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ON POWER MOMENTS OF THE HECKE MULTIPLICATIVE FUNCTIONS

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Abstract

In a recent paper, Soundararajan has proved the quantum unique ergodicity conjecture by getting a suitable estimate for the second order moment of the so-called 'Hecke multiplicative' functions. In the process of proving this he has developed many beautiful ideas. In this paper we generalize his arguments to a general kth power and provide an analogous estimate for the kth power moment of the Hecke multiplicative functions. This may be of general interest.

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1. Introduction

Let f be an arithmetical function which satisfies the Hecke multiplication, that is,

$$f(m)f(n) = \sum_{d|(m,n)} f\left(\frac{mn}{d^2}\right) \quad \text{and} \quad f(1) = 1 \quad (m, n \in \mathbb{N}). \tag{1.1}$$

A typical example is a Fourier coefficient of a Maaß cusp form with respect to $SL_2(\mathbb{Z})$. To prove the quantum unique ergodicity for the modular surface $SL_2(\mathbb{Z})\backslash\mathbb{H}$, Soundararajan [2] proved the following inequality.

THEOREM 1.1 (Soundararajan [2, page 1531, Theorem 3]). *For the above function f and $1 \leq y \leq x$,*

$$\sum_{n\leq x/y} |f(n)|^2 \leq 10^8\left(\frac{1+\log y}{\sqrt{y}}\right)\sum_{n\leq x} |f(n)|^2. \tag{1.2}$$

This is the main tool used by Soundararajan to prove the quantum unique ergodicity result. He further suggested that one can obtain an analogue of (1.2) with $|f(n)|$ in place of $|f(n)|^2$. However, in this paper we shall obtain an inequality in the case of the kth power, that is, for $|f(n)|^k$, which may be of independent interest.

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THEOREM 1.2. *Fix $k \geq 1$. Let f be an arithmetical function which satisfies the Hecke multiplication property and $1 \leq y \leq x$; we obtain*

$$\sum_{n \leq x/y} |f(n)|^k \leq 10^{s(k)} \Big(\frac{1 + \log y}{\sqrt{y}} \Big) \sum_{n \leq x} |f(n)|^k,$$

where

$$s(k) = \Big[\frac{30\,103}{100\,000} (6k + 8) \Big] + 2.$$

This estimate may be of more interest from the viewpoint of Maaß forms. The proof of this theorem is based on many beautiful ideas due to Soundararajan and the Hölder inequality.

2. Lemmas

To prove Theorem 1.2, we shall generalize Lemma 3.1 and Propositions 3.2 and 3.3 in [2, pages 1533–1536]. Let f be a function satisfying (1.1) and $k, y, x \geq 1$ (with $1 \leq y \leq x$). We define a decreasing function $\mathcal{F}_k(y)$ by

$$\mathcal{F}_k(y) = \mathcal{F}_k(y; x) := \frac{\sum_{n \leq x/y} |f(n)|^k}{\sum_{n \leq x} |f(n)|^k}.$$

We begin with the following result.

LEMMA 2.1 (cf. [2, page 1533, Lemma 3.1]).

(1) *For any positive integer $m \leq x$,*

$$|f(m)| \leq \frac{\tau(m)}{\mathcal{F}_k(m)^{1/k}},$$

where $\tau(m) = \sum_{d|m} 1$.

(2) *For any prime $p \leq \sqrt{x}$,*

$$|f(p)| \leq \frac{2}{\mathcal{F}_k(p^2)^{1/2k}}.$$

(3) *For any prime $p \leq \sqrt{y}$,*

$$|f(p)| \leq \frac{2}{\mathcal{F}_k(y)^{1/2k}}.$$

PROOF. By the Hecke multiplication (1.1) and the Hölder inequality,

$$|f(m)f(n)| \leq \Big(\sum_{d|m} |f(mn/d^2)|^k \Big)^{1/k} \Big(\sum_{d|m} 1^{k/(k-1)} \Big)^{(k-1)/k}$$

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for $k > 1$. Then[1]

$$|f(m)f(n)|^k \le \tau^{k-1}(m) \sum_{d|m} |f(mn/d^2)|^k$$

for $k \ge 1$.

Applying this inequality to the left-hand side of the equation,

$$|f(m)|^k \sum_{n \le x/m} |f(n)|^k = |f(m)|^k \mathcal{F}_k(m) \sum_{n \le x} |f(n)|^k$$

and we obtain the first assertion.

By (1.1), $f^2(p) = \sum_{d|(p,p)} f(p^2/d^2) = f(p^2) + f(1) = f(p^2) + 1$; then we see that $|f(p)|^2 \le |f(p^2)| + 1$. Together with the above result, one gets the second assertion.

Since $\mathcal{F}_k(p^2) \ge \mathcal{F}_k(y)$ for any prime $p \le \sqrt{y}$, the second assertion leads to the third. □

The next lemma is as follows.

Lemma 2.2 (cf. [2, pages 1533–1534, Proposition 3.2]). *Let d be square free. Then:*

(1) $\sum_{\substack{n \le x/y \\ d|n}} |f(n)|^k \le \tau^{k-1}(d) \prod_{p|d}(1 + |f(p)|^k)\mathcal{F}_k(yd) \sum_{n \le x} |f(n)|^k$;

(2) $\sum_{\substack{n \le x/y \\ d^2|n}} |f(n)|^k \le \tau_3^{k-1}(d) \prod_{p|d}(2 + |f(p^2)|^k)\mathcal{F}_k(yd^2) \sum_{n \le x} |f(n)|^k$,

where τ denotes the divisor function and τ_3 the three divisor function $\tau_3(n) = \sum_{abc=n, a,b,c \in \mathbb{N}} 1$.

Proof. The inequality $|f(md)| \le \sum_{ab=d} |f(a)|\,|f(m/d)|$ is shown in [2, page 1534]. Now, by the Hölder inequality,

$$|f(md)|^k \le \tau^{k-1}(d) \sum_{ab=d} |f(a)|^k |f(m/b)|^k.$$

Hence, by a similar argument to [2, page 1534], we obtain the first assertion.

The second formula is shown by the inequality

$$|f(md^2)|^k \le \tau_3^{k-1}(d) \sum_{abc=d} |f(a^2)|^k |f(m/c^2)|^k$$

and by an analogous method as the above one. □

In order to state the final lemma, we introduce some notation (corresponding to [2, pages 1534–1535]). Let $\mathcal{P}(= \mathcal{P}(y))$ denote the set of primes p satisfying $\sqrt{y}/2 \le p \le \sqrt{y}$. By Dusart's result [1], it is known that

$$|\mathcal{P}| \ge \frac{\sqrt{y}}{2 \log y} \quad \text{for } y \ge 10^{16}. \tag{2.1}$$

[1] If q is not an integer, we regard $f(q) = 0$.

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We shall divide $\mathcal{P}$ into $J+1$ disjoint subsets by

$$\mathcal{P}_0 = \{p \in \mathcal{P} \mid |f(p)| \le 1/2\} \quad \text{and} \quad \mathcal{P}_j = \{p \in \mathcal{P} \mid 2^{j-2} < |f(p)| \le 2^{j-1}\}$$

($j = 1, 2, \ldots, J$). Further, using Lemma 2.1(3), we define the number J by

$$J = \left[\frac{1}{2k\log 2}\log\frac{1}{\mathcal{F}_k(y)}\right] + 3 \quad \text{(cf. [2, page 1535, line 3]).}$$

Moreover, we define $\mathcal{N}_0(l)$ with having elements which are positive integers n and which are of the form $n = p_1^2 \cdots p_r^2 n'$ ($r \le l$), where $p_1, \ldots, p_r$ are distinct primes in $\mathcal{P}_0$. Similarly, we define $\mathcal{N}_j(l)$ ($1 \le j \le J$) by elements which are positive integers of the form $p_1 \cdots p_r n'$ ($r \le l$), where $p_1, \ldots, p_r$ are distinct primes in $\mathcal{P}_j$.

The final lemma is the following assertion.

Lemma 2.3 (cf. [2, page 1535, Proposition 3.3]). *Let us use the above notation and let l be a positive integer. Then*

(1) *For $2 \le l \le |\mathcal{P}_0|/4$,*

$$\sum_{\substack{n \le x/y \\ n \in \mathcal{N}_0(l)}} |f(n)|^k \le \left(\frac{4}{3}\right)^{k+1} \frac{l+1}{|\mathcal{P}_0|} \sum_{n \le x} |f(n)|^k.$$

(2) *For $1 \le l \le |\mathcal{P}_j|/4 - 1$ ($1 \le j \le J$),*

$$\sum_{\substack{n \le x/y \\ n \in \mathcal{N}_j(l)}} |f(n)|^k \le \frac{2^{4(k+1)} l^2}{2^{2kj} |\mathcal{P}_j|^2} \sum_{n \le x} |f(n)|^k.$$

Proof. To show the first assertion, we shall choose a prime $p \in \mathcal{P}_0$. We note that $|f(p)| \le 1/2$. Using (1.1), we see that $f(p^2) = f^2(p) - 1$ and $|f(p^2)| = |f^2(p) - 1| \ge 1 - 1/4 = 3/4$. Thus,

$$\begin{aligned} \sum_{\substack{n \le x/y \\ n \in \mathcal{N}_0(l)}} |f(n)|^k \Big(\sum_{\substack{p \in \mathcal{P}_0 \\ p^2 \nmid n}} |f(p^2)|^k\Big) &\ge \left(\frac{3}{4}\right)^k \sum_{\substack{n \le x/y \\ n \in \mathcal{N}_0(l)}} |f(n)|^k (|\mathcal{P}_0| - l) \\ &\ge \left(\frac{3}{4}\right)^{k+1} |\mathcal{P}_0| \sum_{\substack{n \le x/y \\ n \in \mathcal{N}_0(l)}} |f(n)|^k. \end{aligned} \tag{2.2}$$

Now we remark that $|f(n)f(p^2)| \le |f(np^2)|$ for $p \in \mathcal{P}_0$ and $p^2 \nmid n$ (see [2, page 1535]).

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By this relation, we can obtain an upper bound of the left-hand side of (2.2):

$$\begin{aligned}\text{The left-hand side of (2.2)} &= \sum_{\substack{n\le x/y\\ n\in\mathcal{N}_0(l)}}\Big(\sum_{\substack{p\in\mathcal{P}_0\\ p^2\nmid n}}|f(n)|^k|f(p^2)|^k\Big)\\ &\le \sum_{\substack{n\le x/y\\ n\in\mathcal{N}_0(l)}}\Big(\sum_{\substack{p\in\mathcal{P}_0\\ p^2\nmid n}}|f(np^2)|^k\Big)\\ &\le \sum_{m\le x}|f(m)|^k\Big(\sum_{\substack{m=np^2,\ p\in\mathcal{P}_0\\ p^2\nmid n, n\le x/y,\ n\in\mathcal{N}_0(l)}}1\Big)\\ &\le (l+1)\sum_{m\le x}|f(m)|^k. \qquad (2.3)\end{aligned}$$

From (2.2) and (2.3), we can obtain the first inequality.

Noting the fact that $|f(p_1p_2)| = |f(p_1)f(p_2)| > 2^{2(j-2)}$ for any two distinct primes p_1 and $p_2 \in \mathcal{P}_j$,

$$\begin{aligned}&\sum_{\substack{n\le x/y\\ n\in\mathcal{N}_j(l)}}|f(n)|^k\Big(\sum_{\substack{p_1,\ p_2\in\mathcal{P}_j\\ p_1<p_2, p_i\nmid n(i=1,2)}}|f(n)|^k\Big)\\ &\quad> 2^{2k(j-2)}\sum_{\substack{n\le x/y\\ n\in\mathcal{N}_j(l)}}\Big(\sum_{\substack{p_1,\ p_2\in\mathcal{P}_j\\ p_1<p_2,\ p_i\nmid n(i=1,2)}}1\Big)\ge 2^{2k(j-2)}\sum_{\substack{n\le x/y\\ n\in\mathcal{N}_j(l)}}|f(n)|^k\binom{|\mathcal{P}_j|-l}{2}\\ &\quad\ge 2^{2k(j-2)-5}\cdot 9\sum_{\substack{n\le x/y\\ n\in\mathcal{N}_j(l)}}|f(n)|^k. \qquad (2.4)\end{aligned}$$

On the other hand, since $(n, p_1p_2) = 1$ for the above n, p_1 and p_2,

$$\begin{aligned}\text{The left-hand side of (2.4)} &\le \sum_{m\le x}|f(m)|^k\Big(\sum_{\substack{m=np_1p_2\in\mathcal{N}_j(l+2),\ n\in\mathcal{N}_j(l)\\ p_1<p_2,\ p_i\nmid n(i=1,2)}}1\Big)\\ &\le \binom{l+2}{2}\sum_{m\le x}|f(m)|^k < 3l^2\sum_{m\le x}|f(m)|^k.\end{aligned}$$

We have the second inequality of the lemma by using these inequalities. □

3. Proof of Theorem 1.2

In this final section, following [2], we shall prove Theorem 1.2. Let $M > 2$ be a positive constant. By Dusart's result (2.1), there are two cases $|\mathcal{P}_0| \ge \sqrt{y}/(M\log y)$ or $|\mathcal{P}_j| \ge (M-2)\sqrt{y}/(MJ\log y)$ (for some j $(1 \le j \le J)$). We choose a 'maximal element' y $(\ge 10^{16})$ in the set

$$E := \Big\{y\,\Big|\,\mathcal{F}_k(y) \ge 10^{s(k)}\frac{1+\log y}{\sqrt{y}}\Big\}$$

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satisfying any $y' \,(\geq y+1) \notin E$. Our aim is to show that E is empty by considering the above two cases. Here we shall take $M = 2(4/3)^{k+1}$.

(I) Suppose that $|\mathcal{P}_0| \geq \sqrt{y}/(M\log y)$. We shall put $L = [|\mathcal{P}_0|\mathcal{F}_k(y)/(2M)]$. Since $\mathcal{F}_k(y) \leq 1$, we have $L \leq |\mathcal{P}_0|/4$. Moreover, noting that $s(k) > k+3$, we see that $L \geq 2$. Then we shall use Lemma 2.3(1) and get

$$\sum_{\substack{n\leq x/y\\ n\in\mathcal{N}_0(L)}} |f(n)|^k \leq \frac{1}{2}\mathcal{F}_k(y)\sum_{n\leq x}|f(n)|^k.$$

By this inequality,

$$\sum_{\substack{n\leq x/y\\ n\notin\mathcal{N}_0(L)}} |f(n)|^k \geq \frac{1}{2}\mathcal{F}_k(y)\sum_{n\leq x}|f(n)|^k. \tag{3.1}$$

On the other hand, observing that $n \notin \mathcal{N}_0(L)$ means that n is divisible by at least $L+1$ squares of primes in $\mathcal{P}_0$, we write $n = d^2n'$, $d = p_1\cdots p_{L+1}$ $(p_i \in \mathcal{P}_0)$. By Lemma 2.2(2),

$$\sum_{\substack{n\leq x/y\\ n\notin\mathcal{N}_0(L)}} |f(n)|^k \leq \left(\frac{2^5}{3(2\cdot 5)^{s(k)}}\right) < \frac{\mathcal{F}_k(y)}{10^3}\sum_{n\leq x}|f(n)|^k.$$

(Details are similar to [2].) It contradicts (3.1). Hence, the set E is empty.

(II) Suppose that $|\mathcal{P}_j| \geq (M-2)\sqrt{y}/(2MJ\log y)$ for a j $(1 \leq j \leq J)$. We observe that

$$J < \frac{1}{4k\log 2}\log y - \frac{10^{s(k)}\log 10}{2k\log 2} - \frac{1+\log y}{2k\log 2} + 3 < \frac{\log y}{2k}.$$

By this upper bound of J,

$$|\mathcal{P}_j| \geq k\left(1-\frac{2}{M}\right)\frac{\sqrt{y}}{(\log y)^2} > 4k\cdot 10^5.$$

Now we choose $L = [2^{kj-3(k+1)}|\mathcal{P}_j|\mathcal{F}_k^{1/2}(y)]$ in this case. Since $|\mathcal{P}_j| \geq 8$ and $\mathcal{F}_k^{1/2}(y) \leq 2^{k(3-j)}$, we have $L \leq |\mathcal{P}_j|/4 - 1$. Also, we see easily that $L \geq 1$. By Lemma 2.3(2),

$$\sum_{\substack{n\leq x/y\\ n\in\mathcal{N}_j(L)}} |f(n)|^k < \frac{\mathcal{F}_k(y)}{2^{2(k+1)}}\sum_{n\leq x}|f(n)|^k.$$

Thus,

$$\sum_{\substack{n\leq x/y\\ n\notin\mathcal{N}_j(L)}} |f(n)|^k > \frac{1}{2}\mathcal{F}_k(y)\sum_{n\leq x}|f(n)|^k. \tag{3.2}$$

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On the other hand, by a similar argument as in (1) and (2) of Lemma 2.2,

$$\sum_{\substack{n \le x/y \\ n \notin \mathcal{N}_j(L)}} |f(n)|^k \le \left(\frac{3^2 \cdot 2^{6k+7}}{10^{s(k)}} \right)^{(L+1)/2} \mathcal{F}_k(y) \sum_{n \le x} |f(n)|^k.$$

Now we note that if $2^{6k+7}/10^{s(k)-1} < 1/2$, then

$$\sum_{\substack{n \le x/y \\ n \notin \mathcal{N}_j(L)}} |f(n)|^k < \frac{1}{2} \mathcal{F}_k(y) \sum_{n \le x} |f(n)|^k.$$

This inequality contradicts (3.2). Since the above condition is equivalent to $s(k) > 1 + (6k + 8) \log 2 / \log 10$, we see that our choice of $s(k)$ is suitable. Therefore, we obtain that E is empty and this concludes the proof of the theorem.

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