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SOME EXAMPLES CONCERNING NORMAL AND UNIFORM NORMAL STRUCTURE IN BANACH SPACES

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Abstract

Examples are given that show the following: (1) normal structure need not be inherited by quotient spaces; (2) uniform normal structure is not a self-dual property; and (3) no degree of k-uniform rotundity need be present in a space with uniform normal structure.

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The concept of normal structure, a geometrical property of sets in normed linear spaces, was introduced by M. S. Brodskii and D. P. Milman [3] in 1948 in order to study the existence of common fixed points of certain sets of isometries; uniform normal structure was initially investigated by A. A. Gillespie and B. B. Williams [9] in 1979. Both of these notions are well-known and have been studied as purely geometrical properties and as tools in fixed point theory.

In this paper, four examples of Banach spaces are given that illustrate the permanence (or lack thereof) of normal or uniform normal structure when forming quotient spaces or dual spaces. More specifically, the first example is a renorming of l^1 that has normal structure and has the property that every separable Banach space is isometrically isomorphic to one of its quotient spaces. Consequently, quotient spaces of spaces with normal structure need not have normal structure; this answers a question of B. Sims [10, page 62]. The second example complements the first by showing that not every renorming of l^1 with normal structure has the properties of the first example.

The third example, originally studied by W. L. Bynum [4], is a renorming of l^2 that is shown to have uniform normal structure while its dual space is known to lack even normal structure. Therefore, dual spaces of spaces with uniform normal structure need not have uniform normal structure; this answers another question of B. Sims [10, page 42]. All of the examples in the literature (at least all that are known to the authors) of spaces with uniform normal structure have this property because they have the stronger geometrical property of being k-uniformly rotund for some positive integer k; this includes the third example in this paper. The fourth example given here is an example of a Banach space that has uniform normal structure and yet is not k-uniformly rotund for any positive integer k.

The terminology used in this paper is standard. Some of the basic definitions and notations are now reviewed. For a closed bounded convex subset A of a Banach space X, the *Chebyshev radius* of A, denoted by $r(A)$, is defined to be $\inf\{\sup\{\|x-y\|: y \in A\}: x \in A\}$ and the *diameter* of A, denoted by $\operatorname{diam}(A)$, is defined to be $\sup\{\|x-y\|: x, y \in A\}$. A Banach space X is said to have *normal structure* if $r(A) < \operatorname{diam}(A)$ for every nonsingleton closed bounded convex subset A of X; the space X is said to have *uniform normal structure* if there is an $\varepsilon > 0$ such that the ratio $r(A)/\operatorname{diam}(A) < 1 - \varepsilon$ for all A as above. The *modulus of rotundity* of a Banach space X is the function $\delta_X: [0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\{1 - \|(x+y)/2\|: x, y \in X, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\}$$

and the *characteristic of convexity* of X, denoted $\varepsilon_0(X)$, is defined to be $\sup\{\varepsilon \in [0, 2]: \delta_X(\varepsilon) = 0\}$. A Banach space X is *uniformly rotund* if and only if $\varepsilon_0(X) = 0$. Several generalizations of uniform rotundity are well-known. Those used in this paper include, in decreasing order of strength, weak uniform rotundity, weak* uniform rotundity (in dual spaces), uniform rotundity in every direction (a property that implies normal structure), and rotundity (also known as strict convexity); see [6, Chapter VII, §2] for detailed information. Also needed here is the generalization called k-uniform rotundity (a property that implies uniform normal structure); see [11] for basic information. The k-dimensional volume enclosed by vectors $x_1, \ldots, x_{k+1}$ is denoted by $A(x_1, \ldots, x_{k+1})$ and the affine span of these vectors is denoted by $[x_1, \ldots, x_{k+1}]$; again, see [11] for definitions.

EXAMPLE 1. $(l^1, |||\cdot|||)$. Define $S: l^1 \to l^2$ by $S(\alpha_n) = (\alpha_n 2^{-n/2})$ for (α_n) in l^1. Then S is a continuous linear mapping and hence $|||\cdot|||$ defined, for (α_n) in l^1, by

$$|||(\alpha_n)||| = [\|(\alpha_n)\|_1^2 + \|S(\alpha_n)\|_2^2]^{1/2}$$

is a norm on l^1 that is equivalent to the usual norm. Since S is an injection and since $(l^2, \|\cdot\|_2)$ is, in particular, uniformly rotund in every direction, it

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follows from [13, Proposition 14], that $(l^1, |||\cdot|||)$ is uniformly rotund in every direction and hence, by [13, Proposition 23], $(l^1, |||\cdot|||)$ has normal structure.

THEOREM 1. *Every separable Banach space is isometrically isomorphic to a quotient space of* $(l^1, |||\cdot|||)$. *Consequently, quotient spaces of Banach spaces with normal structure need not have normal structure.*

PROOF. Let X be a separable Banach space and let (x_n) be a sequence that is dense in the unit ball of X. Define $T\colon l^1 \to X$ by $T(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n x_n$ for (α_n) in l^1. Then T is a continuous linear mapping; in fact, since

$$\|T(\alpha_n)\| = \left\|\sum_{n=1}^{\infty} \alpha_n x_n\right\| \le \sum_{n=1}^{\infty} |\alpha_n| = \|(\alpha_n)\|_1 \le ||| (\alpha_n) |||,$$

it follows that $||| T ||| \le 1$. To see that T is a surjection, let z be in X with $\|z\| = 1$ and let $\varepsilon > 0$ be given. Choose n_1 in $\mathbb{N}$ such that

$$1/2^{n_1} < 2\varepsilon + 8\varepsilon^2/3 \quad \text{and} \quad \|z - x_{n_1}\| \le \varepsilon.$$

Inductively choose a subsequence (x_{n_j}) of (x_n) such that, for all j in $\mathbb{N}$,

$$\left\|z - x_{n_1} - \frac{\varepsilon}{2}x_{n_2} - \frac{\varepsilon}{2^2}x_{n_3} - \cdots - \frac{\varepsilon}{2^{j-1}}x_{n_j}\right\| \le \frac{\varepsilon}{2^{j-1}}.$$

Then let

$$y_\varepsilon = e_{n_1} + \frac{\varepsilon}{2}e_{n_2} + \frac{\varepsilon}{2^2}e_{n_3} + \cdots \quad \text{in } l^1,$$

where (e_n) denotes the standard basis in l^1. By the continuity of T, it follows that $Ty_\varepsilon = z$, and by the choice of n_1,

$$\begin{aligned} ||| y_\varepsilon |||^2 &= \|y_\varepsilon\|_1^2 + \|Sy_\varepsilon\|_2^2 \\ &= (1+\varepsilon)^2 + 2^{-n_1} + \varepsilon^2 \sum_{j=2}^{\infty} 4^{-j+1} 2^{-n_j} \\ &\le (1+\varepsilon)^2 + 2^{-n_1} + \varepsilon^2/3 \\ &< (1+2\varepsilon)^2. \end{aligned}$$

Thus T maps onto X and if $\hat{y} = y + \ker T$ in the quotient space $(l^1, |||\cdot|||)/\ker T$ where $Ty = z$, then $||| \hat{y} ||| < 1 + 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $||| \hat{y} ||| \le 1$ and since $\|z\| = 1$ and $||| T ||| \le 1$, it follows that $||| \hat{y} ||| \ge 1$. Hence $||| \hat{y} ||| = 1$. This shows that if $\hat{T}\colon (l^1, |||\cdot|||)/\ker T \to X$ is defined by $\hat{T}(y + \ker T) = Ty$, then $\hat{T}$ is a continuous linear isometry of $(l^1, |||\cdot|||)/\ker T$ onto X. This completes the proof of Theorem 1.

Of course, the proof just given is a very slight modification of the proof of the classical result due to Banach and Mazur [2, page 111] that every separable Banach space is isometrically isomorphic to a quotient space of $(l^1, \|\cdot\|_1)$.

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This same proof works since the norm $|||\cdot|||$ is a small perturbation of the norm $\|\cdot\|_1$ for elements (α_n) whose first nonzero entry occurs at a large value of n in $\mathbb{N}$; this perturbation, although small, is sufficient to guarantee that $(l^1, |||\cdot|||)$ has normal structure. The next example shows that $|||\cdot|||$ is special in that not every renorming of l^1 has the property that every separable Banach space is isometrically isomorphic to one of its quotient spaces.

EXAMPLE 2. $(l^1, \|\cdot\|_S)$. Since c_0 has a separable dual space, by [12, Theorem 2], there is an equivalent norm on c_0 that is simultaneously weakly uniformly rotund and uniformly Gateaux differentiable. Let $\langle c_0\rangle$ denote c_0 with such a norm. Then, from rotundness and smoothness duality theory (see [6, Chapter VII, §2]), it follows that its dual space, denoted $\langle l^1\rangle$, is simultaneously uniformly Gateaux differentiable and weak* uniformly rotund, and its second dual, denoted by $\langle l^\infty\rangle$, is weak* uniformly rotund and hence, in particular, is rotund. Now let Y be any closed subspace of $\langle l^1\rangle$ and consider the quotient space $\langle l^1\rangle/Y$. Since $(\langle l^1\rangle/Y)^* \cong Y^\perp$, which is a closed subspace of $\langle l^\infty\rangle$, it follows that $\langle l^1\rangle/Y$ is smooth. Let $\|\cdot\|_S$ denote the norm on $\langle l^1\rangle$. Then $(l^1, \|\cdot\|_S)$ has the property that each of its quotient spaces is smooth and hence not every separable Banach space is isometrically isomorphic to one of its quotient spaces. Also note that $(l^1, \|\cdot\|_S)$ has normal structure, by [13, Proposition 23], since it is uniformly rotund in every direction, a property implied by weak* uniform rotundity.

EXAMPLE 3. $(l^2, \|\cdot\|_{2,1})$. For x in l^2, define its positive and negative parts x^+ and x^- as usual and then define $\|\cdot\|_{2,1}$ and $\|\cdot\|_{2,\infty}$ by

$$\|x\|_{2,1} = \|x^+\|_2 + \|x^-\|_2$$

and

$$\|x\|_{2,\infty} = \sup\{\|x^+\|_2, \|x^-\|_2\}.$$

Then $\|\cdot\|_{2,1}$ and $\|\cdot\|_{2,\infty}$ are norms on l^2 each of which is equivalent to the usual norm; these norms were introduced and studied by Bynum [4]. Let l^2 equipped with $\|\cdot\|_{2,1}$ be denoted by $l_{2,1}$ and with $\|\cdot\|_{2,\infty}$ by $l_{2,\infty}$. Bynum showed that $l_{2,1}$ has normal structure while its dual space $l_{2,\infty}$ lacks normal structure, thereby establishing that normal structure is not a self-dual property. In fact, as will be shown here, this same example verifies that uniform normal structure is not a self-dual property.

THEOREM 2. *The Banach space $l_{2,1}$ is 2-uniformly rotund and hence has uniform normal structure. Consequently, dual spaces of Banach spaces with uniform normal structure need not have uniform normal structure.*

PROOF. By [1, Proposition 2.14], a Banach space has uniform normal structure if it is k-uniformly rotund for some positive integer k. Thus it

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suffices to show $l_{2,1}$ is 2-uniformly rotund. Toward this end, let (x_n), (y_n) and (z_n) be sequences of norm-one elements in $l_{2,1}$ with

$$\lim_{n\to\infty} \|(x_n + y_n + z_n)/3\|_{2,1} = 1.$$

The goal is to show that $\lim_{n\to\infty} A(x_n, y_n, z_n) = 0$.

The proof is divided into three steps. The first step is to slightly perturb the given sequences in order to obtain new sequences with nicer properties.

STEP 1. There exist sequences (X_n), (Y_n) and (Z_n) of norm-one elements in $l_{2,1}$ such that

$$\lim_{n\to\infty} \|x_n - X_n\|_{2,1} = 0,$$
$$\lim_{n\to\infty} \|y_n - Y_n\|_{2,1} = 0,$$
$$\lim_{n\to\infty} \|z_n - Z_n\|_{2,1} = 0,$$

(and hence $\lim_{n\to\infty} \|(X_n + Y_n + Z_n)/3\|_{2,1} = 1$) and, for all i and n in $\mathbb{N}$,

$$\operatorname{sgn} X_n(i) \cdot \operatorname{sgn} Y_n(i) \neq -1,$$
$$\operatorname{sgn} X_n(i) \cdot \operatorname{sgn} Z_n(i) \neq -1,$$
$$\operatorname{sgn} Y_n(i) \cdot \operatorname{sgn} Z_n(i) \neq -1,$$

and, for all n in $\mathbb{N}$,

$$X_n^+, X_n^-, Y_n^+, Y_n^-, Z_n^+, Z_n^- \text{ are all nonzero elements in } l_{2,1}.$$

In order to establish this step, let $0 < \delta < 1$ be given and assume that x, y and z are norm-one elements in $l_{2,1}$ and w^* is a norm-one element in $l_{2,1}^*$ $(\cong l_{2,\infty})$ such that

$$w^*((x + y + z)/3) = \|(x + y + z)/3\|_{2,1} > 1 - \delta.$$

Note that this implies, by the triangle inequality, that each of w^*x, w^*y and w^*z is greater than $1 - 3\delta$. Let

$$D_x = \{i\colon \operatorname{sgn} w^*(i) \cdot \operatorname{sgn} x(i) = -1 \text{ or } w^*(i) = 0\}.$$

If $D_x = \varnothing$, let $X = x$; otherwise, let

$$X(i) = \begin{cases} x(i) & \text{if } i \notin D_x, \\ 0 & \text{if } i \in D_x. \end{cases}$$

Then

$$1 = \|x\|_{2,1} \geq \|X\|_{2,1} \geq w^*X = \sum_{i \notin D_x} w^*(i) \cdot x(i)$$
$$\geq \sum_{i=1}^{\infty} w^*(i) \cdot x(i) = w^*x > 1 - 3\delta$$

and, for each i in $\mathbb{N}$,

$$\operatorname{sgn} w^*(i) \cdot \operatorname{sgn} X(i) \neq -1.$$

In similar fashion, obtain Y and Z satisfying

$$1 \geq \|Y\|_{2,1} > 1 - 3\delta \quad \text{and} \quad 1 \geq \|Z\|_{2,1} > 1 - 3\delta$$

and, for each i in $\mathbb{N}$,

$$\operatorname{sgn} w^*(i) \cdot \operatorname{sgn} Y(i) \neq -1 \quad \text{and} \quad \operatorname{sgn} w^*(i) \cdot \operatorname{sgn} Z(i) \neq -1.$$

It follows, for each i in $\mathbb{N}$, that each of

$$\operatorname{sgn} X(i) \cdot \operatorname{sgn} Y(i), \quad \operatorname{sgn} X(i) \cdot \operatorname{sgn} Z(i) \quad \text{and} \quad \operatorname{sgn} Y(i) \cdot \operatorname{sgn} Z(i)$$

is not equal to -1.

Now let $u = x|D_x$ and consider $\|x - X\|_{2,1} = \|u\|_{2,1}$. Since u and X have disjoint supports, $x^+ = (u + X)^+ = u^+ + X^+$ and $x^- = (u + X)^- = u^- + X^-$. Using this and the definition of $\|\cdot\|_{2,1}$ combined with the fact that $(\alpha^2 + \beta^2)^{1/2} \geq \alpha + \beta^2/2$ for real numbers α and β satisfying $0 \leq \alpha$, $\beta \leq 1$ and $\alpha^2 + \beta^2 \leq 1$, it follows that

$$\begin{aligned} 1 = \|x\|_{2,1} &= \|u^+ + X^+\|_2 + \|u^- + X^-\|_2 \\ &= [\|u^+\|_2^2 + \|X^+\|_2^2]^{1/2} + [\|u^-\|_2^2 + \|X^-\|_2^2]^{1/2} \\ &\geq \|X^+\|_2 + \|u^+\|_2^2/2 + \|X^-\|_2 + \|u^-\|_2^2/2 \\ &= \|X\|_{2,1} + (\|u^+\|_2^2 + \|u^-\|_2^2)/2 \\ &> 1 - 3\delta + (\|u^+\|_2^2 + \|u^-\|_2^2)/2. \end{aligned}$$

Therefore $\|u^+\|_2^2 + \|u^-\|_2^2 < 6\delta$ and so $\|u^+\|_2 < \sqrt{6\delta}$ and $\|u^-\|_2 < \sqrt{6\delta}$. It now follows that $\|x - X\|_{2,1} \leq 2\sqrt{6\delta}$ and hence

$$\begin{aligned} \left\| \frac{X}{\|X\|_{2,1}} - x \right\|_{2,1} &\leq \left\| \frac{X}{\|X\|_{2,1}} - X \right\|_{2,1} + \|X - x\|_{2,1} \\ &= 1 - \|X\|_{2,1} + \|X - x\|_{2,1} \\ &< 3\delta + 6\sqrt{\delta} < 9\sqrt{\delta}. \end{aligned}$$

Similarly, this last inequality can be established where the letters X and x are replaced by Y and y respectively or by Z and z respectively. Thus, with x, y, z, and δ replaced by x_n, y_n, z_n, and $1/n$ respectively, norm-one elements X_n, Y_n and Z_n (corresponding to $X/\|X\|_{2,1}$, $Y/\|Y\|_{2,1}$ and $Z/\|Z\|_{2,1}$ respectively) have been constructed in $l_{2,1}$ that satisfy the first six conditions in the statement of Step 1.

Finally, if there exists n in $\mathbb{N}$ such that at least one of X_n^+, X_n^-, Y_n^+, Y_n^-, Z_n^+ or Z_n^- is the zero element in $l_{2,1}$, then the triple X_n, Y_n, Z_n must be modified. For such an n, choose distinct indices j and k such that each of $|X_n(j)|$, $|Y_n(j)|$, $|Z_n(j)|$, $|X_n(k)|$, $|Y_n(k)|$, and $|Z_n(k)|$ is less than $1/2^{n+3}$. Define X_n',

Y_n' and Z_n' by having them agree with X_n, Y_n and Z_n respectively at indices different from j and k, having each of them equal to $1/2^{n+3}$ at j, and each of them equal to $-1/2^{n+3}$ at k. Then the triangle inequality yields that

$$\left\| \frac{X_n'}{\|X_n'\|_{2,1}} - X_n \right\|_{2,1} \leq \frac{1}{2^n}$$

as well as the same inequality where the letter X is replaced by Y or Z. Thus, for each such n, replace X_n, Y_n and Z_n by $X_n'/\|X_n'\|_{2,1}$, $Y_n'/\|Y_n'\|_{2,1}$ and $Z_n'/\|Z_n'\|_{2,1}$ respectively. Then the (modified) sequences (X_n), (Y_n) and (Z_n) have all of the desired properties and the statement in Step 1 is established.

The next step is to show that the sequences constructed in Step 1 have the property that the triangles determined by X_n, Y_n and Z_n are almost contained in the intersection of some two-dimensional subspaces of $l_{2,1}$ and the unit ball of $l_{2,1}$.

STEP 2. Let (X_n), (Y_n) and (Z_n) be the sequences constructed in Step 1 and let (M_n) be the sequence $((X_n + Y_n + Z_n)/3)$. Then, if (V_n) is any one of the sequences (X_n), (Y_n) or (Z_n),

$$\lim_{n\to\infty} \operatorname{dist}(V_n^+, \operatorname{span}\{M_n^+\}) = 0$$

and

$$\lim_{n\to\infty} \operatorname{dist}(V_n^-, \operatorname{span}\{M_n^-\}) = 0.$$

To establish this step, consider a fixed positive integer n and let X, Y, Z, and M denote X_n, Y_n, Z_n, and M_n respectively. Note that

$$M^+ = (X^+ + Y^+ + Z^+)/3 \quad \text{and} \quad M^- = (X^- + Y^- + Z^-)/3$$

by the properties of X, Y and Z established in Step 1. Let $0 < \eta < 4/9$ be given and suppose $\|M\|_{2,1} > 1 - \eta$. Then

$$0 \leq (\|X^+\|_2 + \|Y^+\|_2 + \|Z^+\|_2)/3 - \|M^+\|_2 < \eta.$$

By Clarkson's strong triangle inequality [5, Theorem 3],

$$\begin{aligned} 3\eta &> (\|X^+\|_2 + \|Y^+\|_2 + \|Z^+\|_2) - 3\|M^+\|_2 \\ &\geq 2\delta_2(\|(X^+/\|X^+\|_2 - M^+/\|M^+\|_2)\|_2)\|X^+\|_2 \\ &\quad + 2\delta_2(\|(Y^+/\|Y^+\|_2 - M^+/\|M^+\|_2)\|_2)\|Y^+\|_2 \\ &\quad + 2\delta_2(\|(Z^+/\|Z^+\|_2 - M^+/\|M^+\|_2)\|_2)\|Z^+\|_2 \end{aligned}$$

where δ_2 is the modulus of rotundity of $(l^2, \|\cdot\|_2)$. Therefore

$$\delta_2(\|(X^+/\|X^+\|_2 - M^+/\|M^+\|_2)\|_2)\|X^+\|_2 < 3\eta/2.$$

Now, if $\|X^+\|_2 \geq \sqrt{\eta}$, then

$$\begin{aligned} 3\sqrt{\eta}/2 &> \delta_2(\|(X^+/\|X^+\|_2 - M^+/\|M^+\|_2)\|_2) \\ &= 1 - [1 - \|(X^+/\|X^+\|_2 - M^+/\|M^+\|_2)\|_2^2/4]^{1/2}. \end{aligned}$$

Thus, $\|X^+\|_2 \geq \sqrt{\eta}$ implies

$$\|(X^+/\|X^+\|_2 - M^+/\|M^+\|_2)\|_2 < [12\sqrt{\eta} - 9\eta]^{1/2} < 2(9\eta)^{1/4}.$$

Therefore, no matter what the (nonzero) norm of X^+ is, it follows that $\text{dist}(X^+, \text{span}\{M^+\}) < 2(9\eta)^{1/4}$. This shows that

$$\lim_{n\to\infty} \text{dist}(X_n^+, \text{span}\{M_n^+\}) = 0$$

and each of the other five limits follows in an identical manner. This establishes Step 2.

Consequently, if n is large enough, X_n, Y_n and Z_n (and hence x_n, y_n and z_n) are "almost" in the two-dimensional subspace of $l_{2,1}$ spanned by M_n^+ and M_n^-. The next step is to show that $\lim_{n\to\infty} A(x_n, y_n, z_n) = 0$, which will complete the proof of Theorem 2.

Toward this end, choose norm-one elements X_n', Y_n' and Z_n' in the span of M_n^+ and M_n^- such that each of $\|X_n - X_n'\|_{2,1}$, $\|Y_n - Y_n'\|_{2,1}$ and $\|Z_n - Z_n'\|_{2,1}$ tends towards 0 as n increases.

STEP 3. $\lim_{n\to\infty} A(X_n', Y_n', Z_n') = 0$.

To establish this step, consider a fixed positive integer n and let X', Y' and Z' denote X_n', Y_n' and Z_n' respectively. Let $0 < \eta < 1/3$ be given and suppose $\|X' + Y' + Z'\|_{2,1} > 3(1 - \eta)$. Note that this implies that every convex combination of X', Y' and Z' has norm greater than $1 - 3\eta$. Thus 0 is not in the triangle determined by X', Y' and Z'. Let w be a point in this triangle of minimum norm. Without loss of generality, assume w lies on the chord between X' and Z'. Choose w^* in the unit sphere of $l_{2,1}^*$ such that

$$w^*X' = w^*Z' = w^*w = \|w\| > 1 - 3\eta.$$

Observe that $w^*Y' \geq \|w\|$ and hence

$$\text{dist}(Y', [X', Z']) = w^*(Y' - w) < 1 - (1 - 3\eta) = 3\eta.$$

Thus, by [8, Lemma 2],

$$A(X', Y', Z') \leq 2\|X' - Z'\| \,\text{dist}(Y', [X', Z']) < 12\eta.$$

This shows that $\lim_{n\to\infty} A(X_n', Y_n', Z_n') = 0$ and Step 3 is established.

Since the area function is uniformly continuous, it follows that

$$\lim_{n\to\infty} A(x_n, y_n, z_n) = \lim_{n\to\infty} A(X_n, Y_n, Z_n) = \lim_{n\to\infty} A(X_n', Y_n', Z_n') = 0$$

and hence the proof of Theorem 2 is complete.

EXAMPLE 4. $(l^2(X_0), \|\cdot\|_2)$. Let $(X_0, \|\cdot\|_0)$ be the two-dimensional space $\mathbf{R}^2$ normed so that the unit sphere consists of all points (α, β) satisfying one

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of the following:

$$(\alpha - 1/8)^2 + \beta^2 = 1 \quad \text{and} \quad \alpha \geq 1/8,$$
$$|\beta| = 1 \quad \text{and} \quad |\alpha| < 1/8,$$
$$(\alpha + 1/8)^2 + \beta^2 = 1 \quad \text{and} \quad \alpha \leq -1/8.$$

So the unit sphere of X_0 is a "short bathtub". Consider the Banach sequence space $(l^2(X_0), \|\cdot\|_2)$ consisting of all sequences (x_n) in X_0 whose norms form a square-summable sequence in $\mathbb{R}$ and where $\|(x_n)\|_2$ is given by $[\sum_{n=1}^{\infty} \|x_n\|_0^2]^{1/2}$. As will be shown, this space is not k-uniformly rotund for any k in $\mathbb{N}$ and yet this space has uniform normal structure. The first assertion is a consequence of the following result since X_0 is not even rotund (and hence not uniformly rotund).

THEOREM 3. *Let $1 < p < \infty$ and let (X_i) be a sequence of Banach spaces each one of which is not uniformly rotund. Then the Banach sequence space $l^p(X_i)$ is not k-uniformly rotund for any k in $\mathbb{N}$.*

The proof of this result will be facilitated by the following elementary, geometrical fact.

LEMMA. *Let X be a Banach space. For norm-one elements x and y in X,*

$$\operatorname{dist}(y, \operatorname{span}\{x\}) \geq \tfrac{1}{2} \min\{\|x - y\|, \|x + y\|\}.$$

PROOF. Let $\eta = \min\{\|x - y\|, \|x + y\|\}/2$. Note that $\operatorname{dist}(y, \operatorname{span}\{x\}) = \|y - \alpha x\|$ for some α in $\mathbb{R}$. To obtain a contradiction, assume $\|y - \alpha x\| < \eta$. From this and the triangle inequality, it follows that

$$|\alpha| = \|\alpha x\| \geq \|y\| - \|y - \alpha x\| \geq 1 - \eta$$

and

$$|\alpha| = \|\alpha x\| \leq \|y\| + \|y - \alpha x\| \leq 1 + \eta$$

and hence $|(1 - |\alpha|)| \leq \eta$. Now if $\alpha \geq 0$, then

$$\begin{aligned} \|x - y\| &= \|\alpha x + (1 - \alpha)x - y\| \\ &\leq \|\alpha x - y\| + |1 - \alpha| \\ &< \eta + |(1 - |\alpha|)| < 2\eta, \end{aligned}$$

a contradiction. On the other hand, if $\alpha < 0$, then

$$\begin{aligned} \|x + y\| &= \|y - \alpha x + (1 + \alpha)x\| \\ &\leq \|y - \alpha x\| + |1 + \alpha| \\ &< \eta + |(1 - |\alpha|)| < 2\eta, \end{aligned}$$

a contradiction. Therefore $\|y - \alpha x\| \geq \eta$ and the proof is complete.

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PROOF OF THEOREM 3. Since, for each positive integer i, the space X_i is not uniformly rotund, there exist norm-one sequences $(x_n^{(i)})$ and $(y_n^{(i)})$ in X_i and $\varepsilon_i > 0$ such that $\|x_n^{(i)} - y_n^{(i)}\| \geq \varepsilon_i$ for all n in $\mathbb{N}$ and $\lim_{n\to\infty} \|x_n^{(i)} + y_n^{(i)}\| = 2$. Let $k \geq 2$ be given (clearly, $l^p(X_i)$ is not uniformly rotund). Define $k+1$ sequences in $l^p(X_i)$ as follows: for n in $\mathbb{N}$, let

$$\begin{aligned} z_n^{(1)} &= (x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k-1)}, x_n^{(k)}, 0, \dots), \\ z_n^{(2)} &= (y_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k-1)}, x_n^{(k)}, 0, \dots), \\ &\vdots \\ z_n^{(k)} &= (y_n^{(1)}, y_n^{(2)}, \dots, y_n^{(k-1)}, x_n^{(k)}, 0, \dots), \\ z_n^{(k+1)} &= (y_n^{(1)}, y_n^{(2)}, \dots, y_n^{(k-1)}, y_n^{(k)}, 0, \dots). \end{aligned}$$

Then $\|z_n^{(j)}\| = k^{1/p}$ for each $1 \leq j \leq k+1$ and n in $\mathbb{N}$. Moreover,

$$\begin{aligned} \lim_{n\to\infty} \left\| \sum_{j=1}^{k+1} z_n^{(j)} \right\| &= \lim_{n\to\infty} \left[\sum_{i=1}^{k} \|i x_n^{(i)} + (k+1-i) y_n^{(i)}\|^p \right]^{1/p} \\ &= \left[\sum_{i=1}^{k} (k+1)^p \lim_{n\to\infty} \left\| \frac{i}{k+1} x_n^{(i)} + \frac{k+1-i}{k+1} y_n^{(i)} \right\|^p \right]^{1/p} \\ &= (k+1) k^{1/p}. \end{aligned}$$

Now let $w_n^{(j)} = z_n^{(j)} / \|z_n^{(j)}\|$ for each $1 \leq j \leq k+1$ and n in $\mathbb{N}$. Then $\{(w_n^{(j)}) \colon 1 \leq j \leq k+1\}$ is a collection of $k+1$ norm-one sequences in $l^p(X_i)$ such that

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{k+1} w_n^{(j)} \right\| = k+1.$$

The proof will be complete once it is shown, for each n in $\mathbb{N}$, that

$$A(w_n^{(1)}, \dots, w_n^{(k+1)}) \geq \varepsilon$$

where $\varepsilon = (\prod_{j=1}^{k} \varepsilon_j)/(2k^{1/p})^k$. Toward this end, note that the lemma yields

$$\operatorname{dist}(w_n^{(j+1)}, [w_n^{(1)}, \dots, w_n^{(j)}]) \geq \tfrac{1}{2} \|x_n^{(j)} - y_n^{(j)}\| k^{-1/p} \geq \tfrac{1}{2} \varepsilon_j k^{-1/p}$$

for each $1 \leq j \leq k$ and n in $\mathbb{N}$. Also, from [8, Lemma 1],

$$A(w_n^{(1)}, \dots, w_n^{(j+1)}) \geq \operatorname{dist}(w_n^{(j+1)}, [w_n^{(1)}, \dots, w_n^{(j)}]) A(w_n^{(1)}, \dots, w_n^{(j)})$$

for each $1 \leq j \leq k$ and n in $\mathbb{N}$. Now, by repeated applications of these last two inequalities, it follows that, for each n in $\mathbb{N}$,

$$\begin{aligned} A(w_n^{(1)}, \ldots, w_n^{(k+1)}) &\geq \frac{\varepsilon_k}{2k^{1/p}} A(w_n^{(1)}, \ldots, w_n^{(k)}) \\ &\geq \frac{\varepsilon_k \varepsilon_{k-1}}{(2k^{1/p})^2} A(w_n^{(1)}, \ldots, w_n^{(k-1)}) \\ &\ \vdots \\ &\geq \frac{\varepsilon_k \cdots \varepsilon_2}{(2k^{1/p})^{k-1}} A(w_n^{(1)}, w_n^{(2)}) \\ &\geq \frac{\varepsilon_k \cdots \varepsilon_1}{(2k^{1/p})^k} \\ &= \varepsilon. \end{aligned}$$

This completes the proof of Theorem 3.

It remains to be shown that $(l^2(X_0), \|\cdot\|_2)$ has uniform normal structure. To show this, simply combine the following four facts. First, the characteristic of convexity $\varepsilon_0(X_0)$ is $2/9$, as follows immediately from the geometry of the unit sphere of X_0. Second, by [7, Theorem 9], $\varepsilon_0(l^2(X_0)) = \varepsilon_0(X_0)$. Third, for any Banach space X, if $\varepsilon_0(X) < 1$, then $\delta_X(1) > 0$, where δ_X is the modulus of rotundity of X, as follows immediately from the definitions of $\varepsilon_0(X)$ and δ_X. And fourth, by [1, Proposition 2.14], for any Banach space X, if $\delta_X(1) > 0$, then X has uniform normal structure.

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