,

$$\Delta_m(x_1\cdots x_m) = \sum_{1\leqslant i<j\leqslant m} (x_i, x_j)x_1\cdots\widehat{x}_i\cdots\widehat{x}_j\cdots x_m,$$

where $(\cdot,\cdot)$ is the non-degenerated pairing induced from the cup product and the trace: $(x, y) = \mathrm{tr}(x \cup y)$. Define $\mathbb{L}_n$ to be the kernel of Δ_m, and set

$$\mathbb{L}_{n,n} := \mathbb{L}_n \otimes \mathrm{Sym}^n H^1_{\mathrm{et}}(E, \mathbb{Q}_p).$$

The following lemma gives the p-adic étale realization of the motive $\mathcal{D}_n$.

Lemma 6.5. *Consider $\mathcal{D}_n$ as an absolute motive over $\mathbb{Q}$. Let $H_p(-)$ be the p-adic realization functor. Then*

$$H_p(\mathcal{D}_n) \cong H^1_{\mathrm{et}}(\overline{X_M}, \mathbb{L}_{n,n})^G = H^1_{\mathrm{et}}(\overline{X_M}, \mathbb{L}_n)^G \otimes \mathrm{Sym}^n H^1_{\mathrm{et}}(E, \mathbb{Q}_p).$$

Proof. First, note that the p-adic realization of the motive $\mathcal{D}_n^{(M)}$, as thought of as in the derived category, is the complex of $\mathbb{Q}_p$-sheaves

$$\mathbb{L}_n[-n] \otimes \mathrm{Sym}^n H^1_{\mathrm{et}}(E, \mathbb{Q}_p)$$

concentrated in degree $-n$. Then, we just need to compute

$$\begin{aligned} H_p(\mathcal{D}_n) &= (p_G)_*(H^*(\overline{X_M}, \mathbb{L}_n[-n] \otimes \mathrm{Sym}^n H^1_{\mathrm{et}}(E, \mathbb{Q}_p))) \\ &= H^{*-2n}_{\mathrm{et}}(\overline{X_M}, \mathbb{L}_n)^G \otimes \mathrm{Sym}^n H^1_{\mathrm{et}}(E, \mathbb{Q}_p), \end{aligned}$$

which follows from the cohomology of $\mathbb{L}_n$ being concentrated in degree one and from the Künneth formula. □

The Hodge filtration on $H^1_{\mathrm{dR}}(E) := H^1_{\mathrm{dR}}(E, \mathbb{Q}_p)$ induces a filtration on $\mathrm{Sym}^n H^1_{\mathrm{dR}}(E)$. The following lemma follows easily from the definitions.

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Lemma 6.6. *Write H_j for the jth step in the naturally induced filtration on* $\mathrm{Sym}^n H^1_{\mathrm{dR}}(E)$. *Then*

$$H_j = \begin{cases} \mathrm{Sym}^n H^1_{\mathrm{dR}}(E) & \text{if } j \leqslant 0, \\ \mathrm{Sym}^j H^0(E, \Omega^1_E) \otimes \mathrm{Sym}^{2n-j} H^1_{\mathrm{dR}}(E) & \text{if } 1 \leqslant j \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6.7 (Faltings, Iovita–Spieß [IS03, Lemma 5.10]). *There is a canonical isomorphism of filtered isocrystals on $\mathcal{H}_p$:*

$$\pi^* \mathcal{H}^1_{\mathrm{dR}}(\mathcal{A}/X_M) \cong \mathcal{E}(M_2).$$

This isomorphism takes the $\mathcal{B}^\times_{\mathbb{Q}_p^{\mathrm{ur}}}$-action on the left-hand side to the action by ρ_2 on the right-hand side.

Consider the representation (V_n, ρ_1) of GL_2 constructed in § 4, and let ρ_2 be the one-dimensional representation of GL_2 given by $\det^m$. Then the pair (V_n, ρ_1, ρ_2) induces a filtered convergent F-isocrystal $\mathcal{V}_n = \mathcal{E}(V_n\{m\})$ as described in § 4 and in [IS03, § 4]. It turns out that it is regular (see [IS03] after Lemma 4.3). Moreover, a simple computation using the compatibility of the isomorphism of Theorem 6.7 with tensor products gives the following consequence.

Corollary 6.8. *There is a canonical isomorphism of filtered convergent F-isocrystals*

$$\bigcap_{x \in \mathcal{B}^\times} \ker((x - \mathrm{nrd}(x)) : \mathcal{E}(\wedge^2 M_2) \to \mathcal{E}(\wedge^2 M_2)) \cong \mathcal{V}_2.$$

We believe that one has a similar result for odd n, but we refrain from formulating a precise statement for it.

There is a map from the space of modular forms on X_Γ of weight $n+2$ to $\mathrm{Fil}^{n+1} H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_n)$, given by $f(z) \mapsto \omega_f := f(z)\, \mathrm{ev}_z \otimes dz$, where ev_z is the functional that assigns to a polynomial $R(X)$ its evaluation at the point z. Identifying these spaces, one obtains the filtration of $H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_n)$.

Proposition 6.9 [IS03, Proposition 6.1]. *The filtration of $H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_n)$ is given by*

$$\mathrm{Fil}^j H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_n) = \begin{cases} H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_n) & \text{if } j \leqslant 0, \\ M_k(\Gamma) & \text{if } 1 \leqslant j \leqslant n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Define the filtered convergent F-isocrystal $\mathcal{V}_{n,n}$ as

$$\mathcal{V}_{n,n} := \mathcal{V}_n \otimes \mathrm{Sym}^n H^1_{\mathrm{dR}}(E).$$

Understanding the structure of $\mathrm{D}_{\mathrm{st}\,\mathbb{Q}_p^{\mathrm{ur}}}(H_p(\mathcal{D}_n))$ will allow us to compute the Abel–Jacobi map in an explicit way. Write $H^{2n+1}_{\mathrm{dR}}(\mathcal{D}_n)$ for the filtered (ϕ, N)-module $\mathrm{D}_{\mathrm{st},\mathbb{Q}_p^{\mathrm{ur}}}(H_p(\mathcal{D}_n))$. The following key result is a consequence of the facts shown so far.

Theorem 6.10. *The $G_{\mathbb{Q}_p}$-representation $H^{2n+1}_p(\mathcal{D}_n)$ is semistable, and there is a (canonical up to scaling) isomorphism of filtered (ϕ, N)-modules*

$$\mathrm{D}_{\mathrm{st}}(H^{2n+1}_p(\mathcal{D}_n)) = H^{2n+1}_{\mathrm{dR}}(\mathcal{D}_n) \cong H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) = H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_n) \otimes \mathrm{Sym}^n H^1_{\mathrm{dR}}(E).$$

Moreover, writing Fil^j *for* $\mathrm{Fil}^j\, H^{2n+1}_{\mathrm{dR}}(\mathcal{D}_n)$*, we have*

$$\mathrm{Fil}^j = \begin{cases} H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) & \text{if } j \leqslant 0,\\ H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_n) \otimes H_j + M_k(\Gamma) \otimes \mathrm{Sym}^n\, H^1_{\mathrm{dR}}(E) & \text{if } 1 \leqslant j \leqslant n+1,\\ M_k(\Gamma) \otimes H_{j-n-1} & \text{if } n+2 \leqslant j \leqslant 2n+1,\\ 0 & \text{otherwise}. \end{cases}$$

In particular,

$$\mathrm{Fil}^{n+1}\, H^{2n+1}_{\mathrm{dR}}(\mathcal{D}_n) \cong M_k(\Gamma) \otimes \mathrm{Sym}^n\, H^1_{\mathrm{dR}}(E).$$

Proof. To prove semistability, we can extend the base to $\mathbb{Q}_p^{\mathrm{ur}}$. In this case, the curve X is isomorphic to a disjoint union of Mumford curves, and hence it is semistable.

By Corollary 6.8, there is an isomorphism:

$$\bigcap_{x \in \mathcal{B}^\times} \ker(x - \mathrm{nrd}(x) : \mathcal{E}(\wedge^2 M_2) \to \mathcal{E}(\wedge^2 M_2)) \cong \mathcal{V}_2.$$

Applying Theorem 4.1 and functoriality, we see that the filtered (ϕ, N)-module

$$\mathrm{D}_{\mathrm{st}, \mathbb{Q}_p^{\mathrm{ur}}}(H^1_{\mathrm{et}}(\overline{X_M}, \mathbb{L}_n) \otimes \mathrm{Sym}^n\, H^1_{\mathrm{et}}(E, \mathbb{Q}_p))$$

is isomorphic to

$$H^1_{\mathrm{dR}}((X_M)_{\mathbb{Q}_p^{\mathrm{ur}}}, \mathcal{V}_n) \otimes \mathrm{Sym}^n\, H^1_{\mathrm{dR}}(E/\mathbb{Q}_p).$$

This isomorphism can then be descended to $\mathcal{D}_n$ by taking G-invariants.

Putting together Proposition 6.9 with (6.6), we obtain the formula for the filtration. □

7. Geometric interpretation of the values of $L_p'(f, K, s)$

This section contains the main result of this paper. First, we obtain a formula for the values of the derivative of the p-adic L-function, in terms of Coleman integrals on the p-adic upper half-plane. Next, we define a collection of cycles on the motive $\mathcal{D}_n$ introduced in the previous section. Lastly, we calculate this image and give the main result.

Let K be a field of characteristic 0, and let l be a prime. Let X be a smooth projective variety over K, and denote by $\mathrm{CH}^c(X)$ the Chow group of X, consisting of cycles of codimension c with *rational* coefficients. The Chow group has already been introduced in § 6 when discussing a category of relative motives with arbitrary coefficients, but here we work with a simpler setting. Consider the locally constant sheaves $\mathcal{F}_n = \mathbb{Z}/l^n\mathbb{Z}(c)$ as above, and take projective limits with respect to n, to get $\mathbb{Z}_l$-valued cohomology. Inverting l, we get $\mathbb{Q}_l$-valued cohomology, which will be denoted with H_{et} as well. Write $\overline{X}$ for the base change of X to $\overline{K}$, and let $\mathrm{CH}^c_0(X)$ be the kernel of the cycle class map (see [Mil80, § VI.9]).

Consider the *l-adic étale Abel–Jacobi map*

$$\mathrm{AJ}^{\mathrm{et}}_l : \mathrm{CH}^c_0(X) \to \mathrm{Ext}^1(\mathbb{Q}_l, H^{2c-1}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_l(c))),$$

as explained in [BDP09, Definition 4.1]. Assume that K is a p-adic field, and set $l = p$. Bloch and Kato in [BK90] and Nekovář in [Nek93] have defined, for any Galois representation V of G_K, a subspace

$$H^1_{\mathrm{st}}(K, V) := \ker(H^1_{\mathrm{et}}(K, V) \to H^1_{\mathrm{et}}(K, V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})).$$

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This is identified with the group of extension classes of V by $\mathbb{Q}_p$ in the category of semistable representations of G_K. The following result can be found in [Nek00].

LEMMA 7.1. *The image of* $\mathrm{AJ}_p^{\mathrm{et}}$ *is contained in*

$$H^1_{\mathrm{st}}(K, H^{2c-1}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_p(c))) \cong \mathrm{Ext}^1_{\mathrm{Rep}_{st}(G_K)}(\mathbb{Q}_p, H^{2c-1}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_p(c))).$$

As seen in § 3, the Fontaine functors D_{st} and V_{st} give a canonical comparison isomorphism:

$$\mathrm{Ext}^1_{\mathrm{Rep}_{\mathrm{st}}(G_K)}(\mathbb{Q}_p, H^{2c-1}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_p(c))) \cong \mathrm{Ext}^1_{\mathrm{MF}_K^{\mathrm{ad},(\phi,N)}}(K[c], \mathrm{D}_{\mathrm{st}}(H^{2c-1}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_p))),$$

which will make the computations possible.

Consider the generalized Kuga–Sato motive $\mathcal{D}_n = (\mathcal{A}^m \times E^n, \varepsilon)$ as in Definition 6.4. The construction of the p-adic Abel–Jacobi map seen above can be easily extended to the motive $\mathcal{D}_n$, by applying the projector at the appropriate places. This can be done for each realization, but we are specially interested in the de Rham realization of $\mathcal{D}_n$, which we have computed to be $H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n})$. It fits in a short exact sequence as in Theorem 4.1:

$$0 \to H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) \to H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n}) \to \bigoplus_{z \in S} (\mathcal{V}_{n,n})_z[1], \tag{9}$$

where S is a finite set of points in X_Γ lying in distinct residue classes, and U is the complement in X_Γ of S. In § 7, we will define certain cycles on $\mathcal{A}^m \times E^n$ which are supported on a fiber above a point $P \in X$. These cycles are of codimension $n+1$, and therefore sending 1 to their cycle class yields a map

$$K[n+1] \to (\mathcal{V}_{n,n})_z[1].$$

Pulling back the extension (9), we obtain another extension

$$0 \to H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) \to E \to K[n+1] \to 0.$$

Using Lemma 3.3, and the fact that the space

$$\mathrm{Fil}^{n+1} H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n})$$

is self-orthogonal, we obtain

$$H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n})/\mathrm{Fil}^{n+1} H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) \cong (M_{n+2}(\Gamma) \otimes \mathrm{Sym}^n H^1_{\mathrm{dR}}(E))^\vee.$$

The composition map will be denoted AJ_K:

$$\mathrm{AJ}_K : \mathrm{CH}^{n+1}(\mathcal{D}_n) \to (M_{n+2}(\Gamma) \otimes \mathrm{Sym}^n H^1_{\mathrm{dR}}(E))^\vee.$$

We proceed to define a collection of cycle classes in $\mathrm{CH}^{n+1}(\mathcal{D}_n)$, indexed by certain isogenies. Let E be an elliptic curve with complex multiplication by $\mathcal{O}$. Recall that $\mathcal{O} = \mathrm{End}_{\mathcal{R}^{\max}}(E)$ is an order in an imaginary quadratic number field K. Consider an isogeny φ from E to another elliptic curve with complex multiplication E', of degree coprime to N^+M. If there is a level-N^+ structure and full level-M structure on E, we obtain the same structures on E', and also on $A' := E' \times E'$, by putting these level structures only on the first copy. Hence, we obtain a point $P_{A'}$ in X_M, together with an embedding

$$i_{A'} : A' \to \mathcal{A},$$

defined over K. Let Υ_φ be the cycle

$$\Upsilon_\varphi := ({}^t\Gamma_\varphi)^n \subseteq (E' \times E)^n \cong (A')^m \times E^n \hookrightarrow \mathcal{A}^m \times E^n,$$

where the last inclusion is induced from the canonical embedding $i_{A'}$, and Γ_φ is the graph of φ. Finally, apply the projector ε defined in § 6.

DEFINITION 7.2. The *generalized Heegner cycle* attached to the isogeny $\varphi : E \to E'$ is the cycle

$$\Delta_\varphi := \varepsilon \Upsilon_\varphi \in \mathrm{CH}^{n+1}(\mathcal{D}_n).$$

Since $H_p(\mathcal{D}_n)$ is concentrated in degree $2n+1$, the cycle Δ_φ is null-homologous. Therefore, it makes sense to study the image of Δ_φ under the p-adic Abel–Jacobi map discussed above. Let $\tilde{P}_{A'}$ be the point of X_M attached to A' through the isogeny φ. The cycle Δ_φ lies in the $(2n+1)$-dimensional scheme $\mathcal{A}^m \times E^n$, and so it has codimension $n+1$. Consider the map

$$\mathrm{AJ}_K : \mathrm{CH}^{n+1}(\mathcal{D}_n) \to (M_{n+2}(\Gamma) \otimes \mathrm{Sym}^n H^1_{\mathrm{dR}}(E))^\vee,$$

as described above. Let ω_f be the differential form associated to a modular form $f \in M_{n+2}(\Gamma)$ as explained in § 2. Fix $\alpha \in \mathrm{Sym}^n H^1_{\mathrm{dR}}(E)$. We want to compute the value

$$\mathrm{AJ}_K(\Delta_\varphi)(\omega_f \wedge \alpha) \in \mathbb{C}_p.$$

Write $\mathrm{cl}_{P_{A'}}(\Delta_\varphi)$ for the cycle class of Δ_φ on the fiber above $P_{A'} \in X_M$:

$$\mathrm{cl}_{P_{A'}}(\Delta_\varphi) := \mathrm{cl}^{\mathcal{A}^m \times E^n}_{|\overline{\Delta_\varphi}|}(\overline{\Delta_\varphi}) \in H^{2n+2}_{|\overline{\Delta_\varphi}|}(\mathcal{A}^m \times E^n, \mathbb{Q}_p(n+1)).$$

Consider the short exact sequence of filtered (ϕ, N)-modules

$$0 \to H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) \to H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n}) \xrightarrow{\mathrm{res}_{P_{A'}}} (\mathcal{V}_{n,n})_{P_{A'}}[1] \to 0. \tag{10}$$

Remark 7.3. Here is where we need to exclude the case of weight 2, which would correspond to $n=0$: in that case $\mathrm{res}_{P_{A'}}$ is always zero, since the restriction map induces an isomorphism $H^1_{\mathrm{dR}}(X_\Gamma) \cong H^1_{\mathrm{dR}}(U)$. However, in our situation the cokernel of the map $\mathrm{res}_{P_{A'}}$ injects in

$$H^2_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) \cong H^0_{\mathrm{dR}}(X_\Gamma, \mathcal{V}^\vee_{n,n}),$$

where the isomorphism is given by Serre duality. But $\mathcal{V}_{n,n}$ (and therefore $\mathcal{V}^\vee_{n,n}$) does not have Γ-invariants, since it is isomorphic to n copies of the standard representation of Γ.

We argue that the sequence in (10) is exact. Its pull-back under the map $1 \mapsto \mathrm{cl}_{P_{A'}}(\Delta_\varphi)$ yields then a short exact sequence

$$0 \to H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n}) \to E \to K[n+1] \to 0.$$

Lemma 3.3 ensures that if one forgets the filtration, the resulting sequence of (ϕ, N)-modules is split, say by a map $s_1 : K[n+1] \to E$. If we write $([\eta_1], x)$ for $s_1(1)$, note then:

(i) for $([\eta_1], x)$ to be a splitting of the given extension, necessarily $x=1$, and η_1 has to satisfy:
   (a) $N_U([\eta_1]) = 0$; and
   (b) $\Phi([\eta_1]) = p^{n+1}[\eta_1]$;

(ii) for $([\eta_1], 1)$ to be in

$$E = H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n}) \times_{(\mathcal{V}_{n,n})_{P_{A'}}[1]} K[n+1],$$

necessarily $\mathrm{res}_{P_{A'}}(\eta_1) = \mathrm{cl}_{P_{A'}}(\Delta_\varphi)$.

So, let η_1 be a $\mathcal{V}_{n,n}$-valued 1-hypercocycle on U satisfying the conditions

$$\mathrm{res}_{P_{A'}}(\eta_1) = \mathrm{cl}_{P_{A'}}(\Delta_\varphi), \quad N_U([\eta_1]) = 0, \quad \Phi(\eta_1) = p^{n+1}\eta_1 + \nabla G,$$

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where G is a $\mathcal{V}_{n,n}$-valued rigid section over U. Consider next $[\eta_2] \in \mathrm{Fil}^{n+1}\, H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n})$ such that $\mathrm{res}_{P_{A'}}(\eta_2) = \mathrm{cl}_{P_{A'}}(\Delta_\varphi)$. This element exists as well, because it is the image of 1 under the splitting s_2 of Lemma 3.3. Let

$$[\widetilde{\eta}_\varphi] := [\eta_1 - \eta_2] \in H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n}).$$

Then $[\widetilde{\eta}_\varphi]$ can be extended to all of X_Γ. That is, there is $[\eta_\varphi] \in H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n})$ such that

$$j_*([\eta_\varphi]) = [\widetilde{\eta}_\varphi] \equiv [\eta_1] \pmod{\mathrm{Fil}^{n+1}\, H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n})}.$$

Write $[\eta_\varphi] = \iota(c) + t$, with $t \in \mathrm{Fil}^{n+1}\, H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n})$. Then one can replace $[\eta_2]$ by $[\eta_2] + t$ without changing the properties required for $[\eta_2]$, and hence we can assume that $[\eta_\varphi] = \iota(c)$ for some $c \in H^1_{\mathrm{dR}}(X_\Gamma, \mathcal{V}_{n,n})$. Recall the maps I and P_U as appearing in the diagram of (6). We can prove the following proposition.

Proposition 7.4. *With the previous notation, the following equality holds:*

$$\mathrm{AJ}_K(\Delta_\varphi)([\omega_f] \wedge \alpha) = \langle I([\omega_f] \wedge \alpha), P_U([\eta_2])\rangle_\Gamma. \tag{11}$$

Proof. Using the definition of the Abel–Jacobi map and following the recipe given in Lemma 3.3, together with the pairings on $H^1_{\mathrm{dR}}(X, \mathcal{V}_{n,n})$, we obtain the following equality:

$$\mathrm{AJ}_K(\Delta_\varphi)([\omega_f] \wedge \alpha) = \langle [\omega_f] \wedge \alpha, [\eta_\varphi]\rangle_{X_\Gamma}.$$

The assumption of $\eta_\varphi = \iota(c)$ implies that $I(\eta_\varphi)$ is zero. So, the right-hand side can be rewritten, using (4.3) and the diagram of (6), as

$$-\langle I_{U,c}([\omega_f] \wedge \alpha), P_U([\eta_\varphi])\rangle_\Gamma.$$

Now the result follows from observing that on U one can write $[\eta_\varphi] = [\eta_1] - [\eta_2]$, and that $P_U([\eta_1]) = 0$. □

The following result computes a formula for the right-hand side of (11).

Theorem 7.5. *Let F_f be a Coleman primitive of ω_f, and let $z_0' \in \mathcal{H}_p$ be a point in the p-adic upper half-plane such that $P_A' = \Gamma_M z_0'$. Then*

$$\langle I([\omega_f] \wedge \alpha), P_U([\eta_2])\rangle_\Gamma = \langle F_f(z_0') \wedge \alpha, \mathrm{cl}_{z_0'}(\Delta_\varphi)\rangle_{V_{n,n}}.$$

Proof. Observe first that the spaces

$$\mathrm{Fil}^{n+1}\, H^1_{\mathrm{dR,c}}(U, \mathcal{V}_{n,n}) \quad \text{and} \quad \mathrm{Fil}^{n+1}\, H^1_{\mathrm{dR}}(U, \mathcal{V}_{n,n})$$

are orthogonal to each other. Therefore,

$$0 = \langle [\omega_f] \wedge \alpha, [\eta_2]\rangle_U = \langle P_{U,c}([\omega_f] \wedge \alpha), I_U([\eta_2])\rangle_{\Gamma,U} - \langle I([\omega_f] \wedge \alpha), P_U([\eta_2])\rangle_\Gamma,$$

and hence we obtain

$$\langle I([\omega_f] \wedge \alpha), P_U([\eta_2])\rangle_\Gamma = \langle P_{U,c}([\omega_f] \wedge \alpha), I_U([\eta_2])\rangle_{\Gamma,U}.$$

In order to show that the right-hand side of the previous equation coincides with

$$\langle F_f(z_0') \wedge \alpha, \mathrm{cl}_{z_0'}(\Delta_\varphi)\rangle_{V_{n,n}},$$

we use the explicit formula for the pairing as found in Proposition 4.4. Since η_2 has only non-zero residue at $P_{A'}$, the right-hand side of the formula appearing in Proposition 4.4 reduces to pairing the primitive of $\omega_f \wedge \alpha$ with that residue, at the point corresponding to $P_{A'}$, yielding the desired formula. □

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From here on, let z_0 be a point in the p-adic upper half-plane $\mathcal{H}_p$ such that the orbit $P_A := \Gamma z_0$ corresponds to $A = E \times E$ in X. Consider the map

$$g = (\mathrm{Id}_E^n, \varphi^n) : E^n \to E^n \times (E')^n.$$

Then Δ_φ is the projection via ε of the image $g(E^n)$. The functoriality of the cycle class map gives

$$\langle F_f(z_0') \wedge \alpha, \mathrm{cl}_{z_0'}(\Delta_\varphi)\rangle_{V_{n,n}} = \langle \varphi^* F_f(z_0), \alpha\rangle_{V_n},$$

where now the pairing is the natural one in the stalk $V_n = (\mathcal{V}_n)_{z_0}$. To compute this last quantity, we first do it on a horizontal basis for

$$\mathrm{Sym}^n H^1_{\mathrm{dR}}(E/K) = (\mathcal{V}_n)_{z_0}.$$

Let $\{u, v\}$ be a horizontal basis for V_1, normalized so that $\langle u, u\rangle = \langle v, v\rangle = 0$ and $\langle u, v\rangle = -\langle v, u\rangle = 1$. This induces a basis $\{v_i := u^i v^{n-i}\}_{0 \leqslant i \leqslant n}$ of V_n.

Choose a global regular section ω^n in the lowest piece of the filtration which transforms with respect to Γ_M as of weight n, and scale it so that ω^n corresponds to $\sum_{i=0}^n (-1)^i \binom{n}{i} z^i v_i$, which is a regular section in $\mathrm{Fil}^n \mathcal{V}_n$. Note that $\omega^n = (u - zv)^n$, and so

$$\langle \omega^n, v_i\rangle = \langle u - zv, u\rangle^i \langle u - zv, v\rangle^{n-i} = z^i.$$

We want to obtain a formula for the Coleman primitive F_f of ω_f. We proceed by differentiating the section $\langle F_f, v_i\rangle$ and using that $\{v_i\}$ is a horizontal basis:

$$d\langle F_f, v_i\rangle = f(z)\langle \omega^n, v_i\rangle\, dz = z^i f(z)\, dz.$$

One deduces the formula

$$\left\langle F_f(\varphi(z_0)), \sum_i a_i v_i \right\rangle = \sum_{i=0}^n a_i \int_\star^{\varphi(z_0)} f(z) z^i\, dz.$$

From now on, we concentrate on the stalk of $\mathcal{V}_n$ at z_0. The chosen regular differential ω yields a basis element ω_{z_0} for $H^1_{\mathrm{dR}}(E, K)$. Choose η_{z_0} in the line spanned by $\Phi\omega_{z_0}$ and additionally satisfying $\langle \omega_{z_0}, \eta_{z_0}\rangle = 1$. This yields a basis for $\mathrm{Sym}^n H^1_{\mathrm{dR}}(E/K)$, namely $\{\omega_{z_0}^j \eta_{z_0}^{n-j}\}_{0 \leqslant j \leqslant n}$. We express this basis in terms of the horizontal basis $\{v_i\}$. By construction, we have

$$\omega_{z_0}^j \eta_{z_0}^{n-j} = (z_0 - \overline{z}_0)^{j-n} \sum_i P_{i,j,n}(z_0, \overline{z}_0) v_i,$$

where

$$P_{i,j,n}(X, Y) = \sum_k \binom{j}{k}\binom{n-j}{k+i-j} (-1)^{n-i} X^k Y^{n-i-k}.$$

A simple computation yields

$$\sum_i P_{i,j,n}(z_0, \overline{z}_0) z^i = (z - z_0)^j (z - \overline{z}_0)^{n-j},$$

and we obtain a formula for a primitive for ω_f:

$$\langle \varphi^* F_f(z_0), \omega_{z_0}^j \eta_{z_0}^{n-j}\rangle = (z_0 - \overline{z}_0)^{j-n} \int_\star^{\varphi(z_0)} f(z)(z - z_0)^j (z - \overline{z}_0)^{n-j}\, dz.$$

Note that this equality is only defined up to an 'integration constant' in $\mathbb{C}_p$, because in $\mathcal{H}_p$ the sheaf $\mathcal{V}_{n,n}$ is trivial. We can now prove the following application.

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Theorem 7.6. *Let $\varphi : E \to E'$ be an isogeny of elliptic curves with level-N structure, and let $\overline{\varphi}$ be the morphism $E \to \overline{E}'$ obtained from φ by applying to E' the non-trivial automorphism of K. Let $\Delta^-_\varphi := \Delta_\varphi - \Delta_{\overline{\varphi}}$, and write $z'_0 \in \mathcal{H}_p$ for the point in the p-adic upper half-plane which corresponds to the abelian surface $E' \times E'$. Then there exist a constant $\Omega \in K^\times$ such that*

$$\mathrm{AJ}_K(\Delta^-_\varphi)(\omega_f \wedge \omega^j \eta^{n-j}) = \Omega^{j-n} L'_p(f, \Psi_{P_{E'}}, j+1), \quad 0 \leqslant j \leqslant n.$$

Proof. Set Ω to be $z_0 - \overline{z}_0$. Since z_0 does not belong to the boundary of $\mathcal{H}_p$, this quantity is non-zero. Using the previous results, we obtain first

$$\mathrm{AJ}_K(\Delta^-_\varphi)(\omega_f \wedge \omega^j \eta^{n-j}) = \langle F_f(z'_0), \omega^j_{z_0} \eta^{n-j}_{z_0} \rangle - \langle F_f(\overline{z}'_0), \omega^j_{z_0} \eta^{n-j}_{z_0} \rangle.$$

Therefore, the second term in the previous displayed expression becomes

$$\langle F_f(\overline{z}'_0), \omega^{n-j}_{z_0} \eta^j_{z_0} \rangle = \Omega^{j-n} \int_\star^{\overline{z}'_0} f(z)(z - z_0)^j (z - \overline{z}_0)^{n-j}\, dz.$$

Combining this with the formula for $\langle F_f(z'_0), \omega^j_{z_0} \eta^{n-j}_{z_0} \rangle$ yields

$$\mathrm{AJ}_K(\Delta^-_\varphi)(\omega_f \wedge \omega^j \eta^{n-j}) = \Omega^{j-n} \int_{\overline{z}'_0}^{z'_0} f(z)(z - z'_0)^j (z - \overline{z}'_0)^{n-j}\, dz.$$

The result follows now from Theorem 5.3. □

Note that the integral appearing in the previous theorem coincides with the value at $s = j+1$ of the derivative of the partial p-adic L-function described before. We obtain the following corollary.

Corollary 7.7. *Let H/K be the Hilbert class field of K, and consider a set of representatives $\{\Psi_1, \ldots, \Psi_h\}$ for $\mathrm{emb}(\mathcal{O}, \mathcal{R})$. For each Ψ_i, let P_i be the corresponding Heegner point on X_H, and let Δ_{Ψ_i} be the cycle corresponding to P_i. Define $\Delta^- := \sum_i \Delta^-_{\Psi_i}$. There exists a constant $\Omega \in K$ such that for all $0 \leqslant j \leqslant n$,*

$$\mathrm{AJ}_K(\Delta^-)(\omega_f \wedge \omega^j \eta^{n-j}) = \Omega^{j-n} L'_p(f, K, j+1).$$

Proof. This follows immediately from the expression given in Theorem 7.6 for the partial p-adic L-functions:

$$\mathrm{AJ}_K(\Delta^-_{\Psi_i})(\omega_f \wedge \omega^j \eta^{n-j}) = \Omega^{j-n} L'_p(f, \Psi_i, j+1). \qquad □$$

Remark 7.8. There is no canonical choice for the regular differential $\omega \in \Omega^1_{E/K}$. If a given ω is changed to $\omega_\lambda := \lambda\omega$, with $\lambda \in K$, we obtain

$$\mathrm{AJ}_K(\Delta^-)(\omega_f \wedge (\omega^j_\lambda \eta^{n-j}_\lambda)) = \Omega^{j-n} \lambda^{2j-n} L'_p(f, K, j+1).$$

Note in particular that the formula at the central point $j = n/2$ does not depend on the choice of the basis of the differential form ω.

If one wishes to extend the results of this paper to modular forms of odd weight, one needs to work on the two sides of the equation: on the one hand, the motive $\mathcal{D}_n$ needs to be extended to odd n. On the other hand, the anti-cyclotomic p-adic L-function as defined in [BDIS02] does not contemplate possible nebentypes, thus restricting the construction to even-weight modular forms. One should give a more general construction which allowed nebentypes, and these should be incorporated in the definition of the motive $\mathcal{D}_n$ as well.

It would be interesting to compute the image of the Abel–Jacobi map for arbitrary cycles on $\mathcal{D}_n$ supported on CM points of the Shimura curve. This is a more difficult problem than what has been treated in this paper, since some of the techniques used above cannot be used in the general case. However, similar computations have been carried over in the split case in [BDP09], and one should be able to adapt them to the setting of this work.

Finally, let us remark that, although the focus of this paper has been put on the study of the relation of the anti-cyclotomic p-adic L-function to the image of certain cycles under the Abel–Jacobi map, it would be interesting to instead relate the values of the L-function to p-adic analogues of the Néron–Tate heights of the cycles. The investigation of the relation of the p-adic Abel–Jacobi map appearing in this paper with the p-adic height pairings as in the articles of Coleman and Gross [CG89] and of Nekovář [Nek93, Nek95] would certainly be fruitful.

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