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IMPROVED UPPER BOUNDS FOR ODD MULTIPERFECT NUMBERS

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Abstract

In this paper, we prove that, if N is a positive odd number with r distinct prime factors such that $N \mid \sigma(N)$, then $N < 2^{4^r - 2^r}$ and $N\prod_{p|N} p < 2^{4^r}$, where $\sigma(N)$ is the sum of all positive divisors of N. In particular, these bounds hold if N is an odd perfect number.

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1. Introduction

For a positive integer n, let $\sigma(n)$ be the sum of all positive divisors of n. A positive integer N is called a *u/v-perfect number* if $\sigma(N) = uN/v$, where u and v are two integers with $u > v \geq 1$. For an integer $k \geq 2$, a k-perfect number is also called a *multiperfect number*. A 2-perfect number is called a *perfect number*. It is well known that Euler proved that an even perfect number can be written as $2^{p-1}(2^p - 1)$, where both p and $2^p - 1$ are primes. The following is a long-standing problem: Is there any odd perfect number?

Suppose that N is an odd perfect number. In 2007, Nielsen [8] proved that N has at least nine distinct prime factors. Recently, Ochem and Rao [9] proved that N is greater than 10^{1500}. In 2008, Goto and Ohno [4] proved that N has a prime factor exceeding 10^8. In 1913, Dickson [2] proved that there are only finitely many odd perfect numbers with r distinct prime factors. Pomerance [10] gave an explicit upper bound. Heath-Brown [5] proved that $N < 4^{4^r}$, and in 2003 Nielsen [7] improved this bound to $N < 2^{4^r}$. Luca and Pomerance [6] proved that the radical $\prod_{p|N} p$ of N is less than $2N^{17/26}$. Dris and Luca [3] proved that, for any odd perfect number N and $q^\alpha \,\|\, N$, where q is a prime, we have $\sigma(N/q^\alpha)/q^\alpha \geq 6$. Recently, Chen and the first author [1] improved this result by proving that $\sigma(N/q^\alpha)/q^\alpha \neq p_1, p_1^2, p_1^3, p_1^4, p_1p_2, p_1^2p_2$, where p_1, p_2 are distinct primes.

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In this note, we prove the following results.

THEOREM 1.1. *If N is a positive odd number with r distinct prime factors such that $N \mid \sigma(N)$, then*

$$N < 2^{4^r - 2^r}, \quad N \prod_{p \mid N} p < 2^{4^r}.$$

From Theorem 1.1, we have the following corollary.

COROLLARY 1.2. *If N is an odd perfect number with r distinct prime factors, then*

$$N < 2^{4^r - 2^r}, \quad N \prod_{p \mid N} p < 2^{4^r}.$$

2. Proof of the theorem

We will follow the proofs of Heath-Brown [5] and Nielsen [7]. For any positive integer m, let $\omega(m)$ denote the number of distinct prime factors of m. For a set S of integers, let $\prod(S) = \prod_{s \in S} s$. By convention, $\prod_{s \in \emptyset} f(s) = 1$ for any arithmetic function f. We will prove the following stronger result.

THEOREM 2.1. *If u and v are two positive integers with $u > v$ and N is an odd u/v-perfect number with r distinct prime factors, then*

$$N < (v+1)^{4^r - 2^r}, \quad vN \prod_{p \mid N} p < (v+1)^{4^r}.$$

If $N \mid \sigma(N)$ and $N > 1$, then $\sigma(N) = uN$ for an integer u with $u > 1$. Now Theorem 1.1 follows from Theorem 2.1 with $v = 1$.

LEMMA 2.2 [7, Lemma 1]. *Let r, a, b be positive integers and let $x_1, \ldots, x_r$ be integers with $1 < x_1 < \cdots < x_r$. If*

$$\prod_{i=1}^{r} \Big(1 - \frac{1}{x_i}\Big) \le \frac{a}{b} < \prod_{i=1}^{r-1} \Big(1 - \frac{1}{x_i}\Big),$$

then

$$a \prod_{i=1}^{r} x_i < (a+1)^{2^r}.$$

LEMMA 2.3. *If u and v are two positive integers with $u > v$ and N is an odd u/v-perfect number with r distinct prime factors, then*

$$N < \Big(v \prod_{p \mid N} p\Big)^{2^r - 1}.$$

The proof of Lemma 2.3 is similar to that of [7, Proposition 1].

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Lemma 2.4. *Let r, a, b be positive integers and let $x_1, \ldots, x_r$ be integers with $1 < x_1 < \cdots < x_r$. If*

$$\prod_{i=1}^{r}\Big(1-\frac{1}{x_i}\Big) \le \frac{a}{b},$$

then there exists an integer s with $0 \le s \le r$ such that

$$\prod_{i=1}^{s}\Big(1-\frac{1}{x_i}\Big) \le \frac{a}{b}, \quad a\prod_{i=1}^{s} x_i < (a+1)^{2^s}.$$

In particular, if $a < b$, then $s \ge 1$.

Proof. If $a \ge b$, then Lemma 2.4 is true for $s = 0$. Now we assume that $a < b$. Let $y_0 = 1$ and

$$y_j = \prod_{i=1}^{j}\Big(1-\frac{1}{x_i}\Big), \quad j = 1, \ldots, r.$$

Then

$$y_r < y_{r-1} < \cdots < y_1 < y_0 = 1.$$

Since $y_r \le a/b < y_0$, it follows that there exists an integer s with $1 \le s \le r$ such that $y_s \le a/b < y_{s-1}$. Now Lemma 2.4 follows from Lemma 2.2. □

Lemma 2.5. *Let u and v be two positive integers with $u > v$, N an odd u/v-perfect number, and S a set (possibly empty) of prime factors of N. Then there exist two coprime integers U and V with $U \ge 1$ and $V > 1$, and a set T (possibly empty) of prime factors of U with $|T| + \omega(V) \ge |S|$ such that $N = UV$ and*

$$(v_1+1)\prod_{p\mid V} p \prod(T) < \Big((v+1)\prod(S)\Big)^{2^{2\omega(V)+|T|-|S|}},$$

where $v_1 = v\sigma(V)$.

Proof. Let $N = \prod_{p\mid N} p^{e(p)}$ be the standard factorisation of n. Then

$$\frac{uN}{v} = \sigma(N) = \prod_{p\mid N}\frac{p^{e(p)+1}-1}{p-1} < \prod_{p\mid N}\frac{p^{e(p)+1}}{p-1} = N\prod_{p\mid N}\Big(1-\frac{1}{p}\Big)^{-1}.$$

It follows that

$$\prod_{p\mid N}\Big(1-\frac{1}{p}\Big) < \frac{v}{u}.$$

Let $u' = u\prod_{p\in S}(p-1)$ and $v' = v\prod(S)$. Then

$$\prod_{p\mid N, p\notin S}\Big(1-\frac{1}{p}\Big) < \frac{v'}{u'}.$$

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By Lemma 2.4, there exists a subset S' of $\{p : p \mid N, p \notin S\}$ such that

$$\prod_{p\in S'}\Big(1-\frac{1}{p}\Big)\le \frac{v'}{u'},\quad v'\prod(S')<(v'+1)^{2^{|S'|}}\le\Big((v+1)\prod(S)\Big)^{2^{|S'|}}.$$

That is,

$$\prod_{p\in S\cup S'}\Big(1-\frac{1}{p}\Big)\le \frac{v}{u}<1,\quad v\prod(S\cup S')<\Big((v+1)\prod(S)\Big)^{2^{|S'|}}. \tag{2.1}$$

Since the numerator of

$$\prod_{p\in S\cup S'}\Big(1-\frac{1}{p}\Big)$$

is even and the numerator of $v/u=N/\sigma(N)$ is odd when it is written in the lowest terms, it follows that equality in (2.1) cannot hold.

Let $u''=u\prod_{p\in S\cup S'}(p-1)$ and $v''=v\prod(S\cup S')$. By (2.1), $v''>u''$. Since

$$\prod_{p\in S\cup S'}\Big(1-\frac{1}{p^{e(p)+1}}\Big)=\prod_{p\in S\cup S'}\frac{\sigma(p^{e(p)})(p-1)}{p^{e(p)+1}}\le\frac{\sigma(N)}{N}\prod_{p\in S\cup S'}\frac{p-1}{p}=\frac{u''}{v''}<1,$$

it follows from Lemma 2.4 that there exists a nonempty subset $S''\subseteq S\cup S'$ with

$$\prod_{p\in S''}\Big(1-\frac{1}{p^{e(p)+1}}\Big)\le\frac{u''}{v''},\quad u''\prod_{p\in S''}p^{e(p)+1}\le(u''+1)^{2^{|S''|}}\le(v'')^{2^{|S''|}}.$$

As in Heath-Brown's paper, let

$$V=\prod_{p\in S''}p^{e(p)},\quad U=N/V,\quad T=(S\cup S')\setminus S'',\quad v_1=v\sigma(V).$$

Then $|S''|=\omega(V)\ge 1$ and $|S'|=|S''|+|T|-|S|=\omega(V)+|T|-|S|\ge 0$. Since

$$\begin{aligned}
v_1+1\le 2v_1=2v\prod_{p\in S''}\frac{p^{e(p)+1}-1}{p-1}&\le v\prod_{p\in S''}(p^{e(p)+1}-1)\\
&=v\prod_{p\in S''}\Big(1-\frac{1}{p^{e(p)+1}}\Big)\prod_{p\in S''}p^{e(p)+1}\le v\frac{u''}{v''}\frac{1}{u''}(v'')^{2^{|S''|}}\\
&=v(v'')^{2^{|S''|}-1}\\
&=v^{2^{|S''|}}\Big(\prod(S\cup S')\Big)^{2^{|S''|}-1}
\end{aligned}$$

and

$$\prod_{p\mid V}p\prod(T)=\prod(S'')\prod(T)=\prod(S\cup S'),$$

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it follows that

$$(v_1+1)\prod_{p\mid V} p \prod(T) \le \Big(v\prod(S\cup S')\Big)^{2^{|S''|}} < \Big((v+1)\prod(S)\Big)^{2^{|S'|+|S''|}}$$
$$= \Big((v+1)\prod(S)\Big)^{2^{2\omega(V)+|T|-|S|}}.$$

This completes the proof of Lemma 2.5. □

Proof of Theorem 2.1. Let $n_0 = N$, $u_0 = u$, $v_0 = v$, and $S_0 = S = \emptyset$. Let U, V and T be as in Lemma 2.5. Let $n_1 = U$, $V_1 = V$, $u_1 = uV = u_0V_1$, $v_1 = v\sigma(V) = v_0\sigma(V_1)$ and $S_1 = T$. Then $n_0 = n_1V_1$,

$$\frac{\sigma(n_1)}{n_1} = \frac{\sigma(N)}{N}\frac{V_1}{\sigma(V_1)} = \frac{u_0V_1}{v_0\sigma(V_1)} = \frac{u_1}{v_1},$$

and

$$(v_1+1)\prod_{p\mid V_1} p \prod(S_1) \le \Big((v_0+1)\prod(S_0)\Big)^{2^{2\omega(V_1)+|S_1|-|S_0|}}.$$

If $n_1 > 1$, then $u_1 > v_1$. We continue to apply Lemma 2.5. Since $V_1 > 1$, it follows that $n_1 < n_0 = N$. So this procedure must stop in a finite number of steps. Thus, we can obtain sequences $\{n_i\}_{i=0}^t$, $\{V_i\}_{i=1}^t$, $\{u_i\}_{i=0}^t$, $\{v_i\}_{i=0}^t$, and $\{S_i\}_{i=0}^t$ such that $n_t = 1$, $\sigma(n_i) = u_in_i/v_i$, S_i is a set (possibly empty) of prime factors of n_i $(0 \le i \le t)$ and, for all $0 \le i \le t-1$,

$$n_i = n_{i+1}V_{i+1}, \quad (n_{i+1}, V_{i+1}) = 1, \quad V_{i+1} > 1, \tag{2.2}$$
$$u_{i+1} = u_iV_{i+1}, \quad v_{i+1} = v_i\sigma(V_{i+1}), \quad \omega(V_{i+1}) + |S_{i+1}| \ge |S_i|, \tag{2.3}$$
$$(v_{i+1}+1)\prod_{p\mid V_{i+1}} p \prod(S_{i+1}) \le \Big((v_i+1)\prod(S_i)\Big)^{2^{k_{i+1}}}, \tag{2.4}$$

where $k_i = 2\omega(V_i) + |S_i| - |S_{i-1}|$ $(1 \le i \le t)$. It follows from $n_t = 1$ that $S_t = \emptyset$. By (2.2) and (2.3),

$$2\omega(V_{i+1}) + |S_{i+1}| - |S_i| \ge 1, \quad 0 \le i \le t-1,$$
$$N = n_1V_1 = \cdots = V_t\cdots V_1,$$
$$v_t = v_{t-1}\sigma(V_t) = v_{t-2}\sigma(V_{t-1})\sigma(V_t) = \cdots = v_0\sigma(V_1)\cdots\sigma(V_t).$$

So $v_t = v_0\sigma(N) = v_0uN/v = uN$. Let $P_0 = 1$ and

$$P_i = \prod_{p\mid V_1\cdots V_i} p, \quad 1 \le i \le t.$$

Thus, by (2.4),

$$(v_{i+1}+1)P_{i+1}\prod(S_{i+1}) \le \Big((v_i+1)P_i\prod(S_i)\Big)^{2^{k_{i+1}}}, \quad 0 \le i \le t-1.$$

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It follows that

$$\begin{aligned} vN\prod_{p|N} p = vNP_t < uNP_t = v_tP_t < (v_t+1)P_t\prod(S_t) &\\ \le \Big((v_{t-1}+1)P_{t-1}\prod(S_{t-1})\Big)^{2^{k_t}} &\\ \le \cdots &\\ \le \Big((v_0+1)P_0\prod(S_0)\Big)^{2^{k_t+\cdots+k_1}}. & \end{aligned}$$

Noting that $(v_0+1)P_0\prod(S_0) = v+1$ and

$$k_t + \cdots + k_1 = \sum_{i=1}^{t}(2\omega(V_i) + |S_i| - |S_{i-1}|) = 2\omega(N) + |S_t| - |S_0| = 2\omega(N),$$

we have

$$vN\prod_{p|N} p < (v+1)^{4^{\omega(N)}}. \tag{2.5}$$

If $v\prod_{p|N} p \ge (v+1)^{2^{\omega(N)}}$, then, by (2.5),

$$N < (v+1)^{4^{\omega(N)}-2^{\omega(N)}}.$$

If $v\prod_{p|N} p < (v+1)^{2^{\omega(N)}}$, then, by Lemma 2.3,

$$N < \Big(v\prod_{p|N} p\Big)^{2^{\omega(N)}-1} < (v+1)^{4^{\omega(N)}-2^{\omega(N)}}.$$

This completes the proof. □

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