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# BERGMAN-TYPE OPERATORS IN TUBULAR DOMAINS OVER SYMMETRIC CONES

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*Abstract* We study the boundedness properties of Rudin–Forelli-type operators associated to tubular domains over symmetric cones. As an application, we give a characterization of the topological dual space of the weighted Bergman space $A^{p,q}_\nu$.

*Keywords:* Bergman kernel; Bergman space; symmetrical cone; tube domain

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## 1. Introduction

Let $V$ be a real vector space of dimension $n$, endowed with the structure of a simple Euclidean Jordan algebra. We consider an irreducible symmetric cone $\Omega$ inside $V$ and denote by $T_\Omega = V + \mathrm{i}\Omega$ the corresponding tube domain in the complexification of $V$. Here, $V$ is endowed with an inner product $(\cdot \mid \cdot)$ for which the cone $\Omega$ is self-dual. We refer the reader to [**7**] for a complete discussion about symmetric cones. Identifying $V$ with $\mathbb{R}^n$, we have as an example of a symmetric cone the forward light cone given for $n \geqslant 3$ by

$$\Gamma_n = \{y \in \mathbb{R}^n : y_1^2 - y_2^2 - \cdots - y_n^2 > 0,\ y_1 > 0\}.$$

Following the notation in [**7**], we write $r$ for the rank of $\Omega$ and $\Delta(x)$ for the associated determinant function. Light cones have rank 2 and a determinant function given by the Lorentz form

$$\Delta(y) = y_1^2 - y_2^2 - \cdots - y_n^2 \quad \text{for } y = (y_1, y_2, \dots, y_n).$$

We recall that, given $1 \leqslant p, q < \infty$ and $\nu \in \mathbb{R}$, the mixed norm Lebesgue space $L^{p,q}_\nu(T_\Omega)$ is defined by the integrability condition

$$\|f\|_{L^{p,q}_\nu} := \left[\int_\Omega \left(\int_{\mathbb{R}^n} |f(x+\mathrm{i}y)|^p \,\mathrm{d}x\right)^{q/p} \Delta^{\nu-(n/r)}(y)\,\mathrm{d}y\right]^{1/q} < \infty. \tag{1.1}$$

The mixed norm weighted Bergman space $A^{p,q}_\nu(T_\Omega)$ is then the closed subspace of $L^{p,q}_\nu(T_\Omega)$ consisting of holomorphic functions on the tube $T_\Omega$. These spaces are non-null only when $\nu > (n/r)-1$ (see [**6**]). When $p=q$ we shall simply write $A^{p,p}_\nu = A^p_\nu$. The usual Bergman space $A^p$ then corresponds to the case when $\nu = n/r$.

The weighted Bergman projection $P_\nu$ is the orthogonal projection from the Hilbert space $L^2_\nu(T_\Omega)$ onto its closed subspace $A^2_\nu(T_\Omega)$ and it is given by the integral formula

$$P_\nu f(z) = \int_{T_\Omega} B_\nu(z,w) f(w) \Delta^{\nu-(n/r)}(\operatorname{Im} w)\,\mathrm{d}V(w),$$

where

$$B_\nu(z,w) = d_\nu \Delta^{-\nu-(n/r)}\left(\frac{z-\bar{w}}{\mathrm{i}}\right) \tag{1.2}$$

is the weighted Bergman kernel and $\mathrm{d}V$ is the Lebesgue measure on $\mathbb{C}^n$ (see [**6**]). Note that the Bergman kernel is a reproducing kernel on $A^2_\nu(T_\Omega)$, that is, for any $f \in A^2_\nu(T_\Omega)$,

$$f(z) = \int_{T_\Omega} B_\nu(z,w) f(w) \Delta^{\nu-(n/r)}(\operatorname{Im} w)\,\mathrm{d}V(w).$$

The $L^{p,q}_\nu$-boundedness of the Bergman projection $P_\nu$ is still an open problem and has attracted a lot of attention in recent years (see [**1**, **2**, **4**, **5**]). To date, it is only known that this projection extends as a bounded operator on $L^{p,q}_\nu$ for general symmetric cones for the range $1 \leqslant p < \infty$ and $q'_{\nu,p} < q < q_{\nu,p}$, with

$$q_{\nu,p} = \min\{p,p'\} q_\nu, \quad q_\nu = 1 + \frac{\nu}{(n/r)-1} \quad \text{and} \quad \frac{1}{p} + \frac{1}{p'} = 1$$

(see, for example, [**5**]) with slight improvements over this range in the case of light cones (see [**9**]).

The importance of the boundedness of the Bergman projection can be expressed in terms of its consequences, among which the following one is well known: if $P_\nu$ extends to a bounded operator on $L^{p,q}_\nu$, then the topological dual space $(A^{p,q}_\nu)^*$ of the Bergman space $A^{p,q}_\nu$ identifies with $A^{p',q'}_\nu$ under the integral pairing

$$\langle f,g\rangle_\nu = \int_{T_\Omega} f(z)\overline{g(z)} \Delta^{\nu-n/r}(\operatorname{Im} z)\,\mathrm{d}V(z),$$

for $f \in A^{p,q}_\nu$ and $g \in A^{p',q'}_\nu$ (see [**6**]). So, since the range of boundedness of $P_\nu$ on $L^{p,q}_\nu$ is far from being completely known, a natural question is whether there is any way of characterizing the dual space of $A^{p,q}_\nu$ for values of the parameters $p$, $q$, $\nu$ for which $P_\nu$ is not necessarily bounded. To answer this type of question, it seems natural to consider the problem of $L^{p,q}_\nu$-boundedness of a family of operators generalizing the Bergman projection. This family is given by the integral operators $T = T_{\alpha,\beta,\gamma}$ and $T^+ = T^+_{\alpha,\beta,\gamma}$ defined for $C^\infty_{\mathrm{c}}(T_\Omega)$ by the formulae

$$Tf(z) = \Delta^\alpha(\operatorname{Im} z) \int_{T_\Omega} B_\gamma(z,w) f(w) \Delta^\beta(\operatorname{Im} w)\,\mathrm{d}V(w)$$

and

$$T^+ f(z) = \Delta^{\alpha}(\operatorname{Im} z) \int_{T_\Omega} |B_\gamma(z,w)| f(w) \Delta^{\beta}(\operatorname{Im} w)\, \mathrm{d}V(w).$$

Note that the boundedness of $T^+$ on $L^{p,q}_\nu(T_\Omega)$ implies the boundedness of $T$, although the boundedness of $T$ is typically expected in a larger range than $T^+$.

The boundedness of this family of operators on $L^{p,q}_\nu(T_\Omega)$ has been considered in [**4**] for the case when $P_\mu = T_{0,\mu-(n/r),\mu}$ and in [**2**] for $T_{0,\mu-(n/r),\mu+m}$. Both works deal with the case of the light cone. Here, we consider the problem of the boundedness of the operator $T^+$ for general symmetric cones and obtain optimal results for this operator. For this, we systematically make use of the methods of [**2**,**4**], which seem to be appropriate here and, since we are considering general symmetric cones, the general power function defined in the text is also useful in this case. Note that the case $p = q$ for general symmetric cones was implicit in [**3**]. Our results can be stated in the following way.

**Theorem 1.1.** *Suppose that $\nu \in \mathbb{R}$ and $1 \leqslant p, q < \infty$. Then the following conditions are equivalent:*

(a) *the operator $T^+_{\alpha,\beta,\gamma}$ is bounded on $L^{p,q}_\nu(T_\Omega)$;*

(b) *the parameters satisfy $\gamma = \alpha + \beta + (n/r)$, $\alpha + \beta > -1$ and*

$$\max\biggl\{-q\alpha + \left(\frac{n}{r}\right) - 1,\, q\biggl(-\alpha + \left(\frac{n}{r}\right) - 1\biggr) - \left(\frac{n}{r}\right) + 1\biggr\} < \nu < \min\biggl\{q(\beta+1) + \left(\frac{n}{r}\right) - 1,\, q\biggl(\beta + \left(\frac{n}{r}\right)\biggr) - \left(\frac{n}{r}\right) + 1\biggr\}.$$

**Theorem 1.2.** *The operator $T^+_{\alpha,\beta,\gamma}$ is bounded on $L^\infty(T_\Omega)$ if and only if $\alpha > (n/r) - 1$, $\beta > -1$ and $\gamma = \alpha + \beta + (n/r)$.*

The condition

$$\max\biggl\{-q\alpha + \left(\frac{n}{r}\right) - 1,\, q\biggl(-\alpha + \left(\frac{n}{r}\right) - 1\biggr) - \left(\frac{n}{r}\right) + 1\biggr\} < \nu < \min\biggl\{q(\beta+1) + \left(\frac{n}{r}\right) - 1,\, q\biggl(\beta + \left(\frac{n}{r}\right)\biggr) - \left(\frac{n}{r}\right) + 1\biggr\}$$

in Theorem 1.1 is equivalent to

$$\biggl(\beta + \left(\frac{n}{r}\right)\biggr) q - \nu > \max\{1, q-1\}\biggl(\left(\frac{n}{r}\right) - 1\biggr) \quad \text{and} \quad \alpha q + \nu > \max\{1, q-1\}\biggl(\left(\frac{n}{r}\right) - 1\biggr).$$

As an application, we characterize the dual space of Bergman spaces in some cases where the Bergman projection is not necessarily bounded, partially answering the question above.

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## 2. Integral operators on the cone

The aim of this section is to give $L^q_\nu$-continuity properties of a family of operators on the cone $\Omega$ which are closely related to the operators $T_{\alpha,\beta,\gamma}$. Considering $V = \mathbb{R}^n$ as a Jordan algebra, we denote its identity element by $\boldsymbol{e}$ (this corresponds to the point $(1, 0, \dots, 0)$ in the forward light cone). We recall that a generalized power in the symmetric cone $\Omega$ of rank $r$ is defined by

$$\Delta_s(x) = \Delta_1^{s_1-s_2}(x)\Delta_2^{s_2-s_3}(x)\cdots\Delta_r^{s_r}(x), \quad s = (s_1, s_2, \dots, s_r) \in \mathbb{C}^r,$$

where $x \in \Omega$ and $\Delta_k(x)$ are the principal minors of $x$ (see [**7**, p. 122]). The generalized Gamma function of the cone $\Omega$ is given by

$$\Gamma_\Omega(s) = \int_\Omega \mathrm{e}^{-(\xi|\boldsymbol{e})}\Delta_s(\xi)\frac{\mathrm{d}\xi}{\Delta^{n/r}(\xi)}$$

for $s \in \mathbb{C}^r$.

**Lemma 2.1.** *The above integral is absolutely convergent if and only if* $\operatorname{Re} s_j > \frac{1}{2}(j-1)d$ *for* $j \in \{1, \dots, r\}$ *and in this case*

$$\Gamma_\Omega(s) = (2\pi)^{(n-r)/2}\prod_{j=1}^{r}\Gamma(s_j - \tfrac{1}{2}(j-1)d),$$

*where* $\Gamma$ *is the usual Euler function on* $\mathbb{R}^+$, $\operatorname{Re} s_j$ *is the real part of* $s_j$ *and* $\frac{1}{2}(r-1)d = (n/r) - 1$.

**Proof.** See [**7**, p. 123]. □

We now recall the integrability properties of powers of $\Delta$ and Schur's lemma. For this we use the following notation:

$$g_0 = (\dots, \tfrac{1}{2}d(j-1), \dots) \quad \text{for } j = 1, \dots, r,$$

and

$$t^* = (t_r, \dots, t_1), \quad \text{where } t = (t_1, \dots, t_r).$$

For $t \in \mathbb{R}^r$ and $k \in \mathbb{R}^r$, $t < k$ is equivalent to $t_j < k_j$ for $j = 1, \dots, r$. The following result, which can be deduced from the previous lemma, is due to Gindikin [**10**].

**Lemma 2.2.** *Let* $s \in \mathbb{C}^r$ *with* $\operatorname{Re} s_j > (j-1)\frac{1}{2}d$, $\alpha \in \mathbb{C}^r$ *and* $t \in \Omega$. *Then the integral*

$$I_{s,\alpha}(t) = \int_\Omega \Delta_\alpha(y+t)\Delta_s(y)\frac{\mathrm{d}y}{\Delta^{n/r}(y)}$$

*is absolutely convergent if and only if* $\operatorname{Re}(s+\alpha) < -g_0^*$, *and in this case*

$$I_{s,\alpha}(t) = C_{s,\alpha}\Delta_{s+\alpha}(t).$$

**Lemma 2.3 (Schur's lemma).** *Let $\mu$ be a positive measure on a measure space $X$, let $H(x,y)$ be a positive measurable function on $X\times X$, and let $q>1$, with $1/q+1/q'=1$.*

*If there exist a positive measurable function $h(x)$ on $X$ and a positive constant $C$ such that*

$$\int_X H(x,y)h^q(x)\,\mathrm{d}\mu(x)\leqslant Ch^q(y)$$

*and*

$$\int_X H(x,y)h^{q'}(y)\,\mathrm{d}\mu(y)\leqslant Ch^{q'}(x)$$

*for all $x$ and $y$ in $X$, then the integral operator*

$$Hf(x)=\int_X H(x,y)f(y)\,\mathrm{d}\mu(y)$$

*is bounded on $L^q(X,\mu)$ with $\|H\|\leqslant C$.*

**Proof.** See [**11**, Theorem 3.2.2]. □

For the real parameters $\alpha$, $\beta$ and $\gamma$, we now consider the integral operators $S=S_{\alpha,\beta,\gamma}$ which are defined on the cone $\Omega$ by

$$Sg(y)=\int_\Omega \Delta^\alpha(y)\Delta^{-\gamma}(y+v)g(v)\Delta^\beta(v)\,\mathrm{d}v.$$

The following lemmas give continuity properties of the operators $S_{\alpha,\beta,\gamma}$ on $L^q_\nu(\Omega)=L^q(\Omega,\Delta^{\nu-(n/r)}(y)\,\mathrm{d}y)$, $\nu\in\mathbb{R}$.

**Lemma 2.4.** *Let $\nu\in\mathbb{R}$, $1<q<\infty$, $\gamma=\alpha+\beta+(n/r)$ and*

$$\max\biggl\{-q\alpha+\biggl(\frac{n}{r}\biggr)-1,\ q\biggl(-\alpha+\biggl(\frac{n}{r}\biggr)-1\biggr)-\biggl(\frac{n}{r}\biggr)+1\biggr\}$$
$$<\nu<\min\biggl\{q(\beta+1)+\biggl(\frac{n}{r}\biggr)-1,\ q\biggl(\beta+\biggl(\frac{n}{r}\biggr)\biggr)-\biggl(\frac{n}{r}\biggr)+1\biggr\}.$$

*Then the operator $S=S_{\alpha,\beta,\gamma}$ is bounded on $L^q(\Omega,\Delta^{\nu-(n/r)}(y)\,\mathrm{d}y)$.*

**Proof.** We can write the integral $S$ as

$$Sg(y)=\int_\Omega H(y,v)g(v)\Delta^{\nu-(n/r)}(v)\,\mathrm{d}v,$$

where $H(y,v)=\Delta^\alpha(y)\Delta^{-\gamma}(y+v)\Delta^{\beta-\nu+(n/r)}(v)$ is a positive kernel with respect to the measure $\Delta^{\nu-(n/r)}(v)\,\mathrm{d}v$. By Schur's lemma, it is sufficient to find a positive function $h$ on $\Omega$ such that

$$\int_\Omega H(x,y)h^{q'}(y)\,\mathrm{d}\mu(y)\leqslant Ch^{q'}(x)$$

and

$$\int_{\Omega} H(x,y)h^q(x)\,\mathrm{d}\mu(x) \leqslant Ch^q(y)$$

for $q > 1$, and $\mathrm{d}\mu(y) = \Delta^{\nu-(n/r)}(y)\,\mathrm{d}y$. We take $h(y) = \Delta_s(y)$, where $s = (s_1, \dots, s_r)$ and $s_j$, $j = 1, \dots, r$, are real numbers to be determined.

Straightforward computations with the use of the given choice of $h$ and Lemma 2.2 yield

$$\frac{-\alpha - \nu + \frac{1}{2}d(j-1)}{q'} < s_j < \frac{-\alpha - \nu + \gamma - (n/r) + 1 + \frac{1}{2}d(j-1)}{q'}$$

and

$$\frac{-\beta - (n/r) + \frac{1}{2}d(j-1)}{q} < s_j < \frac{-\beta + \gamma + \frac{1}{2}d(j-1) - 2(n/r) + 1}{q}.$$

Thus, each $s_j$ must belong to an intersection of two intervals. This intersection is not empty, by hypothesis, since the condition

$$\max\left\{ -q\alpha + \left(\frac{n}{r}\right) - 1,\ q\left( -\alpha + \left(\frac{n}{r}\right) - 1\right) - \left(\frac{n}{r}\right) + 1\right\} \\ < \nu < \min\left\{ q(\beta+1) + \left(\frac{n}{r}\right) - 1,\ q\left(\beta + \left(\frac{n}{r}\right)\right) - \left(\frac{n}{r}\right) + 1\right\}$$

is equivalent to

$$\max_{1\leqslant j\leqslant r}\left\{ q\left( -\alpha + \left(\frac{n}{r}\right) - 1 - \tfrac{1}{2}d(j-1)\right) + (j-1)d - \left(\frac{n}{r}\right) + 1\right\} \\ < \nu < \min_{1\leqslant j\leqslant r}\left\{ q\left(\beta + \left(\frac{n}{r}\right) - \tfrac{1}{2}d(j-1)\right) - \left(\frac{n}{r}\right) + 1 + (j-1)\,\mathrm{d}\right\}.$$

It follows that $S$ is bounded on $L^q_\nu(\Omega)$ for every $q > 1$ and the proof is complete. □

**Lemma 2.5.** *Suppose that $1 < q < \infty$, $\nu \in \mathbb{R}$ and that $S = S_{\alpha,\beta,\gamma}$ is bounded on $L^q(\Omega, \Delta^{\nu-(n/r)}(y)\,\mathrm{d}y)$. Then*

$$\max\left\{ -q\alpha + \left(\frac{n}{r}\right) - 1,\ q\left(\beta - \gamma + 2\left(\frac{n}{r}\right) - 1\right) - \left(\frac{n}{r}\right) + 1\right\} \\ < \nu < \min\left\{ q(\gamma - \alpha) - \left(\frac{n}{r}\right) + 1,\ q(\beta+1) + \left(\frac{n}{r}\right) - 1\right\}.$$

**Proof.** Let us take the characteristic function of the Euclidean ball $b_1(\underline{e})$ of radius 1 centred at $\underline{e}$ as a test function $g$. By continuity, $\Delta(v)$ is almost constant on the support of $g$. Let us estimate $\Delta(v+y)$ on the support of $g(v)$ for fixed $y \in \Omega$. For this, note that, for all $y$ and $t$ in the cone $\Omega$ and $\lambda > (n/r) - 1$, we can write

$$\Delta^{-\lambda}(y+v) = c\int_{\Omega} \mathrm{e}^{-(y+v|\xi)}\Delta^{\lambda-(n/r)}(\xi)\,\mathrm{d}\xi \tag{2.1}$$

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(see [**7**, Chapter VII]). By [**6**, Theorem 2.45] there exists a constant $C = C(\Omega) \geqslant 1$ such that, for all $\xi \in \Omega$,

$$\frac{1}{C} \leqslant \frac{(v \mid \xi)}{(\underline{\boldsymbol{e}} \mid \xi)} \leqslant C, \quad \text{whenever } v \in b_1(\underline{\boldsymbol{e}}). \tag{2.2}$$

Note that, for $C > 1$,

$$\frac{1}{C}(y+v \mid \xi) \leqslant \frac{1}{C}(v \mid \xi) + (y \mid \xi) \leqslant (y+v \mid \xi) \leqslant C(v \mid \xi) + (y \mid \xi) \leqslant C(y+v \mid \xi).$$

Thus, using the estimates (2.2), formula (2.1) and the fact that the determinant function is homogeneous of degree $r$ (see [**7**]) we obtain that, for $v$ in the support of $g$ and $y \in \Omega$, the following hold:

$$\begin{aligned} \left(\frac{1}{C}\right)^r \Delta(\underline{\boldsymbol{e}}+y) = \Delta\left(\frac{1}{C}(\underline{\boldsymbol{e}}+y)\right) &\leqslant \Delta\left(\frac{1}{C}\underline{\boldsymbol{e}}+y\right) \\ &\leqslant \Delta(v+y) \leqslant \Delta(C\underline{\boldsymbol{e}}+y) \leqslant \Delta(C(\underline{\boldsymbol{e}}+y)) \\ &= C^r \Delta(\underline{\boldsymbol{e}}+y). \end{aligned}$$

We conclude that there exists a constant $C = C(\Omega) \geqslant 1$ such that, for all $y \in \Omega$,

$$\frac{1}{C}\Delta(\underline{\boldsymbol{e}}+y) \leqslant \Delta(v+y) \leqslant C\Delta(\underline{\boldsymbol{e}}+y), \quad \text{whenever } v \in b_1(\underline{\boldsymbol{e}}).$$

It follows that

$$Sg(y) = S\chi_{b_1(\underline{\boldsymbol{e}})}(y) \approx C\Delta^{\alpha}(y)\Delta^{-\gamma}(y+\underline{\boldsymbol{e}}).$$

So, if $S$ is bounded on $L^q(\Omega, \Delta^{\nu-(n/r)}(y)\,\mathrm{d}y)$, then the function $\Delta^{\alpha}(y)\Delta^{-\gamma}(y+\underline{e})$ is in $L^q(\Omega, \Delta^{\nu-(n/r)}(y)\,\mathrm{d}y)$, which means that

$$\int_{\Omega} \Delta^{q\alpha+\nu-(n/r)}(y)\Delta^{-q\gamma}(y+\underline{\boldsymbol{e}})\,\mathrm{d}y < \infty.$$

By Lemma 2.2 we necessarily have $q\alpha + \nu - (n/r) > -1$ and $-q\gamma + q\alpha + \nu - (n/r) < -2(n/r) + 1$, which is equivalent to $\nu > -q\alpha + (n/r) - 1$ and $\nu < q(\gamma - \alpha) - (n/r) + 1$ with $1 \leqslant q < \infty$. This gives half of the conditions.

By duality, the boundedness of $S$ on $L^q(\Omega, \Delta^{\nu-(n/r)}(y)\,\mathrm{d}y)$ implies the boundedness of its adjoint $S^*$ on $L^{q'}(\Omega, \Delta^{\nu-(n/r)}(y)\,\mathrm{d}y)$, where $1/q + 1/q' = 1$. It is easy to see that

$$S^*g(y) = \int_{\Omega} \Delta^{\beta-\nu+(n/r)}(y)\Delta^{-\gamma}(y+v)g(v)\Delta^{\alpha+\nu-(n/r)}(v)\,\mathrm{d}v.$$

Using the same reasoning as before we obtain that the function $\Delta^{\beta-\nu+(n/r)}(y)\Delta^{-\gamma}(y+\underline{e})$ must belong to $L^{q'}(\Omega, \Delta^{\nu-(n/r)}(y)\,\mathrm{d}y)$. Again by Lemma 2.2, we must have $(\beta - \nu + (n/r))q' + \nu - (n/r) > -1$ and $-q'\gamma + (\beta - \nu + (n/r))q' + \nu - (n/r) < -2(n/r) + 1$, which is equivalent to $\nu < q(\beta+1) + (n/r) - 1$ and $\nu > q(\beta - \gamma + 2(n/r) - 1) - (n/r) + 1$. This completes the proof of the lemma. □

**Lemma 2.6.** *For $\nu \in \mathbb{R}$, the operator $S = S_{\alpha,\beta,\gamma}$ is bounded on $L^1(\Omega, \Delta^{\nu-(n/r)}(y)\,\mathrm{d}y)$ if and only if $\gamma = \alpha + \beta + (n/r)$ and $-\alpha + (n/r) - 1 < \nu < \beta + 1$.*

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**Proof.** We first show the sufficient condition. For any function $g$ in $L^1_\nu(\Omega)$, using Fubini's theorem we have that

$$\begin{aligned}\int_\Omega |Sg(y)|\Delta^{\nu-(n/r)}(y)\,\mathrm{d}y &\leqslant \int_\Omega \biggl(\int_\Omega \Delta^{\alpha}(y)\Delta^{-\gamma}(y+v)|g(v)|\Delta^{\beta}(v)\,\mathrm{d}v\biggr)\Delta^{\nu-(n/r)}(y)\,\mathrm{d}y \\ &= \int_\Omega |g(v)|\biggl(\int_\Omega \Delta^{-\gamma}(y+v)\Delta^{\alpha+\nu-(n/r)}(y)\,\mathrm{d}y\biggr)\Delta^{\beta}(v)\,\mathrm{d}v \\ &= C\int_\Omega |g(v)|\Delta^{\nu-(n/r)}(v)\,\mathrm{d}v,\end{aligned}$$

where the last equality follows from Lemma 2.2, since $\gamma = \alpha+\beta+(n/r)$ and $-\alpha+(n/r)-1<\nu<\beta+1$.

To prove the necessary condition, we proceed as in the proof of Lemma 2.5. This means that the operator

$$S^*g(y) = \int_\Omega \Delta^{\beta-\nu+(n/r)}(y)\Delta^{-\gamma}(y+v)g(v)\Delta^{\alpha+\nu-(n/r)}(v)\,\mathrm{d}v$$

must be bounded on $L^\infty(\Omega)$. As a test function, we take $g(v)=1$. Then

$$|S^*g(y)| = \int_\Omega \Delta^{\beta-\nu+(n/r)}(y)\Delta^{-\gamma}(y+v)\Delta^{\alpha+\nu-(n/r)}(v)\,\mathrm{d}v.$$

It follows from Lemma 2.2 that we must necessarily have $\alpha+\nu-(n/r)>-1$ and $-\gamma+\alpha+\nu-(n/r)<-2(n/r)+1$. In this case, $|S^*g(y)| = C\Delta^{\beta-\gamma+\alpha+(n/r)}(y)$, which belongs to $L^\infty(\Omega)$ if and only if $\beta-\gamma+\alpha+(n/r)=0$. This completes the proof of the lemma. □

## 3. Positive integral operators on the tube $T_\Omega$

In this section, we give some boundedness conditions for the family of integral operators $T_{\alpha,\beta,\gamma}$ defined on the tube $T_\Omega$. We begin by recalling some results.

**Lemma 3.1 (Békollé *et al.* [6, Lemma 4.11]).** *There are constants $C_\nu>0$ and $\delta>0$ such that, for all $z=x+\mathrm{i}y\in T_\Omega$, $v\in\Omega$ with $|x|\leqslant\frac{1}{2}$, $|v|,|y|<\delta$,*

$$\int_{|u|\leqslant 1}|B_\nu(z,u+\mathrm{i}v)|\,\mathrm{d}u \geqslant C_\nu\Delta^{-\nu}(y+v).$$

**Lemma 3.2.** *Let $\alpha$ be real. Then we have the following.*

(i) *The integral*

$$J_\alpha(y) = \int_{\mathbb{R}^n}\left|\Delta^{-\alpha}\left(\frac{x+\mathrm{i}y}{\mathrm{i}}\right)\right|\mathrm{d}x$$

*converges if and only if $\alpha>2(n/r)-1$. In this case, $J_\alpha(y)=C_\alpha\Delta^{-\alpha+(n/r)}(y)$, where $C_\alpha$ is a constant that depends only on $\alpha$.*

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(ii) *The function*

$$f(z) = \Delta^{-\alpha}\left(\frac{z+\mathrm{i}t}{\mathrm{i}}\right),$$

*with* $t \in \Omega$, *belongs to* $A_\nu^{p,q}$ *if and only if*

$$\nu > \left(\frac{n}{r}\right) - 1 \quad \text{and} \quad \alpha > \max\left(\frac{2(n/r)-1}{p}, \frac{n}{rp} + \frac{\nu + (n/r) - 1}{q}\right).$$

*In this case,*

$$\|f\|^q_{A_\nu^{p,q}} = C_{\alpha,p,q}\Delta^{-q\alpha+(nq/rp)+\nu}(t).$$

**Proof.** See [**6**, Lemma 3.20]. □

**Theorem 3.3.** *Suppose that* $\nu \in \mathbb{R}$ *and* $1 \leqslant p, q < \infty$. *Then the following conditions are equivalent:*

(a) *the operator* $T^+$ *is bounded on* $L_\nu^{p,q}(T_\Omega)$;

(b) *the parameters satisfy* $\gamma = \alpha + \beta + (n/r)$, $\alpha + \beta > -1$ *and*

$$\max\left\{-q\alpha + \left(\frac{n}{r}\right) - 1,\, q\left(-\alpha + \left(\frac{n}{r}\right) - 1\right) - \left(\frac{n}{r}\right) + 1\right\}$$
$$< \nu < \min\left\{q(\beta+1) + \left(\frac{n}{r}\right) - 1,\, q\left(\beta + \left(\frac{n}{r}\right)\right) - \left(\frac{n}{r}\right) + 1\right\}.$$

**Proof.** The ideas of the proof are the same as those in [**2**, **6**]. Let us first prove the sufficient condition. For $f : T_\Omega \to \mathbb{C}$, we write $f_y(x) = f(x + \mathrm{i}y)$. Then

$$\begin{aligned} T^+f(x+\mathrm{i}y) &= (T^+f)_y(x) \\ &= d_\gamma \Delta^\alpha(y)\left(\int_\Omega \int_{\mathbb{R}^n} |\Delta_{y+v}^{-(\gamma+(n/r))}(x-u)| f_v(u)\,\mathrm{d}u\right)\Delta^\beta(v)\,\mathrm{d}v \\ &= d_\gamma \Delta^\alpha(y) \int_\Omega (|\Delta_{y+v}^{-(\gamma+(n/r))}| * f_v)(x)\Delta^\beta(v)\,\mathrm{d}v. \end{aligned}$$

Without loss of generality, we may assume that $f$ is non-negative. By the Minkowski inequality, the Young inequality and Lemma 3.2 (i) we obtain

$$\begin{aligned} \|(T^+f)_y\|_{L^p(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} |(T^+f)_y(x)|^p\,\mathrm{d}x\right)^{1/p} \\ &= d_\gamma \Delta^\alpha(y)\left(\int_{\mathbb{R}^n}\left(\int_\Omega (|\Delta_{y+v}^{-(\gamma+(n/r))}| * f_v)(x)\Delta^\beta(v)\,\mathrm{d}v\right)^p \mathrm{d}x\right)^{1/p} \\ &\leqslant d_\gamma \Delta^\alpha(y) \int_\Omega \|\,|\Delta_{y+v}^{-(\gamma+(n/r))}| * f_v\|_p \Delta^\beta(v)\,\mathrm{d}v \\ &\leqslant d_\gamma \Delta^\alpha(y) \int_\Omega \|\Delta_{y+v}^{-(\gamma+(n/r))}\|_1 \|f_v\|_p \Delta^\beta(v)\,\mathrm{d}v \\ &= C_\alpha \int_\Omega \Delta^\alpha(y)\Delta^{-\gamma}(y+v)\|f_v\|_p\Delta^\beta(v)\,\mathrm{d}v \\ &= C_\alpha S(\|f_v\|_p)(y), \end{aligned}$$

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where $\|\Delta_{y+v}^{-(\gamma+(n/r))}\|_1$ is given by part of Lemma 3.2 (i). The sufficient condition then follows from Lemmas 2.4 and 2.6.

We now prove the necessary condition. We first show that if the operator $T^+$ is bounded on $L^{p,q}_\nu(T_\Omega)$, then the equality $\gamma = \alpha + \beta + (n/r)$ necessarily holds. We recall that the determinant function is homogeneous of degree $r$ (see [**7**]). For $f \in L^{p,q}_\nu(T_\Omega)$, we define $f_R$, $R > 0$, by $f_R(z) = f(Rz)$ for any $z \in T_\Omega$. The function $f_R$ belongs to $L^{p,q}_\nu(T_\Omega)$. Using the homogeneity of the determinant function we obtain

$$\|f_R\|^q_{L^{p,q}_\nu} = R^{-r(\nu-(n/r))-n(q/p)-n}\|f\|^q_{L^{p,q}_\nu}$$

and

$$\|T^+(f_R)\|^q_{L^{p,q}_\nu} = R^{r(\gamma+(n/r))q-r\alpha q-r(\nu-(n/r))-n(q/p)-n-q(r\beta+2n)}\|Tf\|^q_{L^{p,q}_\nu}.$$

It follows from the hypotheses that there exists a positive constant $C$ such that $\|T^+(f_R)\|_{L^{p,q}_\nu} \leqslant C\|f_R\|_{L^{p,q}_\nu}$. This is equivalent to $R^{r(\gamma-\alpha-\beta)-n}\|T^+f\|_{L^{p,q}_\nu} \leqslant C\|f\|_{L^{p,q}_\nu}$ for all $R > 0$, which necessarily implies that $\gamma = \alpha + \beta + (n/r)$. The condition $\alpha + \beta > -1$ is naturally necessary, since otherwise the range of $\nu$ would be empty. To obtain the other necessary conditions, we test $T^+$ on the functions $f(x + \mathrm{i}y) = \chi_{|x|<1}(x)g(y)$, with $g$ a positive function compactly supported in the intersection of the cone with the Euclidean ball of radius $\delta$ centred at 0. Using Lemma 3.1, it follows that, for $x$ and $y$ with $|x| < \frac{1}{4}$, $|y| < \delta$, the following inequality holds:

$$T^+f(x+\mathrm{i}y) \geqslant C\Delta^\alpha(y)\int_\Omega \Delta^{-\gamma}(y+v)g(v)\Delta^\beta(v)\,\mathrm{d}v.$$

Then, by assumption, there exists a constant $C$ independent of $g$ such that

$$\int_{y\in\Omega,|y|<\delta}\left(\Delta^\alpha(y)\int_\Omega \Delta^{-\gamma}(y+v)g(v)\Delta^\beta(v)\,\mathrm{d}v\right)^q \Delta^{\nu-(n/r)}(y)\,\mathrm{d}y \leqslant C\int_\Omega g^q(v)\Delta^{\nu-(n/r)}(v)\,\mathrm{d}v.$$

By homogeneity of the kernel, we can replace the constant $\delta$ by an arbitrary positive constant $K$. It follows that, for every positive function $g$ on $\Omega$, we have the inequality

$$\int_{y\in\Omega,|y|<K}\left(\Delta^\alpha(y)\int_\Omega \Delta^{-\gamma}(y+v)g(v)\Delta^\beta(v)\,\mathrm{d}v\right)^q \Delta^{\nu-(n/r)}(y)\,\mathrm{d}y \leqslant C\int_{v\in\Omega,|v|<K} g^q(v)\Delta^{\nu-(n/r)}(v)\,\mathrm{d}v.$$

Then, by density of compactly supported functions, we have the same inequality without any bound on the integrals. The other necessary condition of the theorem is then a consequence of the necessary conditions in Lemmas 2.5 and 2.6 and the relation $\gamma = \alpha + \beta + (n/r)$, obtained previously. This completes the proof of the theorem. □

**Corollary 3.4.** *Let $1 \leqslant p, q < \infty$ and $\nu \in \mathbb{R}$. If $\gamma = \alpha + \beta + (n/r)$, $\alpha + \beta > -1$ and*

$$\max\left\{-q\alpha + \left(\frac{n}{r}\right) - 1,\, q\left(-\alpha + \left(\frac{n}{r}\right) - 1\right) - \left(\frac{n}{r}\right) + 1\right\} < \nu < \min\left\{q(\beta + 1) + \left(\frac{n}{r}\right) - 1,\, q\left(\beta + \left(\frac{n}{r}\right)\right) - \left(\frac{n}{r}\right) + 1\right\},$$

*then the operator $T_{\alpha,\beta,\gamma}$ is bounded on $L^{p,q}_\nu(T_\Omega)$.*

We define the Berezin transform on $T_\Omega$ as the operator defined on $L^1(T_\Omega)$ by the pairing

$$\langle f\tilde{B}_z, \tilde{B}_z\rangle, \quad z \in T_\Omega,$$

where $\tilde{B}_z = \Delta^{n/r}(\operatorname{Im} z)B_{(n/r)}(\cdot\,, z)$ is the normalized reproducing kernel of $A^2(T_\Omega)$ (see, for example, [**11**] for more on the Berezin transform).

**Corollary 3.5.** *Let $1 \leqslant p, q < \infty$. Then the Berezin transform defined on $T_\Omega$ by*

$$B(f)(z) = \Delta^{2(n/r)}(\operatorname{Im} z)\int_{T_\Omega} |B_{3(n/r)}(z, w)|f(w)\,\mathrm{d}V(w), \quad z \in T_\Omega,$$

*is bounded on $L^{p,q}(T_\Omega, \mathrm{d}V(z))$, if and only if $q > 2 - (r/n)$.*

Note that the above corollary was proved in [**8**] in the setting of the light cones and for $p = q$.

**Corollary 3.6.** *Let $1 \leqslant p, q < \infty$. If $\nu$ and $m$ are real numbers such that $\nu + m > (n/r) - 1$, then the positive operator $Q^+$ defined by*

$$Q^+ f(z) = \int_{T_\Omega} |B_{\nu+m}(z, w)|f(w)\Delta^{\nu-(n/r)}(\operatorname{Im} w)\,\mathrm{d}V(w)$$

*is bounded from $L^{p,q}_\nu(T_\Omega)$ to $L^{p,q}_{\nu+mq}(T_\Omega)$ if and only if the following conditions are satisfied:*

$$\max\left\{-mq + \left(\frac{n}{r}\right) - 1,\, q\left(-m + \left(\frac{n}{r}\right) - 1\right) - \left(\frac{n}{r}\right) + 1\right\} < \nu < \min\left\{q\left(\nu - \left(\frac{n}{r}\right) + 1\right) + \left(\frac{n}{r}\right) - 1,\, q\nu - \left(\frac{n}{r}\right) + 1\right\}.$$

**Proof.** The operator $K$ defined by $K(f)(z) = \Delta^{-m}(\operatorname{Im} z)f(z)$ is an isometric isomorphism of $L^{p,q}_\nu(T_\Omega)$ to $L^{p,q}_{\nu+mq}(T_\Omega)$. Since, for every $f$ in $L^{p,q}_\nu(T_\Omega)$,

$$\begin{aligned} Q^+ f(z) &= \int_{T_\Omega} |B_{\nu+m}(z, w)|\Delta^{-m}(\operatorname{Im} w)f(w)\Delta^{\nu+m-(n/r)}(\operatorname{Im} w)\,\mathrm{d}V(w) \\ &= T^+_{0,\nu+m-(n/r),\nu+m}(Kf)(z), \end{aligned}$$

the corollary follows from Theorem 1.1. □

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Note that the above corollary in the case when $r=2$ is [**2**, Proposition 3.5]. We recall that $P_\mu^+ = T^+_{0,\mu-(n/r),\mu}$. The boundedness of $P_\mu^+$ has been obtained in [**4**] for the case of the light cone. The following corollary is its generalization.

**Corollary 3.7.** *Let $\mu,\nu\in\mathbb{R}$ and $1\leqslant p,q<\infty$. Then $P_\mu^+$ is bounded in $L_\nu^{p,q}(T_\Omega)$ if and only if $\mu,\nu>(n/r)-1$ and*

$$\max\left\{\frac{\nu-((n/r)-1)}{\mu-((n/r)-1)},\frac{\nu+(n/r)-1}{\mu}\right\}<q<\frac{\nu+(n/r)-1}{(n/r)-1}.$$

Recall that the Bergman projection $P_\mu$ is defined for $f\in L_\mu^2(T_\Omega)$ by

$$P_\mu f(z)=\int_{T_\Omega}B_\mu(z,w)f(w)\Delta^{\mu-n/r}(\operatorname{Im}w)\,\mathrm{d}V(w),$$

where the Bergman kernel $B_\mu$ is given by (1.2). $P_\mu f(z)$ defines a holomorphic function in $T_\Omega$ whenever the above integral is absolutely convergent. This is also the case if we consider $P_\mu f(z)$ with $f\in L_\nu^{p,q}(T_\Omega)$. Using the notation

$$\tilde q_{\nu,p}=\frac{\nu+(n/r)-1}{((n/rp')-1)_+}$$

with $\tilde q_{\nu,p}=\infty$ if $n/r\leqslant p'$, we have the following proposition (see also [**5**, Lemma 4.23] for the case $\mu=\nu$).

**Proposition 3.8.** *Let $\mu,\nu\in\mathbb{R}$, and $1\leqslant p,q<\infty$. If $P_\mu$ extends as a bounded operator on $L_\nu^{p,q}(T_\Omega)$, then $B_\mu(z,\mathrm{i}\underline{e})\in L_\nu^{p,q}$ and $\Delta^{\mu-\nu}(\operatorname{Im}z)B_\mu(z,\mathrm{i}\underline{e})\in L_\nu^{p',q'}$. The latter is equivalent to the following conditions: $\nu>(n/r)-1$ and $p((n/r)-1-\mu)<2(n/r)-1<p((n/r)+\mu)$,*

$$\max\left\{\frac{\nu-(n/r)+1}{(\mu-(n/r)+1)_+},\frac{\nu+(n/r)-1}{(\mu+(n/rp'))_+}\right\}<q<\tilde q_{\nu,p}.$$

**Proof.** Let $P_\mu^*$ be the adjoint operator of $P_\mu$ with respect to the pairing $\langle\cdot,\cdot\rangle_\nu$. We have

$$P_\mu^*f(z)=\Delta^{\mu-\nu}(\operatorname{Im}z)\int_{T_\Omega}B_\mu(z,w)f(w)\Delta^{\nu-n/r}(\operatorname{Im}w)\,\mathrm{d}V(w),\quad f\in L_\nu^{p',q'}.$$

Testing $P_\mu$ with

$$f_1(z)=\chi_{B_1(\mathrm{i}\underline{e})}(z)\Delta^{-\mu+(n/r)}(\operatorname{Im}z)$$

and $P_\mu^*$ with

$$f_2(z)=\chi_{B_1(\mathrm{i}\underline{e})}(z)\Delta^{-\nu+(n/r)}(\operatorname{Im}z),$$

where $B_1(\mathrm{i}\underline{e})$ is the Euclidean ball of radius 1 centred at $\mathrm{i}\underline{e}$, it follows from the mean-value property that $P_\mu f_1(z)=CB_\mu(z,\mathrm{i}\underline{e})$ and $P_\mu^*f_2(z)=C\Delta^{\mu-\nu}(\operatorname{Im}z)B_\mu(z,\mathrm{i}\underline{e})$. Consequently, we have $B_\mu(z,\mathrm{i}\underline{e})\in L_\nu^{p,q}$ and $\Delta^{\mu-\nu}(\operatorname{Im}z)B_\mu(z,\mathrm{i}\underline{e})\in L_\nu^{p',q'}$. Thus, by Lemma 3.2

this is equivalent to $\nu + (\mu - \nu)q' > (n/r) - 1$, $\nu > (n/r) - 1$,

$$\mu + \left(\frac{n}{r}\right) > \left(2\left(\frac{n}{r}\right) - 1\right) \max\left(\frac{1}{p'}, \frac{1}{p}\right)$$

and

$$\mu + \left(\frac{n}{r}\right) > \max\left\{\frac{n}{rp'} + \frac{\nu + (\mu - \nu)q' + (n/r) - 1}{q'}, \frac{n}{rp} + \frac{\nu + (n/r) - 1}{q}\right\}.$$

That is,

$$\nu > \left(\frac{n}{r}\right) - 1, \quad \mu + \left(\frac{n}{r}\right) > \left(2\left(\frac{n}{r}\right) - 1\right) \max\left(\frac{1}{p'}, \frac{1}{p}\right)$$

and

$$\max\left\{\frac{\nu - (n/r) + 1}{(\mu - (n/r) + 1)_+}, \frac{\nu + (n/r) - 1}{(\mu + (n/rp'))_+}\right\} < q < \tilde{q}_{\nu,p}.$$

□

**Theorem 3.9.** *The operator $T^+$ is bounded on $L^\infty(T_\Omega)$ if and only if $\alpha > (n/r) - 1$, $\beta > -1$ and $\gamma = \alpha + \beta + (n/r)$.*

**Proof.** We first prove the sufficiency. For any $f \in L^\infty(T_\Omega)$, we have

$$\begin{aligned} |T^+ f(x + \mathrm{i}y)| &\leqslant \Delta^\alpha(y) \int_{T_\Omega} |B_\gamma(x + \mathrm{i}y, u + \mathrm{i}v)|\,|f(u + \mathrm{i}v)|\Delta^\beta(v)\,\mathrm{d}u\,\mathrm{d}v \\ &\leqslant \|f\|_\infty \Delta^\alpha(y) \int_{T_\Omega} \left|\Delta^{-(\gamma+(n/r))}\left(\frac{x - u + \mathrm{i}(y + v)}{\mathrm{i}}\right)\right| \\ &\qquad \times \Delta^{(\beta+(n/r))-(n/r)}(v)\,\mathrm{d}u\,\mathrm{d}v \\ &\leqslant C\|f\|_\infty \Delta^{\alpha-\gamma+\beta+(n/r)}(y) \\ &= C\|f\|_\infty, \end{aligned}$$

where the third inequality follows from Lemma 3.2 under the hypotheses.

We now prove the necessary condition. First, we show that if $T^+$ is bounded on $L^\infty(T_\Omega)$, then the equality $\gamma = \alpha + \beta + (n/r)$ holds. For $f \in L^\infty(T_\Omega)$, we define the function $f_R$, $R > 0$, by $f_R(z) = f(Rz)$ for any $z \in T_\Omega$. The function $f_R$ belongs to $L^\infty(T_\Omega)$ and we have $\|f_R\|_\infty \leqslant \|f\|_\infty$. Using the homogeneity of the determinant function we obtain

$$\|T^+(f_R)\|_\infty = R^{r(\gamma+(n/r))-r\alpha-r\beta-2n}\|T^+ f\|_\infty.$$

It follows from the hypotheses that there exists a positive constant $C$ such that $\|T^+(f_R)\|_\infty \leqslant C\|f_R\|_\infty$. This implies that $R^{r(\gamma+(n/r))-r\alpha-r\beta-2n}\|T^+ f\|_\infty \leqslant C\|f\|_\infty$ for all $R > 0$, which necessarily implies that $\gamma = \alpha + \beta + (n/r)$. Now, we test $T^+$ on the function $f(x + \mathrm{i}y) = \chi_{|x|<1} g(y)$, where $g$ is a positive function compactly supported on the intersection of the cone with the Euclidean ball of radius $\delta$ centred at 0. From Lemma 3.1, we have that, for $x, y$ with $|x| < \frac{1}{4}, |y| < \delta$, the following inequality holds:

$$\Delta^\alpha(y) \int_{v\in\Omega, |v|<\delta} \Delta^{-\gamma}(y + v) g(v) \Delta^\beta(v)\,\mathrm{d}v \leqslant CT^+ f(x + \mathrm{i}y) \leqslant C\|f\|_\infty.$$

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We already know that by homogeneity of the kernel we can replace $\delta$ by an arbitrary positive constant $K$. Thus, by density of compactly supported functions, we can just write the left-hand side of the above inequality without any bound on the integral. Taking $g(v)=1$, it follows that we should have

$$\Delta^{\alpha}(y)\int_{\Omega}\Delta^{-\gamma}(y+v)\Delta^{\beta}(v)\,\mathrm{d}v<\infty.$$

It follows easily from Lemma 2.2 that we should have $\beta>-1$ and $-\gamma+\beta<-2(n/r)+1$. Thus, using the equality previously obtained, we deduce that $\alpha>(n/r)-1$. This completes the proof of the theorem. □

Although the conditions for the boundedness of $T^{+}$ are generally only sufficient for the boundedness of $T$, in the case of $L^{\infty}(T_{\Omega})$ they are also necessary, as we show in the next result.

**Theorem 3.10.** *The operator $T$ is bounded on $L^{\infty}(T_{\Omega})$ if and only if $\alpha>(n/r)-1$, $\beta>-1$ and $\gamma=\alpha+\beta+(n/r)$.*

**Proof.** We only have to show the necessity. Let $T$ be bounded on $L^{\infty}(T_{\Omega})$. The condition $\gamma=\alpha+\beta+(n/r)$ follows from the proof of Theorem 3.9. Let $w=\xi+\mathrm{i}t\in T_{\Omega}$ be fixed and consider the function $f_w$ given by

$$f_w(x+\mathrm{i}y)=\frac{|B_{\gamma}(\xi+\mathrm{i}t,x+\mathrm{i}y)|}{B_{\gamma}(\xi+\mathrm{i}t,x+\mathrm{i}y)}\chi_{|x|<1}g(y),$$

where $g$ is a positive function compactly supported on the intersection of the cone with the Euclidean ball of radius $\delta$ centred at 0. Testing $T$ with $f_w$ and taking $x+\mathrm{i}y=w$ yields (using the same reasoning as in the proof of Theorem 3.9) that we should have $\beta>-1$ and $-\gamma+\beta<-2(n/r)+1$ and consequently that $\alpha>(n/r)-1$. □

## 4. The topological dual of $A_{\nu}^{p,q}(T_{\Omega})$, $1<q<q_{\nu}$

We recall the following notation:

$$\tilde{q}_{\nu,p}=\frac{\nu+(n/r)-1}{((n/rp')-1)_{+}},\quad q_{\nu,p}=\min\{p,p'\}q_{\nu}\quad\text{and}\quad q_{\nu}=1+\frac{\nu}{(n/r)-1}$$

with $\tilde{q}_{\nu,p}=\infty$, if $n/r\leqslant p'$. It is clear that $1<q_{\nu}<q_{\nu,p}<\tilde{q}_{\nu,p}$. By the density of the intersection $A_{\nu}^{p,q}\cap A_{\mu}^{2}$ in $A_{\nu}^{p,q}$, we have the following reproducing formula for all $\alpha>(n/r)-1$ and $f\in A_{\nu}^{p,q}$ with $1\leqslant p<\infty$ and $1\leqslant q<\tilde{q}_{\nu,p}$:

$$f(z)=\int_{T_{\Omega}}B_{\alpha}(z,w)f(w)\Delta^{\alpha-(n/r)}(\operatorname{Im}w)\,\mathrm{d}V(w),\quad z\in T_{\Omega}. \tag{4.1}$$

The following theorem characterizes the topological dual space of the Bergman space $A_{\nu}^{p,q}$ for some values of $p$, $q$ and $\nu$ for which the Bergman projection is not necessarily bounded.

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**Theorem 4.1.** *Let $\nu > (n/r) - 1$ be real, $1 < p < \infty$ and $1 < q < q_\nu$. If $\mu$ is a sufficiently large real number so that $\mu > (n/r) - 1$ and $1 < q' < q_\mu$, then the topological dual space $(A_\nu^{p,q})^*$ of the Bergman space $A_\nu^{p,q}$ identifies with $A_\mu^{p',q'}$ under the integral pairing*

$$\langle f, g\rangle_\alpha = \int_{T_\Omega} f(w)\overline{g(w)}\Delta^{\alpha-(n/r)}(\operatorname{Im} w)\,\mathrm{d}V(w),$$

*where*

$$\alpha = \frac{\nu}{q} + \frac{\mu}{q'}, \qquad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1.$$

**Proof.** We have the equality

$$\begin{aligned}\int_{T_\Omega} & f(z)\overline{g(z)}\Delta^{\alpha-(n/r)}(\operatorname{Im} z)\,\mathrm{d}V(z) \\ &= \int_{T_\Omega} (\Delta^{(\nu-(n/r))/q}(\operatorname{Im} z)f(z))(\Delta^{(\mu-(n/r))/q'}(\operatorname{Im} z)\overline{g(z)})\,\mathrm{d}V(z).\end{aligned}$$

Since, for every $f \in A_\nu^{p,q}$, the function $\Delta^{(\nu-(n/r))/q}(\operatorname{Im} z)f(z)$ is in $L^{p,q}(T_\Omega, \mathrm{d}z)$ and, for every $g \in A_\mu^{p',q'}$, the function $\Delta^{(\mu-(n/r))/q'}(\operatorname{Im} z)g(z)$ is in $L^{p',q'}(T_\Omega, \mathrm{d}z)$, it follows that the given form is well defined and every $g \in A_\mu^{p',q'}$ defines an element of $(A_\nu^{p,q})^*$ given by the above integral pairing. The injectivity of the mapping $g \in A_\mu^{p',q'} \mapsto \langle \cdot\,, g\rangle_\alpha$ follows by testing with $f = B_\alpha(\cdot\,, w)$ (which belongs to $A_\nu^{p,q}$ by Lemma 3.2 since $\alpha > (n/r) - 1$ and $q > q'_\mu > (\mu + (n/r) - 1)/(\mu + (n/rp')))$ and using the reproducing formula (4.1).

Now let us show that every element $\mathcal{M}$ of $(A_\nu^{p,q})^*$ can be represented by an element $g$ of $A_\mu^{p',q'}$. By the Hahn–Banach theorem, there exists a function $h \in L_\nu^{p',q'}$ satisfying $\|h\|_{L_\nu^{p',q'}} = \|\mathcal{M}\|$ such that, for any $f \in A_\nu^{p,q}$,

$$\mathcal{M}(f) = \int_{T_\Omega} F(z)\overline{h(z)}\Delta^{\nu-(n/r)}(\operatorname{Im} z)\,\mathrm{d}V(z).$$

Let us set $k(z) = \Delta^{(\nu-\mu)/q'}(\operatorname{Im} z)h(z)$. Then $k \in L_\mu^{p',q'}$ and we have

$$\int_{T_\Omega} f(z)\overline{h(z)}\Delta^{\nu-(n/r)}(\operatorname{Im} z)\,\mathrm{d}V(z) = \int_{T_\Omega} f(z)\overline{k(z)}\Delta^{\alpha-(n/r)}(\operatorname{Im} z)\,\mathrm{d}V(z).$$

It is easy to see that

$$\mu < \min\left\{q'\left(\alpha - \left(\frac{n}{r}\right) + 1\right) + \left(\frac{n}{r}\right) - 1, q'\alpha - \left(\frac{n}{r}\right) + 1\right\}$$

and

$$\nu < \min\left\{q\left(\alpha - \left(\frac{n}{r}\right) + 1\right) + \left(\frac{n}{r}\right) - 1, q\alpha - \left(\frac{n}{r}\right) + 1\right\}.$$

Thus, $P_\alpha$ is bounded on $L_\mu^{p',q'}$ and on $L_\nu^{p,q}$. If we set $g = P_\alpha(k)$, $g$ belongs to $A_\mu^{p',q'}$ and we clearly have

$$\mathcal{M}(f) = \langle f, k\rangle_\alpha = \langle P_\alpha f, k\rangle_\alpha = \langle f, P_\alpha k\rangle_\alpha = \langle f, g\rangle_\alpha.$$

We have used the fact that, since $P_\alpha$ is bounded on $L^{p,q}_\nu$, it reproduces functions of $A^{p,q}_\nu$. The proof is complete. □

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