ON THE "ESSENTIAL METRIZATION" OF UNIFORM SPACES

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Introduction. The notion of "essential metrization" was introduced in **(8)** where it was used to obtain some extensions of the theorems of Egoroff and Lusin on measurable functions. In the present note we shall further develop the theory of essential metrization, in its own right.

As is well known, sets of measure zero ("null sets") may be disregarded in many problems of measure theory. Hence the usual topological prerequisites of such problems are actually too restrictive and may be replaced by what could be called "topology modulo null-sets," imitating such notions as "approximate continuity," "essential supremum," etc. Thus we consider spaces in which certain topological properties (such as metrizability, separability, etc.) hold not in the usual sense but only "essentially," i.e. to within some null sets, as defined below. The "null sets" could be chosen independently of a measure; however, we limit ourselves to the case where they are "null" under a numerical measure, either prescribed or to be constructed. Only uniform spaces will be considered in this note. It will be seen that many spaces that are not pseudometrizable in the usual sense become "essentially pseudometrizable" when a suitable measure ($\not\equiv 0$) is defined in them. Essentially metrizable spaces have properties similar to those of metric spaces, with the ordinary topological notions replaced by their "essential" counterparts; this yields various generalizations, cf. **(8**, §§ 3, 4**)**.

While in **(8)** we considered separable or "essentially separable" spaces only, we shall now take up other "essentially metrizable" cases, e.g., those involving the Lindelöf property, and those in which neighbourhoods are never "null" under the given measure. As every metric space has a nested base, it is clear that the existence of such a base (at least "essentially") is a *necessary* condition for essential metrizability. Our aim is to find out what other conditions, when added, ensure sufficiency also.

1. Preliminaries, terminology, and notation. We shall use the terminology and notation of **(5** and **7)** with the following changes and additions:

I. $(T, \mathbf{U})$ denotes a uniform space T whose uniformity (filter of entourages) is $\mathbf{U}$. Given $x \in T$ and $U \in \mathbf{U}$ we denote the neighbourhood

$$U[x] = \{y \in T \mid (x, y) \in U\}$$

also briefly by U_x. When *numbering* entourages, we shall use superscripts

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rather than subscripts, e.g. $U^0, U^1, \ldots$; "powers" of entourages will never occur (a similar notation was used for "indexed neighbourhoods" in **(8)**.) Bold-face capitals $\mathbf{V}, \mathbf{W}, \ldots$ will denote other uniformities on T or on subsets of T. If $(T, \mathbf{U})$ has some properties such as metrizability, separability, etc., we also say that $\mathbf{U}$ is metrizable, separable, etc. $(T, \mathbf{U})$ is said to be *uniformly* (*topologically*) pseudometrizable if its uniformity (its topology) is compatible with some pseudometric. Unless otherwise stated, "(pseudo)-metrizable" means "*uniformly* (pseudo)metrizable." (If, however, both kinds of pseudometrizability occur (e.g. in § 3), we prefer to use the explicit term "*uniformly* (pseudo)metrizable." Similarly, we avoid such abbreviations as "$\mathbf{U}$ is separable," etc. if only a subspace of T has the property involved (e.g., in 3.4 and 3.5 below).)

II. We shall denote by μ a non-negative countably additive measure defined on a σ-field $\mathfrak{M}$ of subsets ("measurable sets") of T; μ^* will denote the outer measure defined for all $A \subseteq T$ by

$$\mu^*(A) = \inf\{\mu X \mid A \subseteq X \in \mathfrak{M}\}.$$

Given two uniformities $\mathbf{U}$ and $\mathbf{V}$ on T, we say that $\mathbf{U}$ is *μ-essentially finer* than $\mathbf{V}$ (and that $\mathbf{V}$ is *essentially coarser* than $\mathbf{U}$), if, for every $V \in \mathbf{V}$, there is $U \in \mathbf{U}$ such that $\mu^*(U_x - V_x) = 0$ for all $x \in T$. This implies that $V_x \supseteq U_x - Z$ for some set $Z \in \mathfrak{M}(\mu Z = 0)$ depending on V and x. If the sets Z can be so chosen that they depend only on V (not on x), we say that U is essentially finer in the *stronger sense.* If $\mathbf{U}$ is both essentially finer and coarser than $\mathbf{V}$, we say that $\mathbf{U}$ and $\mathbf{V}$ are *essentially equivalent* (possibly in the stronger sense). (In **(8)** only the stronger notion of essential equivalence was considered, and the term "essentially finer" meant what we now call "essentially finer in the stronger sense." Similarly for the notions of "essential metrization," etc., as defined below. Observe that all these definitions remain valid also with $\mathbf{U}$ and $\mathbf{V}$ replaced by some *bases*, $\mathbf{U}'$ and $\mathbf{V}'$, respectively.) Clearly, if the sets Z mentioned above can be chosen empty, we obtain the ordinary notions of "finer" and "coarser," while the essential equivalence of $\mathbf{U}$ and $\mathbf{V}$ becomes equality: $\mathbf{U} = \mathbf{V}$.

III. We say that $(T, \mathbf{U})$ (or $\mathbf{U}$) is *nested* if the uniformity $\mathbf{U}$ has a base $\mathbf{U}'$ which is linearly ordered by inverse inclusion $\supset$. If $\mathbf{U}$ is essentially equivalent to some nested uniformity T on $\mathbf{V}$, we say that $(T, \mathbf{U})$ (or $\mathbf{U}$) is *essentially nested.* $(T, \mathbf{U})$ and $\mathbf{U}$ are said to be *μ-essentially metrizable* if $\mathbf{U}$ is μ- essentially equivalent to some uniformly metrizable $\mathbf{V}$; similarly for pseudometrizability and other notions. In particular, $(T, \mathbf{U})$ and $\mathbf{U}$ are essentially separable, essentially compact etc., if and only if $\mathbf{U}$ is essentially equivalent to some $\mathbf{V}$ which has this property in the ordinary sense. We note that $(T, \mathbf{U})$ is uniformly pseudometrizable if and only if its uniformity $\mathbf{U}$ is *countably generated*, i.e., has a countable base $\mathbf{U}'$ **(5**, p. 186**)**.

IV. The space $(T, \mathbf{U})$ and its uniformity $\mathbf{U}$, are said to be *totally bounded* (*σ-totally bounded*) if, for each $U \in \mathbf{U}$, there is a finite (at most countable)

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set $A \subseteq T$ such that $T = U[A] = \bigcup_{x \in A} U_x$. The "essential" counterparts of these notions are defined accordingly, as explained in III.

V. The measure μ defined in $(T, \mathbf{U})$ is called *topological* if all Borel sets in T(in particular, all open neighbourhoods U_x) are μ-measurable. μ is said to be *essentially topological* if $\mathbf{U}$ is μ-essentially equivalent to some $\mathbf{V}$ under which μ is topological. We say that μ is *regular* (or $\mathbf{U}$-*regular*) if it is topological and if, for every $\epsilon > 0$ and every μ-measurable set $A \subseteq T$, there is a closed set $F \subseteq A$ such that $\mu(A - F) < \epsilon$. μ is called $\mathbf{U}$-*positive* if $\mu^* U[x] \neq 0$ for all $x \in T$ and $U \in \mathbf{U}$. A set $A \subseteq T$ is μ-*null* if and only if $\mu^*(A) = 0$.

2. Essential metrization by constructing a suitable measure. In order to provide examples of essentially metrizable spaces that are not metrizable (or pseudometrizable) in the ordinary sense, we shall now develop a simple method which makes it possible to construct in *every* uniform space a (in general not unique) measure μ under which the space becomes essentially pseudometrizable. The method will leave considerable freedom in choosing this μ, so as to take care of those cases where μ is supposed to meet some additional requirements also. An actual example will be given at the end of this section.

As is shown in Bourbaki (**1**, IX, § 1), every uniformity $\mathbf{U}$ on T is a union of countably generated uniformities $\mathbf{V} \subseteq \mathbf{U}$. Fix such a $\mathbf{V}$ and an entourage V° $\in \mathbf{V}$. Then, let $\mathbf{D}$ be the collection of the sets: $\emptyset$, all U_x, and all $V_x{}^\circ - U_x$ where $U \in \mathbf{U}$ nad $x \in T$, while V° is fixed as above. We define on $\mathbf{D}$ a set function $s \geqslant 0$ as follows: $s(\emptyset) = 0$, $s(V_x{}^\circ - U_x) = 0$ for all $U \in \mathbf{U}$ and $x \in T$; to other members of $\mathbf{D}$ we assign arbitrary *positive* (non-0) s-values. To fix ideas, let $s(U_x) = 1$ for the rest of the proof, for $U \in \mathbf{U}$ and $x \in T$. We also define, for each set $X \subseteq T$, the function value $m^*(X)$ by setting

$$m^*(X) = \inf\left\{ \sum_{k=1}^{\infty} s(A_k) | X \subseteq \bigcup_{k=1}^{\infty} A_k, A_k \in \mathbf{D}\right\}. \tag{2.1}$$

If X cannot be covered by a countable family $\{A_k\} \subseteq \mathbf{D}$, we put $m^*(X) = +\infty$.

As is well known, the set function m^* so defined is an outer measure on all subsets of T, with $m^*(X) \leqslant s(X)$ for $x \in \mathbf{D}$ so that, in particular, $m^*(V_x{}^\circ - U_x) \leqslant s(V_x{}^\circ - U_x) = 0$. The restriction of m^* to the σ-field $\mathfrak{M}$ of all m^*-measurable sets (in the Carathéodory sense) is a measure μ which, by the above, satisfies

$$\mu(V_x{}^\circ - U_x) = m^*(V_x{}^\circ - U_x) = 0 \qquad \text{for all } U \in \mathbf{U} \text{ and } x \in T. \tag{2.2}$$

This formula implies, by definition, that the uniformity $\mathbf{V}$ is μ-essentially finer than $\mathbf{U}$; and since $\mathbf{V} \subseteq \mathbf{U}$, it follows that $\mathbf{V}$ and $\mathbf{U}$ are μ-essentially equivalent. As $\mathbf{V}$ is pseudometrizable (being countably generated), $\mathbf{U}$ is *essentially* so. In order to obtain an example from this for which μ is nontrivial, i.e., not the zero measure, we consider the case where $\mathbf{U}$ is not σ-totally bounded. Then $\mathbf{V}$ and $V^\circ \in \mathbf{V}$ may be chosen such that T cannot be covered by countably many sets V_x°, $x \in T$, and hence by countably many sets $V_x^\circ - U_x$, $x \in T$. From our definition of the set function s, and from (2.1), it then follows that $\mu T \neq 0$. Thus any non-metrizable space T which is not σ-totally bounded can serve as the required example. Such a space is, however, easily constructed from *any* non-metrizable uniform space, e.g., by adding to this space an uncountable family of "copies" of itself, retaining the same structure in each copy but making the copies disjoint from each other. The union of all these "copies" then is a non-metrizable and not σ-totally bounded uniform space, which can, however, be essentially pseudometrized by the above process.

While this simple procedure gives the basic idea of the proposed method, it is clear that various modifications and improvements are possible. In particular, the set family **D** need not contain all neighbourhoods U_x; these may be replaced by any other suitable subsets of T. If desirable, the measure μ can be made bounded by including T itself in **D**, with $s(T) < \infty$. If, instead, it is desirable to have a "large" family of measurable sets, one can modify the definition of m^* by introducing a pseudometric for $(T, \mathbf{V})$ and using a well-known limit method, called "Method II" in **(6**, pp. 105ff.**)**, in order to make m^* a Carathéodory outer measure, called "metric measure" in **(6)**, so that all Borel sets in $(T, \mathbf{V})$ become m^*-measurable. (It is not necessary to assume, as Munroe does **(6**, pp. 105ff.**)**, that T can be covered by a sequence of sets of arbitrarily small diameter.) Instead of fixing only one $V^\circ \in \mathbf{V}$, one can use any subfamily of $\mathbf{V}$ to form the sets $V_x - U_x$. The condition that T be non-σ-totally bounded is often dispensable, as follows from the example given below, which, in essence, is Kelley's example of a non-metrizable uniform space **(5**, p. 204, Problem C**)**.

Let T be the set of all ordinals $\leqslant \omega_1$ (= least uncountable ordinal). For each ordinal $\alpha < \omega_1$, let

$$U^\alpha = \{(x, y) \mid x = y, \quad \text{or} \quad (x \geqslant \alpha,\ y \geqslant \alpha)\};$$

let $\mathbf{U}$ be the uniformity on T generated by all entourages U^α $(\alpha < \omega_1)$. Clearly, these U^α form a nested base for $\mathbf{U}$, with no countable cofinal subset. Thus $\mathbf{U}$ is not countably generated, hence not pseudometrizable. As is easily seen, $(T, \mathbf{U})$ is σ-totally bounded. We now put $V^\circ = U^\omega$ and take for $\mathbf{V}$ any countably generated uniformity $\subseteq \mathbf{U}$ such that $V^\circ \in \mathbf{V}$. With s, $\mathbf{D}$, m^*, and μ defined as before, it follows immediately that none of the neighbourhoods U_x $(x < \omega)$ can be covered by sets of the form $V_z^\circ - U_z$. (Indeed, if $z > \omega$, then the definition of U^α yields $V_z^\circ = U_z^\omega = \{z\}$ whence $V_z^\circ - U_z = \emptyset$. If, however, $z \geqslant \omega$, then $U_z^\omega = \{y \in T \mid y \geqslant \omega\}$. In either case $x \notin V_z^\circ - U_z$, if $x < \omega$.) Thus the procedure outlined above yields a non-trivial countably additive measure μ under which $(T, \mathbf{U})$ is essentially pseudometrizable. Moreover, it is easy to see that we can make under this measure every set $X \subseteq T$ measurable by modifying s slightly, namely by setting $s(U_x^\alpha) = 1$ if $x < \alpha$ and $s(U_x^\alpha) = +\infty$ if $x \geqslant \alpha$. With this definition if all elements of X are less

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than ω, then $\mu(X)$ equals the cardinal number of X (thus there are many sets of finite non-zero measure). Sets consisting of elements $\leqslant\omega$ only have measure 0. All this works also when ω_1 is dropped from T (referee's remark); this yields Kelley's discrete space, whereas our space T is a *non-discrete* nested space.

We conjecture that every nested space $(T, \mathbf{U})$ is either pseudometrizable, or essentially pseudometrizable under some countably additive measure μ which admits *infinitely many measurable sets of finite non-zero measure.* It is a less certain conjecture that this is true of every uniform space.

We now take up the case where the measure μ is not to be constructed but is *prescribed.* Then, in order to ensure μ-essential pseudometrizability, some conditions must be imposed on μ and on the space $(T, \mathbf{U})$. An especially simple procedure applies to Lindelöf spaces, which we shall now consider.

3. Lindelöf spaces. U-positive measures. Throughout this section, the term "Lindelöf space" stands for *hereditary Lindelöf space*, i.e., a space T in which every open covering of any set $A \subseteq T$ has a countable subcovering. A uniform space $(T, \mathbf{U})$ is said to be *pseudodiscrete* or *discretely pseudometrizable* if its separated quotient space is discrete (this does not imply *uniform* pseudometrizability). The "essential "counterparts of these notions are defined accordingly, as in § 1, III. We shall need two propositions proved in **(8)**, (where they appear as 2.1 and Note to Theorem 2.2); we state them here without proof. (These propositions are stated in **(8)** in a slightly different notation, and for essential equivalence in the *stronger* sense only. However, their proofs remain valid also if "essential equivalence," "essential metrization," etc. are defined as in the present paper. Hence also all the following theorems are valid *both* ways.)

3.1. LEMMA. *Let μ be a σ-finite essentially topological measure in a uniform space $(T, \mathbf{U})$. If $\mathbf{U}$ is μ-essentially nested but not countably generated, then, for each $p \in T$, there is an open entourage $V \in \mathbf{U}$ (depending on p) such that*

$$(3.1.1) \qquad \mu^*(V_p - U_p) = 0 \text{ for all } U \in \mathbf{U}.$$

3.2. LEMMA. *If $(T, \mathbf{U})$ is μ-essentially pseudometrizable, then $\mathbf{U}$ is μ-essentially equivalent to some countably generated uniformity $\mathbf{V}$ for T, with $\mathbf{V} \subseteq \mathbf{U}$.*

We start with a simple proposition on $\mathbf{U}$-positive measures.

3.3. THEOREM. *If a uniform space $(T, \mathbf{U})$ is essentially nested under a σ-finite $\mathbf{U}$-positive essentially topological measure μ, then $\mathbf{U}$ is either countably generated or pseudodiscrete (i.e., T is either uniformly or discretely pseudometrizable).*

Proof. Suppose that $\mathbf{U}$ is not countably generated. Then, given any $p \in T$, Lemma 3.1 yields an open $V \in \mathbf{U}$ such that, for all $U \in \mathbf{U}$, $\mu^*(V_p - U_p) = 0$ and certainly $\mu^*(V_p - \bar{U}_p) = 0$ where $\bar{U}_p$ is the closure of U_p in the uniform topology of T. As the null sets $V_p - \bar{U}_p$ are open, the $\mathbf{U}$-positivity of μ implies that they are empty, so that $V_p \subseteq \bar{U}_p$ for all $U \in \mathbf{U}$. Hence

$$V_p \subseteq \bigcap \{\overline{U_p} \mid U \in \mathbf{U}\} = \bar{p}$$

where $\bar{p}$ is the closure of the one-point set $\{p\}$ (**5**, p. 179, Theorems 7, 8). Thus, every $p \in T$ has a neighbourhood $V_p = \bar{p}$, i.e., T is pseudodiscrete.

Note. This is one of the rare cases where *essential* nestedness implies ordinary topological (though not uniform) pseudometrizability. It is clear from the proof that the $\mathbf{U}$-positivity of μ alone would suffice if property (3.1.1) were given *a priori* (replacing essential nestedness, σ-finiteness, etc.).

3.4. THEOREM. *If a uniform Lindelöf space* $(T, \mathbf{U})$ *is essentially nested under a σ-finite essentially topological measure μ, then, by dropping a set Z of measure zero, T can be made separable and uniformly pseudometrizable.*

Proof. Let G be the union of all open μ-null sets i n$(T, \mathbf{U})$, By the Lindelöf property, G is covered by at most countably many such sets, and thus is itself μ-null. As follows from the definition of μ^* (see § 1, II), there is a *measurable* set $Z \supseteq G$, with $\mu Z = 0$. Then, clearly, the set $T' = T - Z$ contains no open null sets other than $\emptyset$. Thus μ becomes a $\mathbf{U}$-positive measure when restricted to T'. By 3.3, the subspace $(T', \mathbf{U})$ is either uniformly or discretely pseudometrizable. (Here and in the following, $\mathbf{U}$ will also denote the *relativized* uniformities inherited from $(T, \mathbf{U})$ by subspaces of T.) Assume the latter. Then all closures $\bar{p}$ of one-point sets $\{p\}$ are open under the relativized topology of T'. By the Lindelöf property, T' is covered by at most countably many such sets; say,

$$T = \bigcup_{n=1}^{\infty} \bar{p}_n$$

where $\bar{p}_n = U^n[p_n]$ for some $U^n \in \mathbf{U}$, $n = 1, 2, \ldots$. Now, if $\mathbf{U}$ is not countably generated, the sequence $\{U^n\}$ cannot be cofinal with $\mathbf{U}$. Hence, if $\mathbf{U}$ is nested, there must be a $V \in \mathbf{U}$ such that $V \subseteq \bigcap_n U^n$ and $V[p_n] \subseteq U^n[p_n] = \bar{p}_n$ for all n. If $\mathbf{U}$ is only μ-essentially nested (as is our assumption), a simple argument shows that $V[p_n] \subseteq \bar{p}_n \cup Z_n$ for some measurable null-sets $Z_n \subseteq T'$ $(n = 1, 2, \ldots)$. Thus, dropping from T' the null-set

$$Z' = \bigcup_{n=1}^{\infty} Z_n,$$

we obtain a subspace $T'' = T' - Z' = T - (Z \cup Z')$ such that $V[p] = \bar{p}$ for all $p \in T''$, under the relative topology of T''. (The equality $V[p] = \bar{p}$ holds not only for $p = p_n$ $(n = 1, 2, \ldots)$ but also for any $p \in \bar{p}_n$, since we may "identify" such p with p_n, by passing to the separated quotient space of T''. As

$$T'' = \bigcup_{n=1}^{\infty} \bar{p}_n$$

(under the same relative topology), the required equality holds for *all* $p \in T''$.) This means, however, that the discrete pseudometrization of T'' preserves also its uniform structure. Thus in all cases T can be made uniformly pseudometrizable by dropping a set of measure zero. Since every pseudometric Lindelöf space is separable, all our assertions are proved.

The main result of this section now easily follows:

3.5. THEOREM. *Let μ be a σ-finite essentially topological measure in a uniform space $(T, \mathbf{U})$. If $(T, \mathbf{U})$ is μ-essentially nested and μ-essentially Lindelöf, it can be made μ-essentially pseudometrizable and μ-essentially separable by dropping from T a set Z of measure 0. More precisely, $\mathbf{U}$ is μ-essentially equivalent to some $V \subseteq \mathbf{U}$ such that $(T - Z, \mathbf{V})$ is separable and uniformly pseudometrizable $(\mu Z = 0)$.*

Proof. By assumption, $\mathbf{U}$ is essentially equivalent to some uniformity $\mathbf{W}$ such that $(T, \mathbf{W})$ is a Lindelöf space. The transitivity of the essential equivalence relation easily implies that the measure μ remains essentially topological in $(T, \mathbf{W})$, and that $(T, \mathbf{W})$ is μ-essentially nested, as is $(T, \mathbf{U})$. Thus, by 3.4, $(T, \mathbf{W})$ has a separable and uniformly pseudometrizable subspace $(T', \mathbf{W})$ where $T' = T - Z$ $(\mu Z = 0)$. It already follows that $(T', \mathbf{U})$ is μ-essentially separable and μ-essentially pseudometrizable, and it only remains to replace $\mathbf{W}$ by a suitable $\mathbf{V} \subseteq \mathbf{U}$. Now, by 3.2, there is a countably generated uniformity $\mathbf{V} \subseteq \mathbf{U}$ which, when relativized to T', is essentially equivalent to $\mathbf{U}$ and hence also to $\mathbf{W}$. Thus the proof will be complete if we show that the subspace $(T', \mathbf{V})$ can be made separable by dropping from it another null set. Let q_n $(n = 1, 2, \ldots)$ be a dense sequence in the separable space $(T', \mathbf{W})$ so that

$$T' = \bigcup_{n=1}^{\infty} W[q_n]$$

for each $W \in \mathbf{W}$. The essential equivalence of $\mathbf{W}$ and $\mathbf{V}$ then implies that, for each $V \in \mathbf{V}$ and q_n, there is a measurable null set Z_V^n such that

$$T' = \bigcup_{n=1}^{\infty} (V[q_n] \cup Z_V^n)$$

for each $V \in \mathbf{V}$. We now choose a countable base $\mathbf{V}'$ for the (countably generated) uniformity $\mathbf{V}$ and put

$$Z' = \bigcup_{V \in \mathbf{V}'} \bigcup_{n=1}^{\infty} Z_V^n.$$

Then $\mu(Z') = 0$ and

$$T' - Z' = \bigcup_{n=1}^{\infty} (V[q_n] - Z')$$

for all $V \in \mathbf{V}'$. Hence, unless $\mu(T) = 0$, some of the sets $V[q_n] - Z'$ $(V \in \mathbf{V}')$ are non-empty. Choosing a point from each such $V[q_n] - Z'$, we obtain a

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countable dense subset of $T' - Z' = T - (Z \cup Z')$. Thus $(T' - Z', \mathbf{V})$ is separable. This completes the proof.

Note 1. In all the theorems of this section (as well as in those proved in **(8)**), the assumption that $(T, \mathbf{U})$ is essentially nested may be replaced by the following:

3.6. *For every sequence of points* $p_n \in T$ $(n = 1, 2, \ldots)$ *and every non-cofinal countable subset* $\{V^n\} \subseteq \mathbf{U}$, *there is a* $\mathbf{U} \in U$ *such that*

$$\mu^*(U[b_n] - V^n[p_n]) = 0, \qquad n = 1, 2, \ldots .$$

Indeed, as is easily verified, it is this property that is actually used in our proofs. Taken separately, 3.6 is weaker than μ-essential nestedness; but, when combined with other assumptions, it implies even μ-essential pseudometrizability (hence also μ-essential nestedness), as follows from our proofs.

Note 2. The proof of 3.1 given in **(8)**, and that of 3.3, work even under a still weaker version of 3.6, namely:

3.7. *For every point* $p \in T$ *and every non-cofinal countable subset* $\mathbf{V} \subseteq \mathbf{U}$, *there is a* $U \in \mathbf{U}$ *such that* $\mu^*(U[p] - V[p]) = 0$ *for all* $V \in \mathbf{V}$.

However, the conclusions of 3.4 and 3.5 must then be weakened to say that, by dropping a null set $Z \in \mathfrak{M}$, T can be made separable and either uniformly or discretely pseudometrizable (or essentially separable and either essentially pseudometrizable or essentially pseudodiscrete).

We now turn to some other cases of essential metrizability.

4. Compact and separable spaces. Regular measures. We shall need a preliminary proposition describing the structure of non-metrizable nested spaces. Part (a) of it was, in essence, proved by Doss **(3**, pp. 117, 119**)**. (Doss states his theorem only for M. Fréchet's *espaces à écart*, and constructs the base $\mathbf{U}'$ of our Theorem 4.1 only for the topology (not the uniformity) of T. However, the theorem applies to nested uniform spaces as well; cf. **(2**, p. 144, Theorem 3.4**)**. The "uniform" version of Doss's theorem can be derived from Isbell's propositions 26 and 27 **(4**, p. 133**)**. We prove a more general theorem in **(9)**, which contains 4.1(a) as a special case.) Thus we shall only prove parts (b) and (c), assuming (a) as given.

4.1. Theorem. (a) *If a nested uniformity* $\mathbf{U}$ *on* T *is not countably generated, then it has an open base* $\mathbf{U}'$ *which is well ordered by* $\supset$ *and consists of equivalence relations* (*so that each* $U \in \mathbf{U}'$ *induces a partition of* T *into disjoint sets* $U[x]$ *which are both open and closed*).

If, in addition, $(T, \mathbf{U})$ *is* μ*-essentially totally bounded* ($\mu \not\equiv \emptyset$), *then the base* $\mathbf{U}'$ *can be so chosen that, for some integer* $r > 0$, *it also satisfies:*

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(b) *For each* $U \in \mathbf{U}'$, *there is a set* $F_U \subseteq T$ *of exactly* r *points such that* $\mu^* U[x] \neq 0$ *for all* $x \in F_U$, *and* $\mu^*(T - U[F_U]) = 0$;

(c) *Moreover, if* $U, V \in \mathbf{U}'$ *and* $U \subseteq V$, *there is a one-to-one correspondence between* F_V *and* F_U *such that* $U[x] \subseteq V[x']$ *and* $\mu^*(V[x'] - U[x]) = 0$ *for corresponding* $x \in F_U$ *and* $x' \in F_V$.

Proof. By hypothesis, $\mathbf{U}$ has a well-ordered open base $\mathbf{U}'$, with no countable cofinal subsets, and $\mathbf{U}$ is μ-essentially equivalent to a totally bounded uniformity $\mathbf{W}$. Because of the latter, there is, for each $U \in \mathbf{U}'$, a $W \in \mathbf{W}$ such that $0 = \mu^*(W[x] - U[x])$ for all $x \in T$, and there are finite sets F_U such that $W[F_U] = T$, and thus $\mu^*(T - \mathbf{U}[F_U]) = 0$. Now, for each $U \in \mathbf{U}'$, let r_U be the least positive integer occurring as the cardinality of such F_U. Then, for any $U, V \in \mathbf{U}'$, $U \supseteq V$ implies $r_U \leqslant r_V$, and since $\mathbf{U}'$ is well ordered with no cofinal countable subset, the map $U \rightarrow r_U$ is constant from some $U \in \mathbf{U}'$ onward (a well-known fact). We may therefore assume that $r_U = r$ (fixed) for all $U \in \mathbf{U}'$. It follows that, whenever $\mu^*(T - U[F]) = 0$ for a set F of r elements, then $\mu^* U[Fx] \neq 0$ for all $x \in F$.

Now consider any $U, V \in \mathbf{U}'$ with $U \subseteq V$, and let $F, G \subseteq T$ be any sets of r elements such that $\mu^*(T - U[F]) = \mu^*(T - V[G]) = 0$. Then

$$U[F] \subseteq V[G];$$

for otherwise there would be an $x = F$ with

$$U[x] \cap (T - V[G]) \neq \emptyset.$$

Since U and V are equivalence relations, this would imply that $U[x] \subseteq T - V[G]$ and hence $\mu^* U[x] = 0$, a contradiction. Hence each of the r sets $U[x]$, $x \in F$, is contained in exactly one of $V[y]$, $y \in G$, and the difference of any two such sets is a null set. This proves our assertions (b) and (c).

4.2. Theorem. *If a compact uniform space* $(T, \mathbf{W})$ *is essentially nested under a regular measure* μ, *then it can be made* μ-*essentially pseudometrizable by dropping a measurable null set* Z. *Moreover,* $\mathbf{W}$ *is* μ-*essentially equivalent* (*on* $T - Z$) *to some* $\mathbf{V} \subseteq \mathbf{W}$ *such that* $(T - Z, \mathbf{V})$ *is uniformly pseudometrizable and compact.*

Proof. By assumption, $\mathbf{W}$ is essentially equivalent to some nested uniformity $\mathbf{U}$. We shall assume that $\mathbf{U}$ is not countably generated (otherwise everything is trivial). Then, since $\mathbf{W}$ is totally bounded, 4.1 yields a base $\mathbf{U}'$ for $\mathbf{U}$ satisfying 4.1, (a, b, c), with r and the sets F_U henceforth regarded as fixed. Consider any point $p \in T - U[F_U]$, $U \in \mathbf{U}'$. As U is an equivalence relation, the fact that $p \notin U[F_U]$ implies that $U[p] \subseteq T - U[F_U] = Z_U$ (a *null set*, by 4.1(b)). Also, the essential equivalence of $\mathbf{W}$ and $\mathbf{U}$ yields, for each $U \in \mathbf{U}'$, an open $W^U \in \mathbf{W}$ such that

$$\mu^*(W^U[p] - U[p]) = 0 \qquad \text{for all } p \in T.$$

For $p \in Z_U$, as we have seen, $U[p]$ is null; hence so is $W^U[p]$. Thus *every* $p \in Z_U$ $(U \in \mathbf{U}')$ *is in some* $\mathbf{W}$-*open null set* $W^U[p]$. Let

$$Z' = \bigcup \{Z_U \mid U \in \mathbf{U}'\} \quad \text{and} \quad Z = \bigcup_{U \in U'} \bigcup_{p \in Z'} W^U[p].$$

Then Z is $\mathbf{W}$-open, hence μ-measurable (by regularity), and

$$(4.2.1) \qquad Z' \subseteq Z = \bigcup_{U \in U'} \bigcup_{p \in Z'} W^U[p], \qquad \mu^* W^U[p] = 0.$$

Next we show that $\mu(Z) = \mu^*(Z') = 0$. (This is not immediate from (4.2.1) since the covering by the null sets $W^U[p]$ may be uncountable.) Indeed, suppose that $\mu(Z) > \epsilon > 0$. Then, by the $\mathbf{W}$-regularity of μ, there is a closed set $A \subseteq Z$ with $\mu(Z - A) < \frac{1}{2}\epsilon$. As T is $\mathbf{W}$-compact, so is A. Hence by (4.2.1.), A is covered by a *finite* number of the open null sets $W^U[p]$, and thus $\mu A = 0$. But this implies that $\mu Z = \mu(Z - A) < \frac{1}{2}\epsilon$, contrary to our assumption that $\mu Z > \epsilon$. (Note that this part of the proof works also with compactness replaced by the *plain* (non-hereditary) Lindelöf property. It easily follows that the *plain* Lindelöf property suffices also in 3.4, if μ is σ-finite and *regular*.) Thus, indeed, $\mu Z = 0$.

We now prove that $(T - Z, \mathbf{U})$ (hence also $(T - Z, \mathbf{W})$) is μ-essentially pseudometrizable. From the base $\mathbf{U}'$ we can certainly select an ω-type subset $\mathbf{U}''$ (containing the first member U° of $\mathbf{U}'$) such that $\mathbf{U}''$ is a base for some (countably generated) uniformity $\mathbf{U}^* \subseteq \mathbf{U}$. It then suffices to show that

$$(4.2.2) \qquad \mu^*(U_p{}^\circ - U_p) = 0 \qquad \text{for all } p \in T - Z \text{ and } U \in \mathbf{U}'.$$

Now, if $p \in T - Z$, then $p \notin Z'$ (for $Z' \subseteq Z$, by (4.2.1)); hence, by the definition of Z', $p \notin Z_U = T - U[F_U]$ for all $U \in \mathbf{U}'$. Thus, for all $U \in \mathbf{U}'$, $p \in U[F_U]$; i.e., $p \in U[x]$ for some (unique) $x \in F_U$, and hence $U[p] = U[x]$ (since U is an equivalence relation, by 4.1). In particular, $U^\circ[p] = U^\circ[x']$ for some (unique) $x' \in F_U{}^\circ$. As U° is, by definition, the first (i.e., largest) member of $\mathbf{U}'$, we have $U^\circ \supseteq U$ $(U \in \mathbf{U}')$ and it easily follows that the points x and x' found above correspond to each other in the sense of 4.1(c). Thus, by 4.1(c), we obtain $\mu^*(U_p{}^\circ - U_p) = \mu^*(U_{x'}{}^\circ - U_x) = 0$, proving (4.2.2) and establishing the essential pseudometrizability of $(T - Z, \mathbf{W})$. By 3.2, it now follows that $\mathbf{W}$, when relativized to $T - Z$, is μ-essentially equivalent to some countably generated $\mathbf{V} \subseteq \mathbf{W}$ (on $T - Z$), so that $(T - Z, \mathbf{V})$ is uniformly pseudometrizable. Moreover, since T is by assumption compact and Z is open, $T - Z$ is compact under the $\mathbf{W}$-topology, hence certainly so under the coarser $\mathbf{V}$-topology. Thus all our assertions are proved.

Note. It is worth noting that the entourages W^U $(U \in \mathbf{U}')$ and the countable subset $\mathbf{U}'' = \{U^\circ, U^1, \ldots,\} \subseteq \mathbf{U}'$ mentioned above can always be so chosen that the sequence $\{W^{U_n}\}$ $(n = 0, 1, \ldots)$ is a base for some uniformity $\mathbf{W}^* \subseteq \mathbf{W}$ (this is easily done by finite induction). We shall use this remark later.

We can now strengthen a previous result **(8**, 2.2**)** as follows:

4.3. Theorem. *A μ-essentially separable space $(T, \mathbf{V})$ is μ-essentially pseudometrizable if and only if it is μ-essentially nested, and then $\mathbf{V}$ is μ-essentially* *equivalent to a countably generated uniformity* $\mathbf{V}^* \subseteq \mathbf{V}$ *such that* $(T - Z, \mathbf{V}^*)$ *is separable for some null set* Z.

Remark. In **(8)**, this was proved only for σ-finite essentially topological measures μ.

Proof. The necessity of the essential nestedness condition is obvious. To prove its sufficiency, suppose that $\mathbf{V}$ is essentially equivalent to a nested $\mathbf{U}$ and also to some separable $\mathbf{W}$, so that there is a (fixed) countable set $F = \{p_1, p_2, \ldots\} \subseteq T$, with $T = W[F]$ for all $W \in \mathbf{W}$. As before, we assume that $\mathbf{U}$ is not countably generated, and thus has a well-ordered base $\mathbf{U}'$ satisfying (4.1(a)). The essential equivalence of $\mathbf{W}$ and $\mathbf{U}$ yields, for each $U \in \mathbf{U}'$, a $W^U \in \mathbf{W}$ such that $\mu^*(W_x{}^U - U_x) = 0$ for all $x \in T$. We fix these W^U and also note, from $T = W[F]$, that $Z_U = T - U[F]$ is a null set for all $U \in \mathbf{U}'$, where $U[F]$ is a *disjoint* union of all $U[p_n]$ $(p_n \in F)$, $n = 1, 2, \ldots$. (We may safely assume this. For if p_n and p_m satisfy $U[p_m] = U[p_n]$ for all $U \in \mathbf{U}'$, we simply drop p_m or p_n from F. Thus suppose that, whenever $n \neq m$, there is $U = U^{mn} \in \mathbf{U}'$ such that $U[p_n] \neq U[p_m]$ and hence $U[p_n] \cap U[p_m] = \emptyset$, by 4.1(a). Then all such U^{mn} form a countable subset of $\mathbf{U}'$, *not cofinal* with $\mathbf{U}'$ (for $\mathbf{U}$ is not countably generated). Thus there is a $U^\circ \in \mathbf{U}'$ with $U^\circ \subseteq \bigcap_{m,n} U^{mn}$, Dropping from $\mathbf{U}'$ all $U \supset U^\circ$, we are left with a base satisfying the desired disjointness condition. We may then assume that U° is the largest member of $\mathbf{U}'$ from the outset.)

Thus, if $U \supseteq U'$ $(U, U' \in \mathbf{U}')$, we have

$$T = \bigcup_{n=1}^{\infty} U[p_n] \cup Z_U = \bigcup_{n=1}^{\infty} U'[p_n] \cup Z_{U'},$$

with all terms disjoint, and with $Z_U \subseteq Z_{U'}$ and $U[p_n] \supseteq U'[p_n]$. This clearly implies that $\mu^*(U[p_n] - U'[p_n]) = 0$, $n = 1, 2, \ldots$, and the latter is also trivially true if $U \subseteq U'$; thus it holds for *all* $U, U' \in \mathbf{U}'$. Moreover, for any $x = p_n \in F$ and any $U \in \mathbf{U}'$, the sets U_x and $W_x{}^U$ differ by a null set at most, as easily follows from the formulae

$$T - Z_U = W[F] - Z_U = U[F] - Z_U \text{ and } W_x{}^U - Z \subseteq U_x - Z \quad (\mu^* Z = 0),$$

on noting that the sets $W_x{}^U - Z$ are disjoint, as are the U_x $(x \in F)$. Thus we obtain, for all $U, U' \in \mathbf{U}'$ (replacing x by p_n):

$$\text{(4.3.1)} \qquad \mu^*(U[p_n] - U'[p_n] = \mu^*(W^U[p_n] - W^{U'}[p_n]) = 0, \quad n = 1, 2, \ldots .$$

As has been noted above, we may assume that the well-ordered base $\mathbf{U}'$ contains a sequence $\{U^n\}$, $n = 0, 1, 2 \ldots$, such that the corresponding entourages W^{U^n} (selected above) form a base for a countably generated uniformity $\mathbf{W} \subseteq \mathbf{W}$. The fact that $\mathbf{W}$ is also essentially coarser than $\mathbf{W}^*$ (hence essentially pseudometrizable) can now be established as in the original proof of **(8**, 2.2**)**, namely by using the following formula (proved in **(8)** with purely notational changes) in which, for brevity, we write W^n for W^{U^n}:

$$(4.3.2) \qquad \bigcup_{x \in T} (W_x^{\,1} - W_x^{\,U}) \subseteq \bigcup_{n=1}^{\infty} (\mathbf{W}^0[p_n] - W^{U'}[p_n]), \qquad U \in U',$$

where $U' \in \mathbf{U}'$ is such that $U'\,U' \subseteq U$. (It is assumed here that such a U' has been fixed in advance, for each $U \in \mathbf{U}'$.) Indeed, (4.3.2) combined with (4.3.1) yields $\mu^*(W_x{}^1 - W_x{}^U) = 0$ for all $x \in T$ and $U \in \mathbf{U}'$. By the choice of the W^U, we also have $\mu^*(W_x{}^U - U_x) = 0$ and hence $\mu^*(W_x{}^1 - U_x) = 0$ $(x \in T,\ U \in \mathbf{U}')$. This shows that $\mathbf{W}^*$ is essentially finer than $\mathbf{U}$, hence also essentially finer than the essentially equivalent uniformities $\mathbf{W}$ and $\mathbf{V}$. Thus the essentially pseudometrizability of both is proved, and the first part of the theorem is established. Its second part now easily follows by 3.2 and by using the same procedure as in the last part of 3.5.

Note 1. Theorems 4.2 and 4.3 and their proofs remain valid also if essential equivalence is interpreted in the stronger sense.

Note 2. We can now also slightly strengthen our theorem (**8**, 3.2) by dropping the assumption (contained in the last part of that theorem) as to the measurability of the function f. Indeed, this assumption was only used to show that the measure defined in T is topological. This is, however, no longer required if we replace (**8**, Theorem 3.2) by our present theorem 4.3.

Final remarks. The theorems proved above do not exhaust the subject and are intended only as illustrations of the fact that non-trivial essential metrization theorems can be proved for various types of spaces and measures. Many more such theorems can be derived. Some applications were given in (**8**, §§ 3, 4). Our present theorems entail that these applications remain valid also for Lindelöf spaces and compact spaces of the kind considered in the present paper. The general idea is to extend various propositions, valid in metric spaces, to non-metrizable but "essentially (pseudo)metrizable" spaces, with ordinary topological concepts replaced by their "essential" counterparts. Without going into further details, we conjecture that the theory of essential metrization can be extended to non-uniform spaces and that then, with appropriate definitions, Baire's theory of category and other concepts (e.g., metric density, etc.) can be extended to essentially pseudometrizable spaces. In conclusion, we wish to acknowledge our indebtedness to the referee for his suggestions as to a shorter formulation of our proofs.

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