Compositio Mathematica **136:** 25–59, 2003.

A Theta Lift for Spin_7

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(Received: 23 August 2000; accepted in final form: 25 January 2002)

Abstract. We establish an example of a functorial lift from generic cuspidal representations of a similitude group of the type $A_1 \times C_2$ to generic representations of Spin_7. Our construction uses the theta correspondence associated to the dual pair of the type $(A_1 \times C_2, B_3)$ inside E_7. We also consider another theta correspondence associated to the dual pair of type $(A_1 \times C_2, A_1 \times A_1)$ in D_6 and show that these two pairs fit into a tower and the standard properties of a tower of theta correspondences hold.

Mathematics Subject Classifications (2000). primary 11F70; secondary 11F30, 22E55.

Key words. functorial lifting, minimal representation, theta lifting.

1. Introduction

The theta correspondence method has proved to be very fruitful in establishing many examples of Langlands functorial liftings. In this paper we use this method to construct a lift of cuspidal generic representation of a group of the type $A_1 \times C_2$ to a generic automorphic representation of Spin_7. This lift is proved to be functorial on the level of unramified representations.

To explain the method, we first need the notation of a *dual pair* and of the *minimal theta representation*.

Let H_1, H_2, H be reductive groups defined over a number field F such that H_1 and H_2 embed in H. If H_1 and H_2 inside H commute we say that (H_1, H_2) is a commuting pair inside H. If in addition $H_1 = \mathrm{Cent}_H(H_2)$ and $H_2 = \mathrm{Cent}_H(H_1)$ then (H_1, H_2) is said to be a *dual pair* inside H. For a classification of such pairs, see Rubenthaler [R].

The minimal representation is a generalization of the classical Weil representation of the double cover of the symplectic group. Globally, it is defined as a residue of a degenerate Eisenstein series at some special point. The minimal representation for simply laced groups was constructed over a local non-Archimedian field by Kazhdan and Savin in [KS]. The automorphic theta representations that we use in the present work were constructed in [GRS] using the result of [KS]. By abuse of language, we refer to this representation as the theta representation for the group H.

Using the theta representation, following Howe [H], it is possible to define a *theta correspondence* θ_H between automorphic cuspidal representations of H_1 and automorphic representations of H_2. More precisely, let π be an automorphic cuspidal representation of $H_1(\mathbb{A})$ and f_H be a vector in the space of θ_H. The theta-lift of π is the automorphic representation $\theta_H(\pi)$ of $H_2(\mathbb{A})$ whose space is spanned by the space of functions $h_2 \mapsto \int_{H_1(F)\backslash H_1(\mathbb{A})} f_H(h_1 \cdot h_2)\varphi(h_1)\,\mathrm{d}h_1$, as φ varies in V_π and f_H varies in V_{θ_H}. Experience shows that these liftings occur in what is referred to as *towers of liftings*. To explain this notion let G be a reductive group and let $H_i \subset M_i$, for $i = 1, \ldots, n$, be a set of reductive groups such that each M_i is a Levi subgroup of M_{i+1} for all $i = 1, \ldots, n-1$. Assume that (G, H_i) is a commuting pair inside M_i and the theta representation θ_{M_i} is defined for each i. Let π be an automorphic representation of $G(\mathbb{A})$. Using the above integral, we can define the theta lifting $\theta_{M_i}(\pi)$. We say that all this data fits into a tower of liftings if the following property holds:

> *Let π be a cuspidal automorphic representation of G. The representation $\theta_{M_i}(\pi)$ of H_i is cuspidal if the representation $\theta_{M_k}(\pi)$ of H_k vanishes for all $k < i$.*

It follows that if $\theta_{M_i}(\pi) \neq 0$, then $\theta_{M_k}(\pi)$ is not cuspidal for all $k > i$. This actually means that the obstruction for $\theta_{M_i}(\pi)$ to be cuspidal comes from lower liftings. We say that the tower is *complete from below* if $\theta_{M_1}(\pi)$ is cuspidal for any cuspidal π. Similarly, the tower is said to be *complete from the top* if $\theta_{M_n}(\pi)$ does not vanish for all cuspidal π. Schematically, we draw this tower as follows:

$$\begin{tikzcd}
 & H_n \\
G \arrow[ur, "\theta_{M_n}"] \arrow[r, "\theta_{M_i}"] \arrow[dr, "\theta_{M_1}"'] & H_i \\
 & H_1
\end{tikzcd}$$

There are several known examples for this phenomenon. First there is the tower of (Sp_n, O_{2i}) or $(\widetilde{Sp}_n, O_{2i+1})$ where $\widetilde{Sp}_n$ is the double cover of Sp_n and O_k is an orthogonal group. This tower uses the classical theta representation which is defined on $\widetilde{Sp}_m$ for some appropriate m. The tower property for the case was studied in [Ra]. Another example of tower of liftings for the pairs of type (G_2, H) where H runs over six different types of groups was studied in [GRS1]. This example uses the theta representations which are defined for some exceptional groups.

In the present paper, we consider the theta correspondence associated to the dual pair $((GL_2 \times GSp_4)^0, \mathrm{Spin}_7)$ inside GE_7, and the theta correspondence associated to the dual pair $((GL_2 \times GSp_4)^0, G\mathrm{Spin}_4)$ inside GD_6. Here GE_7 (resp. GD_6) is the similitude group of E_7(resp. D_6). The precise definitions of all groups are contained in Section 2.

The paper is arranged as follows. In Section 2 we introduce notations that will be used throughout the paper.

The next two sections serve as a preparation for defining the theta correspondence. Namely, in Section 3 we show that the groups $(GL_2 \times GSp_4)^0$ and Spin_7 occur in a dual pair inside GE_7. The embeddings are described explicitly. In Section 4 we construct the minimal representations of groups of type E_7 and GE_7 together with their automorphic realization. The automorphic realization of the minimal representation of a group of type E_7 has already been constructed in [GRS]. However, for technical reasons, we realize the representation θ_{E_7} in a slightly different way from the one constructed in [GRS]. Here we realize θ_{E_7} automorphically as a residue of an Eisenstein series attached to the induced representation coming from the Heisenberg parabolic subgroup. We then reprove some of the properties of θ_{E_7} stated in [GRS]. Most of the proofs are quite similar to those in [GRS]. Moreover, we extend the representations θ_{E_7} of E_7 and θ_{D_6} of D_6 to representations θ_{GE_7} of the group GE_7 and θ_{GD_6} of GD_6 respectively.

In Section 5 we define the theta lifting, using θ_{GE_7}, of a cuspidal automorphic representation π of $(GL_2 \times GSp_4)^0$ to an automorphic representation of Spin_7. To make the construction possible we put certain restrictions on the central character of π. Similarly, we define the theta lifting, using θ_{GD_6}, of a cuspidal representation π as above to an automorphic representation of $G\mathrm{Spin}_4$. We do not know whether these theta lifts of π are irreducible representations.

The goal of Section 6 is to prove that these two liftings fit into a tower of liftings. The corresponding tower of liftings in this case is

$$\begin{tikzcd} & \mathrm{Spin}_7 \\ (GL_2 \times GSp_4)^0 \arrow[ur, "\theta_{GE_7}"'] \arrow[dr, "\theta_{GD_6}"] & \\ & G\mathrm{Spin}_4 \end{tikzcd}$$

One of our main results (Theorem 6) is that if $\theta_{GD_6}(\pi)$ is zero, then $\theta_{GE_7}(\pi)$ is cuspidal, in a sense that all constant terms of all vectors in this representation vanish. We call this result 'the first cuspidality property'.

In Section 7 we prove that $\theta_{GD_6}(\pi)$ is cuspidal whenever π is cuspidal. This result is called 'the second cuspidality property'. This implies that the tower is complete from below.

In Section 8 we express the Whittaker model of $\Pi = \theta_{GE_7}(\pi)$ in terms of the Whittaker model of π.

Section 9 is devoted to the proof of a nonvanishing result. Namely, we prove that $\theta_{GE_7}(\pi) \neq 0$ for a *generic* cuspidal representation π. Combining this result with the result of Section 8, we deduce that $\theta_{GE_7}(\pi)$ is a nonzero generic cuspidal representation of Spin_7 whenever π is a generic cuspidal representations of $(GL_2 \times GSp_4)^0$. This implies that the tower is complete from the top when π is assumed to be generic.

Finally, in Section 10 we show that the lifting defined by θ_{GE_7} is functorial for generic representations on the level of the unramified representations. Namely, we prove that if π is a cuspidal irreducible generic representation of $(GL_2 \times GSp_4)^0$ and Π is an irreducible generic representation contained in $\theta_{GE_7}(\pi)$, then Π is a weak Langlands lift of π corresponding to the natural map of L-groups

$$r\colon (GL_2 \times GSp_4)^0/\{\pm 1\}(\mathbb{C}) \mapsto PSp_6(\mathbb{C}).$$

To prove this, we compute a Mellin transform $J(s)$ of the Whittaker model for the representation Π_v. Let Π'_v be a Langlands lift from π_v with respect to the map r. We show that $J(s) = P(q^{-s})L(\Pi'_v, \omega_2, s)$, where $P(q^{-s})$ is a polynomial in q^{-s} which depends on the representation Π'_v, and ω_2 is the second fundamental representation of $PSp_6(\mathbb{C})$ which is the dual group of Spin_7. Using the expression of the Whittaker model of Π_v in terms of the Whittaker model of π_v we get that $J(s) = Q(q^{-s})$ $L(\Pi_v, \omega_2, s)$ where Q is a polynomial in q^{-s}. Using this and the representation $2\omega_1$ of $PSp_6(\mathbb{C})$, we recover the Langlands parameter of Π_v and show that it is equal to the Langlands parameter of Π'_v. This proves that the lifting θ_{GE_7} is functorial on the unramified level.

The present paper is a shortened version of my thesis [G]. All proofs that are omitted or only sketched here, appear in full detail in [G].

2. Notations

We start by setting the notations for the groups used in the paper. By E_7 we denote the simply connected group of the type E_7. We shall label the seven simple roots α_i of E_7 as follows:

$$\begin{array}{ccccccccccc} \alpha_1 & — & \alpha_3 & — & \alpha_4 & — & \alpha_5 & — & \alpha_6 & — & \alpha_7 \\ & & & & | & & & & & & \\ & & & & \alpha_2 & & & & & & \end{array}$$

Given a positive root α, we shall write $(n_1 \dots n_7)$ for $\alpha = \sum_{i=1}^{7} n_i\alpha_i$. Given a root $\alpha = \sum_{i=1}^{7} n_i\alpha_i$ positive or negative, x_α or $x_\alpha(r)$ or $x_{(n_1\dots n_7)}(r)$ will denote the one-dimensional unipotent subgroup corresponding to the root α. We shall denote by w_i the simple reflection in the Weyl group of E_7 corresponding to the simple root α_i. In short, we shall write $w[i_1 \dots i_m]$ for $w_{i_1}w_{i_2}\dots w_{i_m}$. To each simple root, there is an embedding of SL_2 in E_7. Each such embedding gives a one-dimensional torus in E_7 corresponding to the torus $\left(\begin{smallmatrix} t & \\ & t^{-1} \end{smallmatrix}\right)$ of SL_2. We shall denote the image of this torus in E_7, corresponding to the simple root α_i by $h_i(t)$. Thus, a general torus element in E_7 is $\prod_{i=1}^{7} h_i(t_i)$ which we denote by $h(t_1, \dots t_7)$. The action of the torus on the roots can be read from the Cartan matrix. Similarly, one can deduce the action of the Weyl group on the roots.

The similitude group GE_7 is obtained by adding a one-dimensional torus to the group E_7 which acts linearly on the root α_2 and trivially on the rest of the simple roots. We denote this torus by $h_8(t_8)$. Thus, a general torus element in GE_7 is $\prod_{i=1}^{8} h_i(t_i)$ which we denote by $h(t_1, \dots, t_8)$. Moreover, if an element of the torus

$t \in T_H$, in some matrix realization of H is a diagonal matrix with entries $a_1, \ldots, a_n$, we write $t = \mathrm{diag}(a_1, \ldots, a_n)$.

For any parabolic subgroup P of a split reductive group G we have the Levi decomposition $P = M_P U_P$. Here U_P is the unipotent radical of P and M_P is the reductive part. For $\{\alpha_{n_1}, \ldots, \alpha_{n_k}\}$ a subset of simple roots of any split reductive group G we denote by $P(n_1, \ldots, n_k)$ the parabolic subgroup of G whose Levi component contains roots $\{\alpha_{n_1}, \ldots, \alpha_{n_k}\}$. Accordingly, we denote $P(n_1, \ldots, n_k) = M(n_1, \ldots n_k)$ $U(n_1, \ldots n_k)$. We denote by B_G, or just B if there is no confusion, the Borel subgroup of G. Then $B_G = T_G N_G$ where T_G is the torus and N_G is the maximal unipotent radical of G. Similarly, Z_G, or just Z if there is no confusion, denotes the center of the group G. In general, for a split reductive group G we denote by $\phi(G)$ the set of roots, by $\phi^+(G)$ the set of positive roots and by $\Delta(G)$ the set of simple roots. We will denote the highest weight by ρ. We use the standard notation on the Lie algebra side. Namely, $g = \mathrm{Lie}(G)$, g_α denotes the root subspace to the root α spanned by X_α.

Throughout the paper we consider the following parabolic subgroups:

(a) $GE_7 \supset P = MU$ where the semisimple part of M is of type D_6. Thus $M \simeq GSO_{12} \cdot GL_1$. This is the Heisenberg parabolic subgroup, i.e. $R = [U, U]$ is one-dimensional.
(b) $GE_7 \supset Q = M_Q U_Q$ where the semisimple part of M_Q is of type E_6. In this case, the unipotent radical U_Q is Abelian.
(c) $M \supset E = P(2, 4, 5, 6, 7)$. The semisimple part of M_E is of type A_5. This parabolic subgroup is denoted by $P(D_6)$ in [GRS]; sometimes we also use this notation.
(d) By $P_{\mathrm{Heis}}(H)$ we denote the unique maximal parabolic subgroup of a reductive group H whose unipotent radical is a Heisenberg group. It is known to exist for all split simple reductive groups not of type A_n.

By D_6 we denote the subgroup of the group E_7 which is generated by $x_{\pm\alpha_i}$ for $i = 2, \ldots, 7$. This is the simply-connected group isomorphic to Spin_{12}. The maximal torus of D_6 is $h(1, t_2, \ldots, t_7)$. Similarly, the group GD_6 is the subgroup of GE_7 generated by the subgroup D_6 and one-dimensional torus $h_8(t_8)$.

Consider the group GL_4 whose simple roots we denote by $\gamma_1, \gamma_2, \gamma_3$. The group GSp_4 whose long simple root is denoted by β_1 and the short simple root by β_2 is canonically embedded in GL_4 by

$$x_{\beta_1}(r) \mapsto x_{\gamma_2}(r), \qquad x_{\beta_2}(r) \mapsto x_{\gamma_1}(r) x_{\gamma_3}(-r).$$

To simplify notations we denote by P_1 the parabolic of Sp_4 or GSp_4 whose Levi subgroup contains the long simple root β_1 and by P_2 the parabolic whose Levi subgroup contains the short simple root β_2.

Given a subgroup H of a group G and an element $\gamma \in G$ we denote by H^γ the subgroup $\gamma^{-1} H \gamma$ of G.

3. The Dual Pair

Let H_1, H_2, H be reductive groups defined over a number field F such that $H_1 \times H_2$ embeds in H. We say that (H_1, H_2) is a commuting pair inside H. If in addition $H_1 = \mathrm{Cent}_H(H_2)$ and $H_2 = \mathrm{Cent}_H(H_1)$, then (H_1, H_2) is said to be a dual pair inside H. The classification of such pairs was done by Rubenthaler [R]. In this section we describe explicitly the dual pair $(\mathrm{Spin}_7, (GL_2 \times GSp_4)^0)$ inside GE_7.

First we note (see [R]) that in E_7 there is a dual pair of type $(C_2 \times A_1, B_3)$. We shall describe the embedding of the simple roots.

Let us denote the positive roots of C_2 by $\beta_1, \beta_2, \beta_1 + \beta_2, \beta_1 + 2\beta_2$ and the positive root of A_1 by β_3. The embedding of the simple roots of $C_2 \times A_1$ in E_7 is given as follows:

$$x_{\beta_1}(r) \mapsto x_{0001000}(r), \quad x_{\beta_2}(r) \mapsto x_{0100000}(r)x_{0000100}(r),$$
$$x_{\beta_3}(r) \mapsto x_{0000001}(r).$$

The group generated by $x_{\pm\beta_i}(r)$ is the semisimple group $SL_2 \times Sp_4$.

Let us denote the simple roots of B_3 by α, β and γ where γ is a short root. The embedding of the simple roots of B_3 in E_7 is given by

$$x_\alpha(r) \mapsto x_{0112221}(r), \quad x_\beta(r) \mapsto x_{1000000}(r),$$
$$x_\gamma(r) \mapsto x_{0011100}(r)x_{0111000}(r).$$

The group generated by these roots and their negatives is the simply-connected group Spin_7. Using commutation relations we deduce that $(SL_2 \times Sp_4, \mathrm{Spin}_7)$ is a commuting pair inside the simply-connected group of type E_7. Our aim is to enlarge these groups to get a dual pair inside GE_7. Recall that the group GE_7 is generated by the semisimple group of type E_7 and the eight-dimensional torus defined above.

Define

$$H_1 = \{h \in T_{GE_7} : h \text{ commutes with the simple roots of } \mathrm{Spin}_7\}.$$

Then by using the Cartan matrix of E_7, any $h \in H_1$ has the form

$$h = h({s_5}^2{s_2}^{-2}, s_2, {s_5}^4{s_2}^{-4}, s_4, s_5, {s_5}^3{s_7}^{-3}, s_7, {s_5}^{-1}s_2). \tag{3.1}$$

Denote the group generated by H_1 and the embedding of $A_1 \times C_2$ in GE_7 by G.

PROPOSITION 3.1. *The group G is isomorphic to $(GL_2 \times GSp_4)^0$, where*

$$(GL_2 \times GSp_4)^0 = \{(g_1, g_2) \in GL_2 \times GSp_4 : \det(g_1) = s(g_2)\}.$$

Here $s(g)$ means the similitude factor in GSp_4.

Proof. Obviously the simple roots of G can be identified with the simple roots of $(GL_2 \times GSp_4)^0$. Thus it is enough to construct an isomorphism of the tori $s: T_G = H_1 \mapsto T_{(GL_2 \times GSp_4)^0}$ such that $\alpha(s(t)) = \alpha(t)$ for every simple root α and every $t \in H_1$. Any element t of the torus $T_{(GL_2 \times GSp_4)^0}$ has the form

$$t = (\mathrm{diag}(a, a^{-1}\lambda), \mathrm{diag}(b, c, \lambda c^{-1}, \lambda b^{-1})).$$

The isomorphism is given by

$$a = s_2 s_5{}^{-1} s_7, \qquad b = s_5{}^{-3} s_2{}^4, \qquad c = s_2 s_5{}^{-2} s_4, \qquad \lambda = s_2{}^{-1} s_5.$$

One can check that the above condition holds. □

Similarly, we define

$$H_2 = \{h \in T_{GE_7} : h \text{ commutes with the simple roots of } C_2 \times A_1\}.$$

Then any $h \in H_2$ has the form

$$h = h(t_1, t_2, t_2{}^2 t_7{}^{-1}, t_2{}^2, t_2 t_7, t_7{}^2, t_7, 1). \tag{3.2}$$

Denote the group generated by H_2 and by the embedding in GE_7 by $\tilde{G}$.

PROPOSITION 3.2. *The group $\tilde{G}$ is isomorphic to* Spin_7.

Proof. Since both groups are of the type B_3 we identify the simple roots of $\tilde{G}$ and Spin_7. Thus it is enough to construct an isomorphism $s: T_{\tilde{G}} = H_2 \mapsto T_{\mathrm{Spin}_7}$ such that $\alpha(s(t)) = \alpha(t)$ for every simple root α and for every $t \in H_2$. We can write any element of T_{Spin_7} in the form $h(s_\alpha, s_\beta, s_\gamma)$ in the same fashion as we did for E_7. The action of the roots can be read from the Cartan matrix for B_3. Then the claimed isomorphism is given by

$$s_\alpha = t_7, \qquad s_\beta = t_1, \qquad s_\gamma = t_2 t_7^{-1}.$$

One can check that the above condition holds. □

We denote by GD_6 the subgroup of GE_7 generated by $x_{\pm\alpha_i}(r)$ for $i = 2, 3, \dots, 7$ and by the torus $h_8(t_8)$. This is a simply connected group of type D_6 and it is isomorphic to $G\mathrm{Spin}_{12}$. The group G is contained in $M = GD_6 \cdot GL_1$. Obviously $(G, \mathrm{Spin}_7 \cap M)$ is a commuting pair inside M.

PROPOSITION 3.3. *The group* $\mathrm{Spin}_7 \cap M$ *is a simply-connected group isomorphic to* $G\mathrm{Spin}_4$.

Proof. Note that $\mathrm{Spin}_7 \cap M = M(\alpha, \gamma)$ is a Levi subgroup of Spin_7. Thus both groups are of type $A_1 \times A_1$ as before, so it suffices to construct an isomorphism $s: T_{G\mathrm{Spin}_4} \mapsto T_{\mathrm{Spin}_7}$ that preserves the action of the roots α and γ. Writing as before an element of T_{Spin_7} and $T_{G\mathrm{Spin}_4}$ in the form $h(t_\alpha, t_\beta, t_\gamma)$ and $h(s_1, s_2, s_3)$ respectively, where $h(s_2)$ is a similitude factor of $G\mathrm{Spin}_4$, the isomorphism is given by

$$s_1 = t_\alpha, \qquad s_2 = t_\beta, \qquad s_3 = t_\gamma.$$

Obviously the isomorphism preserves the root action. □

Remark. Note that the group $G\mathrm{Spin}_4$ is isomorphic to

$$(GL_2 \times GL_2)^0 = \{(g_1, g_2) \in GL_2 \times GL_2 : \det(g_1) = \det(g_2)\}.$$

4. Minimal Representations θ_{E_7} and θ_{GE_7}

Recall that in [GRS] the minimal representation θ_{E_7} is defined as a residue of an Eisenstein series induced from the parabolic subgroup Q. In this section we shall realize automorphically the same abstract representation as a residue of Eisenstein series induced from the parabolic Heisenberg subgroup P. Then we reprove for this new realization the properties that are studied in [GRS].

Recall that P is the maximal parabolic subgroup whose Levi part has the semisimple part of type D_6. Note that the unipotent subgroup U is a Heisenberg group, i.e. $[U, U] = R$ is one-dimensional.

4.1. LOCAL MINIMAL REPRESENTATION

Let F be a local field of characteristic zero. Then any simply connected, simply laced group $G(F)$ has a distinguished representation known as the minimal representation. These representations for the groups of type D_n and E_n were constructed explicitly in [KS]. In particular, for $G = E_7$ and F non-Archimedian field the minimal representation is the unique irreducible unramified quotient of $\mathrm{Ind}_P^{E_7}\delta_P^{14/17}$. The induction here and elsewhere in this paper is not normalized.

4.2. POLES OF EISENSTEIN SERIES

In this subsection we give some general definitions. Here H denotes any split reductive group and P_H any maximal parabolic subgroup of G. Let F be a number field and $\mathbb{A}$ its ring of adeles. For $s \in \mathbb{C}$ set $I(s) = \mathrm{Ind}_{P_H(\mathbb{A})}^{H(\mathbb{A})}\delta_{P_H}^s$. Consider the corresponding Eisenstein series defined first for $Re(s)$ large, by

$$E_{P_H}(g, f, s) = \sum_{\gamma\in P_H(F)\backslash H(F)} f(\gamma g, s) \tag{4.1}$$

for $g \in H(\mathbb{A})$ and $f \in I(s)$. This series converges absolutely for $Re(s)$ large and admits a meromorphic continuation to the whole complex plane. It has a finite number of poles after suitable normalization.

Let K be the standard maximal compact subgroup of $H(\mathbb{A})$. The function f is standard if it is K-finite and its restriction to K does not depend on s. From now on we consider only a standard section f in $I(s)$.

Given $f = \bigotimes_v f_v \in I(s)$ we denote by S the finite set of places such that f_v is unramified for $v \notin S$. We denote by $\zeta_v(s)$ the local zeta factor at the place v and we denote $\zeta_S(s) = \prod_{v\notin S}\zeta_v(s)$.

Given a Weyl element $w \in W(H)$ we form the intertwining operator $(M_w(s)f)(g, s) = \int_{N_w(\mathbb{A})} f(wng, s)\mathrm{d}n$, where N_w is the group generated by $\{x_\alpha(r) : \alpha > 0, x_{w\alpha}(r) \notin P_H\}$. Thus $M_w(s)$ is factorizable and $M_w(s) = \prod_v M_{w,v}(s)$. If f_v is a K_v-fixed vector, normalized so that $f_v(e, s) = 1$, and $\tilde{f}_v$ is the K_v-fixed vector in the image of $M_{w,v}(s)$ normalized so that $\tilde{f}_v(e, s) = 1$, then we have

$$M_{w,v}(s)f_v = L_v^1(H, P_H, w, s)\tilde{f}_v.$$

Set

$$L_S^1(H, P_H, w, s) = \prod_{v \notin S} L_v^1(H, P_H, w, s).$$

We will also denote

$$A_w(s)f = \left(\prod_{v \in S} M_{w,v}(s)f_v\right) \otimes \prod_{v \notin S} \tilde{f}_v.$$

We define the normalized Eisenstein series by $E^*_{P_H}(g, f, s) = L_S(H, P_H, s)E_{P_H}(g, f, s)$, where the normalizing factor $L_S(H, P_H, s)$ is the denominator of $L_S^1(H, P_H, w_0, s)$, when written as a quotient of products of zeta factors (after cancellation) and w_0 is the representative of the big cell in $P_H \backslash H / N$ with the minimal length.

4.3. SEVERAL COMPUTATIONAL LEMMAS

In this subsection we give some simple computations that will be used throughout the paper.

LEMMA 4.2.1. *We have*

(1) $\delta_P(h(t_1, \ldots, t_7)) = |t_1|^{17}$.
(2) $\delta_{P(D_6)}(h(1, t_2, \ldots, t_7)) = |t_3|^{10}$.
(3) $\delta_{P_{Heis}(D_6)}(h(1, t_2, \ldots, t_7)) = |t_6|^9$.

Remark. We identify the group of type D_6 with the subgroup of E_7 as described in Section 3. Thus the maximal torus of this subgroup is $h(1, t_2, \ldots, t_7)$.

LEMMA 4.2.2. *There are five double cosets $P \backslash E_7 / P$ and the distinguished representatives are: $e, w[1], w[13425431], w[13425436542765431]$ and w_0, which is the shortest representative in the big cell.*

LEMMA 4.2.3. *We have*

(1) $L_S(E_7, P, s) = \zeta_S(17s)\zeta_S(17s-3)\zeta_S(17s-5)\zeta_S(34s-16)$.
(2) $L_S(D_6, P(D_6), s) = \zeta_S(10s)\zeta_S(10s-2)\zeta_S(10s-4)$.
(3) $L_S(D_6, P_{Heis}(D_6), s) = \zeta_S(9s)\zeta_S(9s-1)\zeta_S(9s-3)\zeta_S(18s-8)$.
(4) $L_S^1(E_7, P, w[1], s) = \dfrac{\zeta_S(17s-1)}{\zeta_S(17s)}$.
(5) $L_S^1(E_7, P, w[13425436542765431], s) = \dfrac{\zeta_S(17s-6)\zeta_S(17s-8)\zeta_S(17s-10)\zeta_S(34s-17)}{\zeta_S(17s)\zeta_S(17s-3)\zeta_S(17s-7)\zeta_S(34s-16)}$.
(6) $L_S^1(E_7, P, w[13425431], s) = \dfrac{\zeta_S(17s-4)\zeta_S(17s-7)}{\zeta_S(17s)\zeta_S(17s-3)}$.

Proof. Proved directly using Gindikin–Karpelevich formula for the intertwining operators. □

4.4. THE RESIDUE REPRESENTATION

Now we are ready to prove the main theorem of this section.

THEOREM 4.3. (i) *For any standard section f_s, the Eisenstein series $E_P^*(g,f,s)$ has at most a simple pole at $s = 14/17$. The pole is attained by the spherical section f_s^0.*

(ii) *The space of automorphic forms on $E_7(\mathbb{A})$ spanned by the functions $(s - 14/17)E_P^*(g,f_s)|_{s=14/17}$ is an irreducible square-integrable automorphic representation, and at all finite places the local component of this representation is isomorphic to $\theta_{E_7 v}$.*

Proof. (i) To determine the poles of the Eisenstein series it suffices to determine the poles of its constant term along U. For $g \in D_6 \subset E_7$ we obtain by standard computation

$$\int_{U(F)\backslash U(\mathbb{A})} E_P(ug,f,s) = \sum_w E_{M^w}(g, M_w(s)f, s'). \tag{4.2}$$

Here w runs over $P(F)\backslash E_7(F)/P(F)$. Also E_{M^w} is the Eisenstein series of the group M obtained by inducing from the maximal parabolic $M^w = w^{-1}Pw \cap M$. Finally s' is a linear translation of s and we view $M_w(s)f$ as a section on M by restriction.

Let us take representatives of cosets as in Lemma 4.2.2. We compute the contribution of every w to the constant term along U of $E_P^*(h(a)g,f,s)$, where $h(a) = h(a^2, a^2, a^3, a^4, a^3, a^2, a)$ is the center of M. Using Lemma 4.2.3, we obtain

$$\begin{aligned}
&\int_{U(F)\backslash U(\mathbb{A})} E_P^*(uh(a)g,f,s)\,\mathrm{d}u \\
&\quad= |a|^{34s}L_S(E_7,P,s)f(g,s) + |a|^{51s-3}\zeta_S(34s-16)E^*_{P(D_6)} \\
&\qquad (g, A_{w[1]}(s)f, 17/10s - 1/10) + \\
&\qquad + |a|^{34s-8}E^*_{P_{Heis}(D_6)}(g, A_{w[13425431]}(s)f, 17/9s - 4/9) + \\
&\qquad + |a|^{51s-18}\zeta_S(34s-17)E^*_{P(D_6)}(g, A_{w[13425436542765431]}(s)f, 17/10s - 6/10) + \\
&\qquad + |a|^{34-34s}\zeta(17s-11)\zeta(17s-13)\zeta(17s-16)\zeta(34s-17)(A^*_{w_0}(s)f)(g,s),
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
&(A^*_{w_0}(s)f)(g,s) \\
&\quad= \left(\prod_{v\in S}\zeta_S(17s-11)\zeta_S(17s-13)\zeta_S(17s-16)\zeta_S(34s-17)\right)^{-1} \times (A_{w_0}(s)f)(g,s).
\end{aligned}$$

Provided all the sections $A_w(s)f$ are holomorphic, it is easy to see that first, second and third summands of (4.3) are holomorphic at $s = 14/17$. Let us examine the remaining terms in greater detail:

(1) $w = w[13425436542765431]$. Here for $s = 14/17$ one has $s' = 17/10s - 6/10 = 4/5$. We know from [GRS] that $E^*_{P(D_6)}(g, f, s)$ has at most a simple pole at $s = 4/5$ and that pole is attained for the spherical section f on $D_6(\mathbb{A})$. Thus we conclude that there is a simple pole and the residue is by definition a function from the space of θ_{D_6}.

(2) $w = w_0$. Provided $A_{w_0}(s)f$ is holomorphic the last summand has at most a simple pole at $s = 14/17$ that comes from the factor $\zeta(17s - 13)$. Since $A_{w_0}(s)f^0_s$ is a nonzero spherical vector the pole is attained for the spherical section f^0_s.

Comparing the powers of $|a|$ we see that no cancellations are possible. Hence the Eisenstein series has at most a simple pole at $s = 14/17$ and the pole is attained by the spherical section. To finish the proof of (i) we need the following lemma which is proved similarly to lemma 2.5 in [GRS]. □

LEMMA 4.3.1. *Given $f \in I(s)$ the intertwining operators $A_w(s)$ for $w = w[1]$, $w[13425431]$, $w[13425436542765431]$ and $A^*_{w_0}(s)$ are holomorphic at $s = 14/17$.*

This ends the proof of the first part of the theorem and it remains to prove the square integrability and irreducibility of the residue representation.

To prove (ii) we consider the $E_7(\mathbb{A})$ equivariant map Θ_{E_7} from $\mathrm{Ind}_{P(\mathbb{A})}^{E_7(\mathbb{A})}\delta_P^{14/17}$ to the space of the automorphic forms on $E_7(\mathbb{A})$ sending $f \mapsto (s - 14/17)E^*_P(g, f, s)|_{s=14/17}$. We have seen above that this map is nonzero.

CLAIM. *The image of Θ_{E_7} is contained in $L^2(E_7(F)\backslash E_7(\mathbb{A}))$.*

To prove the claim we use the square integrability criterion of Jacquet [MW1]. Since the elements φ of space of θ_{E_7} are concentrated along the Borel subgroup it suffices to show that the automorphic exponents of φ along B_{E_7} have a real part which is a linear combination of the simple roots with negative coefficients.

First represent an element of T_{E_7} in the form $z(a)t$ where $z(a) = (a, a, a^{3/2}, a^2, a^{3/2}, a, a^{1/2})$ is a center of M acting linearly on the root α_1 and $t \in D_6$. Any exponent thus has a form $\chi_1(z(a))\chi_2(t)$. The automoprhic exponents along B are provided by (4.3). Recall that the only summands that contribute to θ_{E_7} are the two last ones. Contribution from the last term provides exponents $\chi_1(z) = \delta_P^{-1/2}(z)|a|^3 = \delta_P^{-11/34}(z)$ and $\chi_2(t) = \delta_{B_{D_6}}^{-1/2}(t)$.

The one before last term provides $\chi_1(z) = \delta_P^{-1/2}(z)|a|^{12} = \delta_P^{-5/17}(z)$ and $\chi_2(t)$ is an exponent of θ_{D_6} along the Borel subgroup and, hence, is a linear combination of the roots with negative coefficients by Theorem 3.1 of [GRS].

Thus, all the exponents θ_{E_7} along B are linear combinations of simple roots with negative coefficients and this proves the claim.

The claim implies that the image of Θ_{E_7} is a semisimple representation. On the other hand, it is proved in [S] that $\mathrm{Ind}_{P(\mathbb{A})}^{E_7(\mathbb{A})}\delta_P^{14/17}$ has the unique irreducible unramified quotient. Thus the image of Θ_{E_7} is an isotypic representation, and each irreducible summand is an unramified representation. Since $\mathrm{Ind}_{P(\mathbb{A})}^{E_7(\mathbb{A})}\delta_P^{14/17}$ has a unique up to scalars spherical vector f^0 and $\Theta_{E_7}(f^0) \neq 0$ we conclude that the image of Θ_{E_7} is an irreducible unramified representation and every local component of it at a finite place v is isomorphic to the minimal representation $\theta_{E_{7_v}}$. This proves (ii). □

We denote the resulting representation by θ_{E_7}.

4.5. THE SPACE OF CONSTANT TERMS ALONG U

Let us define the space $\theta_{E_7}^U$ of the constant terms along U of θ_{E_7} as space of functions $f^U(g) = \int_{U(F)\backslash U(\mathbb{A})} f(ug)\,\mathrm{d}u$ as f varies in $V_{\theta_{E_7}}$.

Note that in (4.3) $A_{w_0}^*(s)f|_{D_6}$ is a constant at $s = 14/17$. Thus the residue of the last summand at $s = 14/17$ restricted to D_6 gives a constant representation. On the other hand, the residue of the fourth summand in (4.3) at $s = 14/17$ restricted to D_6 gives a small representation defined in [GRS]. All other summands of (4.3) are holomorphic at $s = 14/17$.

THEOREM 4.6. *We have* $\theta_{E_7}^U|_{D_6} = 1 \oplus \theta_{D_6}$.

Proof. First by (4.3) we have $\theta_{E_7}^U|_{D_6} \subset 1 \oplus \theta_{D_6}$. Let us show that the equality holds. By (4.3) we have a $D_6(\mathbb{A})$ equivariant map

$$\Phi\colon \mathrm{Ind}_{P(\mathbb{A})}^{E_7(\mathbb{A})}\delta_P^{14/15} \mapsto L^2(D_6(F)\backslash D_6(\mathbb{A}))$$

sending

$$f \mapsto \mathrm{Res}_{s=14/17} \int_{U(F)\backslash U(\mathbb{A})} E_P^*(ug, f, s)\mathrm{d}u = Cf(e) + Res_{s=4/5} E(g, A_w(14/17)f|_{D_6}, s)$$

for some nonzero constant C, $w = w[13425436542765431]$.

If f is a spherical section then $E(g, A_w(s)f|_{D_6}, s)$ has a simple pole at $s = 4/5$ so the projection of Im Φ on the space of θ_{D_6} is nonzero. Since Φ is D_6 equivaraint and θ_{D_6} is irreducible a projection of Im Φ on the space θ_{D_6} is surjective. Obviously the projection of Im Φ on a trivial representation is also surjective. Denote by w the vector generating the one-dimensional representation. Let $(v_1, c_1 w), (v_2, c_2 w)$ be in Im Φ for some linearly independent vectors v_1, v_2 in the space of θ_{D_6}. Thus $(c_2 v_1 - c_1 v_2, 0)$ is a nonzero vector in Im Φ. Hence θ_{D_6} is a subrepresentation of Im Φ and so Im $\Phi = \theta_{D_6} \oplus 1$. □

4.6. FOURIER COEFFICIENTS OF θ_{E_7}

Let f be a vector in $V_{\theta_{E_7}}$. Then f^R is $R(\mathbb{A})U(F)$ left invariant function and moreover $R(\mathbb{A})U(F)\backslash U(\mathbb{A})$ is Abelian. In this subsection we study the Fourier expansion of f^R along $R(\mathbb{A})U(F)\backslash U(\mathbb{A})$.

THEOREM 4.7. *For $f \in V_{\theta_{E_7}}$ the Fourier expansion of f^R along $R(\mathbb{A})U(F)\backslash U(\mathbb{A})$ contains just one orbit under M_F of nontrivial characters. Namely the orbit of the character*

$$\psi_U(\exp Z) = \psi(B(Z, X_{-\alpha_1})), \quad Z \in \mathrm{Lie}(U)(\mathbb{A}),$$

where B is the Killing form and ψ is a fixed nontrivial character of $F\backslash\mathbb{A}$.

Denote

$$f^R(g) = \int_{R(F)\backslash R(\mathbb{A})} f(rg)\,\mathrm{d}r \quad \text{and} \quad f^{R\psi_U}(g) = \int_{R(\mathbb{A})U(F)\backslash U(\mathbb{A})} \psi_U{}^{-1}(u) f^R(ug)\mathrm{d}u$$

as f varies in $V_{\theta_{E_7}}$.

COROLLARY. *For $f \in V_{\theta_{E_7}}$ we have*

$$f^R(g) = f^U(g) + \sum_{\gamma \in \mathrm{Stab}_M(\psi_U)(F)\backslash M(F)} f^{R\psi_U}(\gamma g). \tag{4.4}$$

Note that $\mathrm{Stab}_M(\psi_U)$ is almost a parabolic subgroup E of M; a one-dimensional torus is missing.

Proof of Theorem 4.7. We outline the steps of the proof. Each step is proved similarly to the proof of theorem 5.2 in [GRS] for the case $G = E_8$. The characters of $R(\mathbb{A})U(F)\backslash U(\mathbb{A})$ have the following form. For any $Y \in \bigoplus_{\alpha>0} g_{-\alpha}$ such that Y has zero projection on the root space which corresponds to the negative of the highest root let us define

$$\psi_Y(\exp Z) = \psi(B(Z, Y)) \quad Z \in \mathrm{Lie}(U)(\mathbb{A}).$$

Denote by $\theta_{E_7}^{\psi_Y}$ the space of functions

$$f^{R\psi_Y}(g) = \int_{R(\mathbb{A})U(F)\backslash U(\mathbb{A})} \psi_Y^{-1}(u) f^R(ug)\mathrm{d}u \tag{4.5}$$

on $E_7(\mathbb{A})$ as f varies in $V_{\theta_{E_7}}$. Assume that $\theta_{E_7}^{\psi_Y}$ is nontrivial and fix a finite place v. Consider the linear functional

$$l_Y(f) = f^{R\psi_Y}(1). \tag{4.6}$$

By restriction to $\theta_{E_{7,v}}$ the functional l_Y defines a linear functional $l_{Y,v}$ on the space $V_{\theta_{E_{7,v}}}$, such that

$$l_{U,v}(\theta_{E_{7,v}}(u)\xi) = \psi_{U,v}(u) l_{U,v}(\xi).$$

The first step is to show that $l_{U,v}$ defines a degenerate Whittaker model of $\theta_{E_{7,v}}$ in the sense of [MW]. The definition and detailed proof can be found in [MW] and [GRS]. The smallness of $\theta_{E_{7,v}}$ means that in the germ expansion of $\theta_{E_{7,v}}$ only one nontrivial nilpotent orbit occurs, namely the coadjoint orbit of highest weight ρ.

The main result of [MW] is that the set of nilpotent orbits that occur in the germ expansion of a representation coincides with the set of nilpotent orbits that contains an element Y such that a representation admits a Whittaker model with respect to Y. Thus $Y = 0$ or Y belongs to the orbit of X_ρ under $E_{7,v}$.

The next step is to show that if $Y \neq 0$, then Y belongs to the orbit under $D_{6,v}$ of $X_{-\alpha_1}$. This is proved similarly to proposition of 5.3 in [GRS].

The last step is to show that Y belongs to the orbit of $X_{-\alpha_1}$ under $D_6(F)$. □

Note that for $\theta_{E_7} \in V_{\theta_{E_7}}$

$$\sum_{\gamma \in Stab_M(\psi_U)(F)\backslash M(F)} \theta_{E_7}^{R\ \psi_U}(\gamma g) = \sum_{\gamma \in E(F)\backslash M(F)} \sum_{\varepsilon \in GL_1(F)} \theta_{E_7}^{R\ \psi_U}(h(\varepsilon)\gamma g),$$

where $h(\varepsilon)$ is any element of the torus T_{E_7} acting on $x_{\alpha_1}(r)$ by multiplication of the parameter r on ε.

We shall denote

$$\tilde{\theta}_{E_7}^{\psi_U}(g) = \sum_{\varepsilon \in GL_1(F)} \theta_{E_7}^{R\ \psi_U}(h(\varepsilon)g). \tag{4.7}$$

To simplify notations we shall write $\tilde{\theta}^{\psi}$ for $\tilde{\theta}_{E_7}^{\psi_U}$. So

$$\theta_{E_7}^R(g) = \theta_{E_7}^U(g) + \sum_{\gamma \in E(F)\backslash M(F)} \tilde{\theta}^{\psi}(\gamma g). \tag{4.8}$$

We shall need the following lemma, whose proof follows from [MW1,1.2.10].

LEMMA 4.7.1. *The series* $\sum_{\gamma \in \mathrm{Stab}_M(\psi_U)(F)\backslash M(F)} \theta_{E_7}^{\psi_U}(\gamma g)$ *is absolutely convergent and defines a function of moderate growth. Moreover, there are* $c > 0$ *and* T *such that*

$$\sum_{\gamma \in \mathrm{Stab}_M(\psi_U)(F)\backslash M(F)} |\theta_{E_7}^{\psi_U}(\gamma g)| \leq c|g|^T$$

for all $g \in E_7(\mathbb{A})$. *Here* $|\cdot|$ *is a norm on* $E_7(\mathbb{A})$ *as in* [MW1,12.2].

4.7. INVARIANCE PROPERTY OF THE FOURIER COEFFICIENTS

The next useful property is the invariance property of Fourier coefficients.

THEOREM 4.8. *For all* $f \in V_{\theta_{E_7}}$

$$f^{R\psi_U}(rg) = f^{R\psi_U}(g)\ \forall r \in \mathrm{Stab}_{M(\mathbb{A})}(\psi_U)U(\mathbb{A}).$$

The proof is very similar to the proof of Theorem 5.4 in [GRS]. □

Next we show that this invariance property uniquely defines θ^{ψ} in the following sense. We show that, for a local place v, the space of linear functionals on $\theta_{E_7\,v}$ such that

$$l(\theta_{E_7\,v}(r)\xi) = \psi_v(r)l(\xi) \tag{4.9}$$

for $r \in \mathrm{Stab}_M \psi_U \times U$ is one-dimensional. By Proposition 4.2 from [GRS] $\theta_{E_{7,v}}$ is also a subquotient of $\mathrm{Ind}_Q^{E_7} \delta_Q^{14/18}$. Thus it is enough to show

PROPOSITION 4.8.1. *The space* $\mathrm{Hom}_{E_7}(\mathrm{Ind}_{Q_v}^{E_{7,v}} \delta_{Q,v}^{14/18}, \mathrm{Ind}_{\mathrm{Stab}_M(\psi_U)U_v}^{E_{7,v}} \psi_v^{-1})$ *is one-dimensional.*

Proof. The proof uses Bruhat theory [W] and is similar to the proof of Theorem 6.2 in [GRS]. The detailed computation can be found in [G]. □

4.8. THE RESIDUE REPRESENTATION OF THE SIMILITUDE GROUPS GE_7 AND GD_6

Recall that there is a natural chain of simply connected split reductive groups

$$D_4 \subset D_5 \subset D_6 \subset E_7. \tag{4.10}$$

For all the groups of type D_i, $i = 4, 5, 6$ their minimal representations were constructed together with their automorphic realizations in [GRS] and the minimal representation θ_{E_7} was constructed in Theorem 4.3. Moreover, the automorphic realization of the minimal representation of every one of these groups depends on a automorphic realization of the minimal representations of the smaller groups in this sequence.

In this subsection our aim is to extend the automorphic minimal representation of E_7 to an automorphic irreducible square integrable representation θ_{GE_7} of GE_7, the similitude group of E_7. The extended representation should have properties similar to those of θ_{E_7}. For this it is necessary first to extend the minimal representation of θ_{D_i} to automorphic minimal representation θ_{GD_i} for $i = 4, 5, 6$.

Similarly to the first sequence there is a natural sequence of similitude groups

$$GD_4 \subset GD_5 \subset GD_6 \subset GD_7. \tag{4.11}$$

Recall that the group GE_7 is obtained from the group E_7 by adding a one-dimensional torus $h_8(t_8)$ such that $\alpha_2(h_8(t_8)) = t_8$ and $\alpha_i(h_8(t_8)) = 1$ for $i = 1, 3, 4, 5, 6, 7$. All other similitude groups GD_i are considered as subgroups of GE_7 generated by D_i and $h_8(t_8)$.

For any group H from the chain (4.10) we denote by GH the corresponding similitude group from the chain (4.11). Denote by $P_{E_7}(P_{GE_7})$ the parabolic subgroup of $E_7(GE_7)$ whose Levi subgroup is of type D_6, and by $P_{D_i}(P_{GD_i})$ the parabolic subgroup of $D_i(GD_i)$ whose Levi subgroup is of type D_{i-1}.

Let us fix a a multiplicative unitary character $\tilde{\sigma}$ of $\mathbb{A}^*$. For any group GH in the sequence (4.11), let σ be a character of $M_{P_{GH}}$ defined by $\sigma(g) = \tilde{\sigma}(s(g))$, where $s(g)$ is the similitude factor of g.

For $H = D_4, D_5, D_6, E_7$ set

$$s(H) = \begin{cases} 2/3, & H = D_4, \\ 3/4, & H = D_5, \\ 4/5, & H = D_6, \\ 14/17, & H = E_7. \end{cases}$$

For $s \in \mathbb{C}$ set $I_{GH}(s, \sigma) = \mathrm{Ind}_{M_{GH}}^{GH} \delta_{P_{GH}}^{s} \sigma$.

We can similarly define the Eisenstein series $E^{*}_{P(GH)}(g, f, s)$ for $f \in I_{GH}(s, \sigma)$.

Since $P_{GH} \backslash GH = P_H \backslash H$ the Eisenstein series $E^{*}_{P_{GH}}(g, f, s)$ has at most a simple pole at $s = s(H)$ and the pole is attained by a section, whose restriction on $H(\mathbb{A})$ gives a spherical function. The proof of this fact is a very slight modification of the proof of Theorem 2.3 in [GRS] for H of the type D_i and of the proof of Theorem 4.3, part (i) for $H = E_7$.

We define the minimal representation of a similitude group GH in the tower (4.11) by

$$\theta_{GH} = \mathrm{Span}\{(s - s(H)) E^{*}_{P_{GH}}(g, f, s)|_{s=s(H)} \text{ such that } f \in I_{GH}(s, \sigma)\}.$$

All these representations are square integrable and, hence, completely reducible, since the unitary character σ does not change the automorphic exponents. It follows from [S] for $G = E_7$ and from [GRS], pp. 91–92 for $G = D_i$, that the representation $\mathrm{Ind}_{P_{GH}}^{GH} \delta_{P_{GH}}^{s(H)} \sigma$ has a unique irreducible quotient. It follows, as in the case of simple groups, that θ_{GH} is an isotypic representation. Note that there is a unique up to scalars vector f in $\mathrm{Ind}_{P_{GH}}^{GH} \delta_{P_{GH}}^{s(H)} \sigma$, whose restriction to E_7 gives a spherical function. The pole is attained for this vector. Thus we conclude that θ_{GH} is an irreducible representation.

Theorems 4.7 and 4.8 are rewritten for θ_{GE_7} without any change. Theorem 4.6 could be rewritten as follows.

THEOREM 4.6′. *We have* $\theta^{U}_{GE_7}|_{GD_6} = \sigma \oplus \theta_{GD_6}$.

The proof follows immediately from the formula analogous to (4.3) for $f \in I_{GE_7}(s, \sigma)$. From now on we shall write P for P_{GE_7}.

The action of the center of $M = GD_6 \cdot GL_1$ on θ_{GD_6} can be read from (4.3). According to this action we extend θ_{GD_6} to the representation of $M = GD_6 \cdot GL_1$.

5. The Definition of the Lifting

Let F be a number field and $\mathbb{A}$ its ring of adèles. Let π be a cuspidal, irreducible representation of $G = (GL_2 \times GSp_4)^0(\mathbb{A})$ with central character ω_π. We realize V_π as a subspace of automorphic forms in $L^2_{\mathrm{cusp}}(G(F) \backslash G(\mathbb{A}))$. From now on let $G = (GL_2 \times GSp_4)^0$ which is a subgroup of $GL_2 \times GSp_4$. For every subgroup H of $GL_2 \times GSp_4$ we denote by H^0 the subgroup $H \cap G$. The center of the group $G(\mathbb{A})$ is one-dimensional. We parametrize it by

$$z(a) = (\mathrm{diag}(a, a), \mathrm{diag}(a, a, a, a)), \;\; a \in \mathbb{A}^{*}.$$

In our construction we consider only those representations π whose central character satisfies the following condition:

$$\omega_\pi(z(a)) = \tilde{\sigma}^2(a) \tag{5.1}$$

for some multiplicative unitary character $\tilde{\sigma}$ of $\mathbb{A}^{*}$.

In the previous section we defined the automorphic theta representation θ_{GE_7} depending on a multiplicative unitary character $\tilde{\sigma}$ and studied some of its properties.

The center of the group GE_7 coincides with the center of G and $z(a)$ when embedded in GE_7 is $h(a^4, a^7, a^8, a^{12}, a^9, a^6, a^3, a^{-2})$. Hence, $\theta_{GE_7}(z(a)g) = \tilde{\sigma}^{-2}(a)$. Thus, for the vector φ from the space of a representation, satisfying condition (5.1) we see that the function $\theta_{GE_7}((g_1, g_2)\varphi(g_1, g_2)$ is $Z_G(\mathbb{A})$-invariant from the left.

It follows from Section 2 that (G, Spin_7) is a commuting pair in GE_7. Thus given $\varphi \in V_\pi$ we may define the lifting of π from $(GL_2 \times GSp_4)^0$ to Spin_7 as

$$\theta_{GE_7}(\varphi)(g) = \int\limits_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta_{GE_7}(g, (g_1, g_2))\varphi\,(g_1, g_2)\mathrm{d}g_1\,\mathrm{d}g_2. \tag{5.2}$$

By the remark above, this integral is well-defined. By the cuspidality of φ this integral converges absolutely. The space of functions $\theta_{GE_7}(\varphi)(g)$ as φ varies in V_π and θ_{GE_7} varies in $V_{\theta_{GE_7}}$ defines an automorphic representation of $\mathrm{Spin}_7(\mathbb{A})$ which we denote by $\theta_{GE_7}(\pi)$.

Similarly one has that $(G, G\mathrm{Spin}_4)$ is a commuting pair in $GD_6 \cdot GL_1$. Thus given $\varphi \in V_\pi$ we may define the lifting of π from $(GL_2 \times GSp_4)^0$ to $G\mathrm{Spin}_4$ as

$$\theta_{GD_6}(\varphi)(g) = \int\limits_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta_{GD_6}(g, (g_1, g_2))\varphi(g_1, g_2)\mathrm{d}g_1\,\mathrm{d}g_2. \tag{5.3}$$

The center of $GD_6 \cdot GL_1$ contains the center of G. Since π satisfies (5.1) the integral (5.3) is well defined.

6. The First Cuspidality Property

In this section we prove the following theorem:

THEOREM 6. *Let π be an irreducible cuspidal representation of G satisfying the condition* (5.1). *If $\theta_{GD_6}(\pi) = 0$, then $\theta_{GE_7}(\pi)$ is cuspidal.*

Proof. We need to show that if

$$\int\limits_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta_{GD_6}(g, (g_1, g_2))\varphi(g_1, g_2)\mathrm{d}g_1\mathrm{d}g_2 \tag{6.1}$$

is zero for all choice of data, then

$$\int\limits_{V(F)\backslash V(\mathbb{A})} \int\limits_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta_{GE_7}(vg, (g_1, g_2))\varphi\,(g_1, g_2)\mathrm{d}g_1\,\mathrm{d}g_2\,\mathrm{d}v \tag{6.2}$$

is zero for all maximal unipotent radicals V of Spin_7, all $\theta_{GE_7} \in V_{\theta_{GE_7}}$ and all $\varphi_i \in V_{\pi_i}$. We may assume $g = 1$.

The proof is computational. We replace $\theta^R_{GE_7}$ by its Fourier expansion along $R(\mathbb{A})U(F)\backslash U(\mathbb{A})$ and then study separately the contribution of each term.

Recall that U is a Heisenberg group with $[U, U] = R$, where R is the one-dimensional unipotent group corresponding to the highest root (2234321). Notice that R is contained in all unipotent radicals of Spin_7. Hence the integral (6.2) equals

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \int_{R(\mathbb{A})V(F)\backslash V(\mathbb{A})} \int_{F\backslash\mathbb{A}} \theta_{GE_7}(x_{2234321}(r)v, (g_1, g_2))\varphi\,(g_1, g_2)\mathrm{d}r\,\mathrm{d}v\,\mathrm{d}g_1\,dg_2. \tag{6.3}$$

By (4.8)

$$\theta^R_{GE_7}(v, (g_1, g_2)) = \theta^U_{GE_7}(v, (g_1, g_2)) + \sum_{\gamma\in E(F)\backslash M(F)} \tilde{\theta}^{\psi}(\gamma(v, (g_1, g_2)).$$

Plugging this into the integral (6.3) we need to show that

$$\begin{aligned}&\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \int_{R(\mathbb{A})V(F)\backslash V(\mathbb{A})} \theta^U_{GE_7}(v, (g_1, g_2))\varphi(g_1, g_2)\mathrm{d}v\,\mathrm{d}g_1\,\mathrm{d}g_2+\\ &+\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \int_{R(\mathbb{A})V(F)\backslash V(\mathbb{A})} \sum_{\gamma\in E(F)\backslash M(F)} \tilde{\theta}^{\psi}(\gamma(v, (g_1, g_2))\varphi(g_1, g_2)\,\mathrm{d}v\,\mathrm{d}g_1\,\mathrm{d}g_2\end{aligned} \tag{6.4}$$

vanishes for all choices of data. We compute each summand separately. To show the first summand vanishes it is enough to show that

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta^U_{GE_7}(g_1, g_2)\varphi(g_1, g_2)\mathrm{d}g_1\,\mathrm{d}g_2$$

vanishes for all choices of data. By Theorem 4.6 we have

$$\theta^U_{GE_7}(g) = \sigma(g) + \theta_{GD_6}(g), \quad \text{for } g \in GD_6.$$

Thus it suffices to show that

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \sigma(g_1, g_2)\varphi(g_1, g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2$$

and

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta_{GD_6}((g_1, g_2))\varphi(g_1, g_2)\mathrm{d}g_1\,\mathrm{d}g_2$$

are both zero for all choices of data. The first integral vanishes since φ is a cusp form. The second one vanishes by our assumption.

Thus to prove that the constant term along V of $\theta_{GE_7}(\pi)$ is zero we need to show that

$$\int_{R(\mathbb{A})V(F)\backslash V(\mathbb{A})} \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \sum_{\gamma\in E(F)\backslash M(F)} \tilde{\theta}^{\psi}(\gamma(v, (g_1, g_2))\varphi(g_1, g_2)\,\mathrm{d}v\,\mathrm{d}g_1\,\mathrm{d}g_2 \tag{6.5}$$

vanishes for all choices of data.

The integral (6.5) equals $\sum_{\gamma\in P(2,4,5,6,7)\backslash M/G}\int_{R(\mathbb{A})V(F)\backslash V(\mathbb{A})} J_{\gamma,\varphi}(v)\mathrm{d}v$, where

$$J_{\gamma,\varphi}(v)=\int\limits_{Z(\mathbb{A})\Gamma_\gamma(F)\backslash G(\mathbb{A})}\tilde{\theta}^{\psi}(\gamma(v,(g_1,g_2))\varphi(g_1,g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2,\tag{6.6}$$

and $\Gamma_\gamma(F)=(P(2,4,5,6,7)^{\gamma}(F)\cap G(F))$.

By straightforward computations we shall show that each $\int_{R(\mathbb{A})V(F)\backslash V(\mathbb{A})} J_\gamma(\varphi)(v)\,\mathrm{d}v$ vanishes for every cuspidal φ. Our proof is similar to the proof of Proposition in [GRS1]. The computations are rather involved so we do not write it here in any detail. The detailed computation can be found in [G]. Here we only state the main reasons for the vanishing of $\int_{R(\mathbb{A})V(F)\backslash V(\mathbb{A})} J_{\gamma,\varphi}(v)\,\mathrm{d}v$:

(1) For some γ the integral $J_\gamma(\varphi)$ happens to contain an inner integral of a cusp form over the group G and, hence, vanishes by cuspidality. A typical example is $\gamma=e$.
(2) The integral $J_\gamma(\varphi)$ contains an inner integral of a cusp form over some unipotent radical of G and hence vanishes by cuspidality. The example for that case is $\gamma=w[3456]$. Note, that in these cases we do not use at all an integration over the unipotent radical of Spin_7.
(3) Fix a representative of the double coset γ. Assume that for some root $\tilde{\alpha}$ in V one has $\gamma x_{\tilde{\alpha}}(r)\gamma^{-1}=x_{\alpha_1}(r)g_0$, where $g_0\in \mathrm{Stab}_{M(\mathbb{A})}\psi\; U(\mathbb{A})$. Thus, using the invariance property of $\tilde{\theta}^{\psi}$ we have $\tilde{\theta}^{\psi}(\gamma x_{\tilde{\alpha}}(r)g)=\sum_{\varepsilon\in F^*}\psi(\varepsilon r)\theta^{\psi}(\gamma g)$.

Hence, the integral of $J_{\gamma,\varphi}$ contains an integral of the character and hence vanishes. We say then that we use the root $\tilde{\alpha}$ in V to show the vanishing of $\int_{R(\mathbb{A})V(F)\backslash V(\mathbb{A})} J_{\gamma,\varphi}(v)\,\mathrm{d}v$.

For example, to show vanishing of $\int_{R(\mathbb{A})V(F)\backslash V(\mathbb{A})} J_{\gamma,\varphi}(v)\,\mathrm{d}v$ for $\gamma=w_0$, where w_0 is a representative of the longest Weyl element of M we use the root (1234321).

(4) To prove the statement for $\gamma=w[345672456]$ we once more use the property of the tower. Namely, the integral $J_\gamma(\varphi)$ contains an inner integral

$$I(\varphi)=\int\limits_{Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A})}\varphi(g,(g,h))\mathrm{d}g\,\mathrm{d}h,\tag{6.7}$$

where

$$G'=\left\{g=\begin{pmatrix} a & b\\ c & d\end{pmatrix},(g,h)=\left(\begin{array}{c|cc|c} a & 0 & 0 & b\\ \hline 0 & & & 0\\ & & h & \\ 0 & & & 0\\ \hline c & 0 & 0 & d\end{array}\right):\det(g)=\det(h)\right\}.$$

We prove the following

PROPOSITION 6.1. *If* $\theta_{GD_6}(\pi) = 0$ *then* $I(\varphi) = 0$ *for any cuspidal* φ.

Proof. The proof is straightforward. We use the Fourier expansion theorem for θ_{GD_6}.

Assume $\theta_{GD_6}(\pi) = 0$. This means

$$\int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta_{GD_6}(g_1, g_2)\varphi(g_1, g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2 \tag{6.8}$$

vanishes for all choices of data. Thus

$$\int_{F\backslash\mathbb{A}} \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta_{GD_6}(x_{0112221}(r)(g_1, g_2))\varphi(g_1, g_2)\psi(r)\,\mathrm{d}g_1\,\mathrm{d}g_2\,\mathrm{d}r. \tag{6.9}$$

vanishes for all choices of data. We replace in (6.9) the function θ_{GD_6} by its Fourier expansion and compute separately the contribution of every term. The contribution of the constant term $\theta^{U'}_{GD_6}$ is zero since it contains $\int_{F\backslash\mathbb{A}} \psi(r)\,\mathrm{d}r$ as the inner integration. Computing the rest of the terms we get that the integral (6.9) contains $I(\varphi)$ as the inner integral. It remains to prove that the vanishing of (6.9) implies the vanishing of $I(\varphi)$. This is also done straightforwardly. The idea of the proof is similar to the idea of the proof of Theorem 9 below. The detailed computations can be found in [G]. □

(5) For $\gamma = w[3452436576]u$ where u belongs to some 10-dimensional unipotent subgroup the proof of the vanishing $\int_{R(\mathbb{A})V(F)\backslash V(\mathbb{A})} J_{\gamma,\varphi}$ requires some work. This is the only case when we use in the sense of (3) *simple* roots to show the vanishing of the contribution of γ. After all, it vanishes for the same reasons as described in (1)–(3) but one has to prove it separately for every maximal radical V of Spin_7. This finishes the proof of Theorem 6. □

7. The Second Cuspidality Property

In this section we prove the following theorem:

THEOREM 7. *Let* π *be an irreducible cuspidal representation of* G *satisfying the condition* (5.1). *Then* $\theta_{GD_6}(\pi)$ *is cuspidal.*

Proof. Recall that $\theta_{GD_6}(\pi)$ is a representation of $G\mathrm{Spin}_4$ which is a Levi subgroup $M(\alpha, \gamma)$ of the group Spin_7. The group $G\mathrm{Spin}_4$ has two maximal parabolics. One of the unipotent radicals of $G\mathrm{Spin}_4$ is generated by $x_\alpha(r)$ and another is generated by $x_\gamma(r)$. Hence, we need to show

$$\int_{F\backslash\mathbb{A}} \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta_{GD_6}(v, (g_1, g_2))\varphi(g_1, g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2\,\mathrm{d}v \tag{7.1}$$

vanishes for all choices of data, where $v = x_\alpha(r)$ or $v = x_\gamma(r)$. We prove this first for $v = x_\alpha(r)$ and the second case is done similarly. The group G contains a subgroup

$\tilde{G} = SL_2 \times Sp_4$. Recall that $\theta_{GD_6}(h) = \theta_{D_6}(h)$ for $h \in SO_{12}(\mathbb{A})$. The integral (7.1) contains as inner integral

$$\int\limits_{F\backslash\mathbb{A}} \int\limits_{\tilde{G}(F)\backslash\tilde{G}(\mathbb{A})} \theta_{GD_6}(x_\alpha(r), (g_1, g_2))\varphi(g_1, g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2\,\mathrm{d}r. \tag{7.2}$$

Hence it is enough to show that (7.2) equals zero.

Here, by the remark following Theorem 6.8 in [GRS], we can replace GD_6 by SO_{12}. Indeed, there is a natural map j: $G\mathrm{Spin}_{12} \mapsto SO_{12}$. Since $(GL_2 \times GSp_4)^0$ imbeds in the image of j over any field we have

$$\theta_{G\mathrm{Spin}_{12}}|_{(GL^2\times GSp_4)^0} = \theta_{G\mathrm{Spin}_{12}} \circ j^{-1}|_{(GL_2\times GSp_4)^0}.$$

So $\theta_{G\mathrm{Spin}_{12}}(\pi) = 0$ if and only if $\theta_{SO_{12}}(\pi) = 0$.

By Theorem 6.9 of [GRS] automorphic forms of $\theta_{SO_{2m}}$ are obtained by a regularized theta lift of identity representation of SL_2. To be more precise, let us set some notation.

Fix an additive character ψ on $F\backslash\mathbb{A}$. Consider the Weil representation ω_ψ of $\widetilde{Sp}_{4n}(\mathbb{A})$ and restrict it on the dual pair $SL_2(\mathbb{A}) \times SO_{2n}(\mathbb{A})$. Recall that ω_ψ acts on the Schwartz space $S(\mathbb{A}^{2n})$. Let $\phi \in S(\mathbb{A}^{2n})$ and consider the theta-series

$$\theta_{2n}(g, h; \phi) = \sum_{\xi\in F^{2n}} \omega_\psi(g, h)\phi(\xi)$$

for $g \in SO_{2n}(\mathbb{A})$ and $h \in SL_2(\mathbb{A})$.

THEOREM (Theorem 6.9, [GRS]).

$$\theta_{SO_{2n}} = \mathrm{Span}\left\{ \int\limits_{SL_2(F)\backslash SL_2(\mathbb{A})} \theta_{Sp_{4n}}(g, h; \omega_\psi(1, \Omega'_m)\phi)\mathrm{d}h | \phi \in S(\mathbb{A}^{2m}) \right\}.$$

Here $\omega_\psi(1, \Omega'_m)\phi$ is a regularization of ϕ at one fixed Archimedean place, its presence it necessary to make the last integral absolutely convergent. For more details we refer the reader to [GRS].

Thus to show that (7.2) vanishes for all data that it is enough to show

$$\int\limits_{F\backslash\mathbb{A}} \int\limits_{\tilde{G}(F)\backslash\tilde{G}(\mathbb{A})} \int\limits_{SL_2(F)\backslash SL_2(\mathbb{A})} \theta_{Sp_{24}}(x_\alpha(r), (g_1, g_2), h; \phi)\varphi(g_1, g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2\,\mathrm{d}r\,\mathrm{d}h$$

vanishes for all data. Note that this integral makes sense even without the presence of Ω'_6. We compute this integral using the definition and the properties of the standard Weil representation. For reasons similar to those in Subsections 6.1 and 6.2, this integral vanishes. Thus, the lift is cuspidal. The full details can be found in [G]. □

8. Whittaker Model for $F \in V_{\mathrm{II}}$

The aim of this section is to express the Whittaker model of $\theta_{GE_7}(\pi)$ in terms of the Whittaker model of π.

Recall the definition of the Whittaker model of an automorphic representation π of a reductive group $H(\mathbb{A})$. Let Ψ be a nondegenerate character of $N_H(\mathbb{A})$. This means that Ψ is nontrivial on the space defined by each simple root. We consider the space generated by functions

$$W_f(g) = \int\limits_{N_H(F)\backslash N_H(\mathbb{A})} f(ng)\Psi(n)\,\mathrm{d}n$$

as f varies in V_π. If this space is not zero, we say that the representation π has a Whittaker model. Under right action this space becomes a representation of H which is equivalent to π. For $F \in V_{\theta_{GE_7}(\pi)}$ the Whittaker model of F is defined by

$$W_F(g) = \int\limits_{N_{\mathrm{Spin}_7}(F)\backslash N_{\mathrm{Spin}_7}(\mathbb{A})} F(ng)\Psi(n)\,\mathrm{d}n, \tag{8.1}$$

where Ψ is the nondegenerate character on N_{Spin_7} defined using the character ψ of $F\backslash\mathbb{A}$. We compute

$$W_F(g) = \int\limits_{N_{\mathrm{Spin}_7}(F)\backslash N_{\mathrm{Spin}_7}(\mathbb{A})} \int\limits_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta_{GE_7}(ng, (g_1, g_2))\varphi(g_1, g_2)\Psi(n)\,\mathrm{d}g_1\,\mathrm{d}g_2\,\mathrm{d}n. \tag{8.2}$$

Using (4.8) we represent

$$F^R(g) = J_{U,\varphi}(g) + \sum_{\gamma\in E(F)\backslash M(F)/G(F)} J_{\gamma,\varphi}(g),$$

where $J_{\gamma,\varphi}(g)$ is defined in Section 6 and

$$J_{U,\varphi}(g) = \int\limits_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \theta^U_{GE_7}(g, (g_1, g_2))\varphi(g_1, g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2.$$

Let us compute the contribution of every summand to $W_F(g)$.

(1) The contribution from $J_{U,\varphi}$ vanishes since $J_{U,\varphi}(x_\beta(r)g) = J_{U,\varphi}(g)$ for any g, r and, hence, the contribution from it contains $\int_{F\backslash\mathbb{A}} \psi(r)\,\mathrm{d}r$.

(2) In the proof of Theorem 6 we showed that

$$\int\limits_{R(\mathbb{A})V(F)\backslash V(\mathbb{A})} J_{\gamma,\varphi}(vg)\,\mathrm{d}v$$

vanishes for all choices of data and all γ. Actually, in the proof of Theorem 6 for all γ but the case $\gamma = w[3452436576]u$ in subsection 6.4 we do not use simple roots of the radical of Spin_7 to show the vanishing of the corresponding integral. This means that we proved a stronger result for those γ. Namely

$$\int\limits_{R(\mathbb{A})V'(F)\backslash V'(\mathbb{A})} J_{\gamma,\varphi}(v'g)\,\mathrm{d}v' \tag{8.3}$$

vanishes for all choices of data, where $V' \subset V$ is the subgroup generated by all roots of V but the simple one. But the integral (8.3) is an inner integral of $W_F(g)$. Thus it remains to compute the contribution of $\gamma = wu$. Computing carefully we get

$$\begin{aligned} W_F(g) = \int\limits_{\mathbb{A}^2} \int\limits_{Z(\mathbb{A})N_G(\mathbb{A})\backslash G(\mathbb{A})} &\theta^{\psi}(wn(1)x_{\alpha}(r_1)x_{\gamma}(r_2)g, (g_1, g_2))\times \\ &\times \psi(r_1 + r_2)W_{\varphi}(g_1, g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2\,\mathrm{d}r_1\,\mathrm{d}r_2, \end{aligned} \tag{8.4}$$

where $n(1) = x_{0000111}(1)x_{0011111}(1)x_{0111110}(1)$. The integral (8.4) gives the required expression of W_F in terms of the Whittaker model of π. We shall show now that this expression is Eulerian, i.e. the integrand is factorizable. By the uniqueness of the Whittaker model we have that W_{φ} is factorizable for φ corresponding to a factorizable vector. It remains to show that θ^{ψ} is factorizable.

PROPOSITION 8.1. *Let θ be a function in the space of θ_{GE_7} corresponding to a factorizable vector and $g = \otimes g_v \in GE_7(\mathbb{A})$. Then $\theta^{\psi}(g) = \prod_v \theta_v^{\psi}(g_v)$. Moreover, for almost all v there exists $f_v \in \mathrm{Ind}_{P_v}^{GE_{7,v}} \delta_{P_v}^{3/17}\sigma_v$ such that*

$$\theta_v^{\psi}(g_v) = \int\limits_{F_v} f_v(w_1 x_{\alpha_1}(r)g_v)\psi(r)\,\mathrm{d}r$$

Proof. Since θ_{GE_7} is irreducible, we have $V_{\theta_{GE_7}} = \bigotimes_v V_{\theta_{GE_7},v}$. Let $\theta = \otimes_v \theta_v$ be a factorizable vector in $V_{\theta_{GE_7}}$. As explained in 4.7, the functional $l_U = l_{X_{-\alpha_1}}$ sending vector θ to $\theta^{\psi}(1)$, defines by restriction the local functional $l_{U,v}$ satisfying the condition (4.9). The space of functionals satisfying this condition is one-dimensional by Proposition 4.8.1. Thus

$$\theta^{\psi}(g) = g.\theta^{\psi}(1) = l_U(g.\theta) = \Pi_v l_{U,v}(g_v, \theta_v)$$

and so is factorizable.

Let us explicitly construct such a local functional $l_{U,v}$ for the places where $\theta_{GE_{7}v}$ is unramified. Recall that $\theta_{GE_{7}v} \subset \mathrm{Ind}_{P_v}^{GE_{7},v}\delta_{P_v}^{3/17}\sigma_v$. Thus it is sufficiant to construct this functional on the induced space, nontrivial on the space of θ_{GE_7}. For $f_v \in \mathrm{Ind}_{P_v}^{GE_{7v}}$ $\delta_{P_v}^{3/17}\sigma_v$ define $l_{U,v}(f_v) = \int_{F_v} f_v(w_1 x_{\alpha_1}(r))\psi(r)\,\mathrm{d}r$. The functional l_v obviously satisfies (4.24). Moreover, it is a GL_2-Whittaker type integral and so is known to be converged. Recall that if $\theta_{GE_{7}v}$ is unramified, then it is generated by the spherical vector $f_v^0 \in \mathrm{Ind}_{P_v}^{GE_{7v}}\delta_{P_v}^{3/17}\sigma_v$. For such a spherical f_v^0 it is easy to compute $l_{U,v}(f_v^0)$ explicitly and to see it does not vanish. Thus $l_{U,v}$ is nontrivial on the space $\theta_{GE_{7}v}$.

So for every vector θ_v from the space of $\theta_{GE_{7v}}$ there exists $f_v \in \mathrm{Ind}_{P_n u}^{GE_{7v}}\delta_{P_v}^{3/17}\sigma_v$ such that

$$\theta_v^{\psi}(g_v) = l_{U,v}(g_v.\theta) = \int_{F_v} f_v(w_1 x_{\alpha_1}(r) g_v)\psi(r)\,\mathrm{d}r.$$

Proposition 8.1 follows. □

The proposition above implies that the expression (8.4) for $W_F(g)$ is Eulerian. This enables us to define a local theta-correspondence.

More precisely, let $\pi = \otimes\pi_v$ be a cuspidal irreducible generic representation of $G = (GL_2 \times GSp_4)^0$. Then obviously any π_v is an irreducible generic representation. Let $\varphi \in V_\pi$ be an automoprhic function corresponding to a factorizable vector $\otimes_v v_v$. Then $W_\varphi(g) = \Pi_v W_{v_v}(g_v)$ for $g = \otimes g_v \in G(\mathbb{A})$. Let θ be as in the proposition, so $\theta^{\psi}(g) = \Pi_v \theta_v^{\psi}(g_v)$. We define the local theta lifting of π_v which we denote by $\theta_{GE_7}(\pi_v)$. The space of $\theta_{E_7}(\pi_v)$ is defined to be span of function of the form:

$$\int_{F^2} \int_{Z(F_v)N_G(F_v)\backslash G(F_v)} \theta_v^{\psi}(x_\alpha(r_1)x_\gamma(r_2)g, (g_1, g_2)) \times \\ \times \psi(r_1 + r_2)\psi(r) W_{v_v}(g_1, g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r \tag{8.5}$$

for all $\theta_v \in \theta_{GE_{7\,v}}$ and $v_v \in V_{\pi_v}$.

Moreover, assume that F_v is a local field such that the local representation $\theta_{GE_{7\,v}}$ is unramified. Then by the proposition the local theta-correspondence of π_v is spanned by all functions of the form

$$\int_{F^2} \int_{Z(F_v)N_G(F_v)\backslash G(F_v)} \int_{F_v} f_v(w_1 x_{\alpha_1}(r) w n(1) x_\alpha(r_1) x_\gamma(r_2) g, (g_1, g_2)) \times \\ \times \psi(r_1 + r_2)\psi(r) W_{v_v}(g_1, g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}r \tag{8.6}$$

for all f runs over $\theta_{GE_{7\,v}} \subset \mathrm{Ind}_{P_v}^{GE_{7v}} \delta_{P_v}^{3/17}\sigma_v$, and $v \in V_{\pi_v}$.

9. A Nonvanishing Result

In this section we prove the following theorem.

THEOREM 9. *Let* π *be an irreducible cuspidal generic representations of G satisfying condition (5.1). Then the Whittaker model of* $\theta_{GE_7}(\pi)$ *is not zero. In particular* $\theta_{GE_7}(\pi) \neq 0$.

COROLLARY. *The local theta lifting of* π_v *as defined in* (8.5) *is nontrivial.*

Proof. Let φ and θ be functions in the space of π and θ respectively such that $W_\varphi(1) = 1$ and $\theta^{\psi}(1) = 1$. In the previous section we showed that $W_{\theta_{GE_7}(\varphi)}(g) = \otimes_v W_{\theta_{GE_7}(\varphi)_v}(g_v)$ where $W_{\theta_{GE_7}(\varphi)_v}(g_v)$ is given by (8.5) for all v and by (8.6) for all v outside a finite set S.

LEMMA 9.1. $\prod_{v\notin S} W_{\theta_{GE_7}(\varphi)_v}(1) = \prod_{v\notin S} \theta^{\psi}(1)$. *In particular this product is not zero.*

Proof. This is a direct computation. Note that the computation of the left-hand side is a special case $t_a = 1$ of Proposition 10.1. $\square$

It remains to show that for $v \in S$, $W_{\theta(\varphi)_v}$ does not vanish identically. Assume the integral (8.5) is zero for all choices of data. Consider

$$u = x_{1122221}(u_1)x_{1123221}(u_2)x_{1123321}(u_3)x_{1223321}(u_4)x_{1224321}(u_5).$$

By abuse of notations we also denote by u the vector $(u_1, u_2, .., u_5)$. Let Φ be a function in the Schwarz space $S(F_v{}^5)$ and write $\Phi(u)$ for $\Phi(u_1, \dots, u_5)$. Then

$$\int\limits_{Z(F_v)N(F_v)\backslash G(F_v)} \int\limits_{F_v{}^2} \int\limits_{F_v{}^5} \theta^{\psi}(wn(1)x_{0112221}(r_1)x_{0111000}(r_2)(g_1, g_2)u) \times \\ \times \psi(r_1 + r_2)W_{\varphi}(g_1, g_2)\Phi(u)\,\mathrm{d}u\,\mathrm{d}r_i\,\mathrm{d}g_i$$

vanishes for all choices of data. We conjugate u to the left. The group G acts on u as follows: $(g_1, g_2)u(g_1, g_2)^{-1} = \rho(g_2).u$, where ρ is the 5-dimensional irreducible representation of GSp_4. After changing variables $u \mapsto \rho(g_2{}^{-1}).u$ and using the invariance properties of θ^{ψ} we get

$$\int\limits_{Z(F_v)N(F_v)\backslash G(F_v)} \int\limits_{F_v{}^2} \theta^{\psi}(wn(1)x_{0112221}(r_1)x_{0111000}(r_2)(g_1, g_2)) \times \\ \times \psi(r_1 + r_2)W_{\varphi}(g_1, g_2) \int\limits_{F_v{}^5} \Phi(\rho(g_2)^{-1}.u)\psi(u_1)\,\mathrm{d}r_i\,\mathrm{d}g_i\,\mathrm{d}u$$

vanishes for all choices of data. Note that

$$\int\limits_{F_v{}^5} \Phi(\rho(g_2)^{-1}u)\psi(u_1)\mathrm{d}u = \widehat{\Phi}(\rho(g_2)^{-1}(1, 0, \dots, 0)),$$

where $\widehat{\Phi}$ is a Fourier transform of Φ.

The orbit of the vector $e = (1, 0, \dots, 0)$ is open in its Zarissky closure. Hence as Φ varies over all Schwarz functions on F_v^5, the function $\widehat{\Phi}(\rho(g^{-1}.e)$ varies over all Schwarz functions on $\mathrm{Stab}_G(e)\backslash G(F_v)$. The stabilizer in G of the vector $(1, 0, \dots, 0)$ is

$$\left\{(g_1, g_2) \in Z_G N_G\backslash G{:}\, g_2 = \begin{pmatrix} g & * \\ 0 & g^* \det(g_1) \end{pmatrix}\right\}$$

where $g \in N_{SL_2}\backslash SL_2$. Hence one has

$$\int\limits_{Z(F_v)N(F_v)\backslash GL_2(F_v)\times SL_2(F_v)} \int\limits_{F_v{}^2} \theta^{\psi}(wn(1)x_{0112221}(r_1)x_{0111000}(r_2)(g_1, g_2)) \times \\ \times \psi(r_1 + r_2)W_{\varphi(g_1, g_2)}\,\mathrm{d}r_i\,\mathrm{d}g_i$$

vanishes for all choices of data.

Let us consider

$$u = x_{1122110}(u_1)x_{1122111}(u_2)x_{1122210}(u_3)x_{1122211}(u_4).$$

Note that $GL_2 \times SL_2$ acts on $u = (u_1, u_2, u_3, u_4)$ by the tensor product. Arguing as above, we obtain

$$\int_{Z(F_v)\backslash T_1(F_v)\times T_2(F_v)} \int_{F_v{}^2} \theta^{\psi}(wn(1)x_{0112221}(r_1)x_{0111000}(r_2)(t_1, t_2))\times$$
$$\times\, \psi(r_1 + r_2)W_{\varphi(t_1,t_2)}\, \mathrm{d}r_i\, \mathrm{d}t_i$$

is zero for any choice of data, where

$$T_G \supset T_1 \times T_2 \ni t = \mathrm{diag}(a, a^{-1}\lambda, \mathrm{diag}(a^{-1}, a, a^{-1}\lambda, a\lambda).$$

Applying the same argument for $u = x_{1011111}(u_1)$ and then for $u = x_{1010000}(u_1)$ we finally get $\theta^{\psi}(wn(1))W_{\varphi}(1)$ is zero for all choices of data. Since θ^{ψ} does not vanish and π is generic we get a contradiction. Hence $\theta_{GE_7}(\pi) \neq 0$, as required. □

10. The Unramified Computation

In this section F denotes a local non-Archimedian field with the uniformizing element p and cardinality of the residue field is q. In this section we write H for $H(F)$ for any algebraic group H. We omit the subscription v if there is no confusion. By ψ we denote an additive character on F whose conductor is the ring of integers O_F.

10.1. PRELIMINARIES.

By definition of the L-group of any reductive group H there are isomorphisms

$$X^*(T_H) \overset{\rho_*}{\simeq} X_*({}^L T_H), \quad X_*(T_H) \overset{\rho^*}{\simeq} X^*({}^L T_H). \tag{10.1}$$

Recall that every unramified representation σ of a group H is a unique unramified subquotient of $\mathrm{Ind}_{B_H}^H \chi$ for some unramified character χ of the torus T_H. To indicate this, we write σ_χ for such σ. And to every unramified character χ there corresponds the Langlands parameter t_χ in the torus of the dual group ${}^L H$.

Given an unramified representation σ of a reductive group H and a finite dimensional representation r of the dual group ${}^L H$, the L-function of a complex variable s is defined as:

$$L(\sigma_\chi, r, s) = \left(\det(I - r(t_\chi)q^{-s})\right)^{-1}. \tag{10.2}$$

Here I stands for the identity matrix.

If H is a complex reductive group of rank k we denote by $[n_1, n_2, ..n_k]$ the finite-dimensional representation with the highest weight $\sum_{i=1}^{k} n_i\omega_i$ where ω_i denotes the ith fundamental weight.

The values of the unramified Whittaker function on the torus can be computed using the famous Casselman–Shalika formula [Ca]. Assume Π is an unramified representation of a reductive group H corresponding to the unramified character χ

and W_χ is the unique K-fixed vector in its Whittaker model normalized such that $W_\chi(e) = 1$. Here K denotes the maximal compact subgroup of H. Let us define

$$T_H \supset A_H = \{t : |\alpha(t)| \leqslant 1 \quad \forall\alpha \in \phi^+(H)\}. \tag{10.3}$$

Then

$$W_\chi(t) = (tr\rho^*(t))(\rho_*(\chi))\delta_B{}^{1/2}(t) \tag{10.4}$$

for $t \in A_H$ and zero otherwise. Applying the Weyl character formula to the right-hand side we get the second version of the Casselman–Shalika formula:

$$W_\chi(t) = C(H) \sum_{w \in W(H)} \mathrm{d}_w(\chi)(w\chi)(t)\delta_B^{1/2}(t) \tag{10.5}$$

for $t \in A_H$ and zero otherwise. For precise definitions of $C(H)$ and $\mathrm{d}_w(\chi)$ see[Ca]. Denote

$$K_\chi(g) = W_\chi(g)\delta_B{}^{-1/2}(g). \tag{10.6}$$

10.2. REPRESENTATIONS AND THEIR UNRAMIFIED CHARACTERS

Let $\pi_{\nu,\mu}$ and Π_χ be unramified representations of the groups $(GL_2 \times GSp_4)^0(F)$ and $\mathrm{Spin}_7(F)$ respectively, where the unramified characters ν, μ and χ are defined as follows:

$$(\nu, \mu)(\mathrm{diag}(a, a^{-1}\lambda), \mathrm{diag}(b, c, \lambda c^{-1}, \lambda b^{-1})) = \nu(a)\mu_1(b)\mu_2(c)\mu_3(\lambda),$$

$$\chi\big(\mathrm{diag}(abc, a, b, c, c^{-1}, b^{-1}, a^{-1}, (abc)^{-1}\big) = \chi_1(a)\chi_2(b)\chi_3(c),$$

where ν, μ_i, χ_i are unramified characters of F^*.

To simplify the notations we omit below the argument p from the expressions $\nu(p), \mu_i(p), \chi_i(p)$. The Langlands parameters corresponding to these representations are given by

$$t_{\nu,\mu} = \big(\mathrm{diag}(\nu(\mu_1\mu_2)^{1/2}\mu_3, (\mu_1\mu_2)^{1/2}\mu_3), \mathrm{diag}(\nu^{1/2}\mu_1\mu_2\mu_3,$$
$$\nu^{1/2}\mu_1\mu_3, \nu^{1/2}\mu_2\mu_3\nu^{1/2}\mu_3)\big),$$

where $t_{\nu,\mu} \in (GL_2 \times GSp_4)^0/\pm 1(\mathbb{C})$ which is the L-group of G. Next t_χ equals

$$\mathrm{diag}\left(\left(\frac{\chi_1\chi_2}{\chi_3}\right)^{1/2}, \left(\frac{\chi_1\chi_3}{\chi_2}\right)^{1/2}, \left(\frac{\chi_2\chi_3}{\chi_1}\right)^{1/2}, \left(\frac{\chi_2\chi_3}{\chi_1}\right)^{-1/2}, \left(\frac{\chi_1\chi_3}{\chi_2}\right)^{-1/2}, \left(\frac{\chi_1\chi_2}{\chi_3}\right)^{-1/2}\right),$$

where $t_\chi \in PSp_6(\mathbb{C}) = {}^L\mathrm{Spin}_7(F)$.

The map of L-groups $r: (GL_2 \times GSp_4)^0(\mathbb{C})/\pm 1 \mapsto PSp_6(\mathbb{C})$ is defined in the obvious way. We see that

$$r(t_{\nu,\mu}) = \mathrm{diag}\left(\nu^{1/2}, (\mu_1\mu_2)^{1/2}, \left(\frac{\mu_1}{\mu_2}\right)^{1/2}, \left(\frac{\mu_2}{\mu_1}\right)^{1/2}, (\mu_1\mu_2)^{-1/2}, \nu^{-1/2}\right).$$

Let $\pi = \otimes_v \pi_v$ be a cuspidal irreducible generic representation of $G(\mathbb{A})$ and $\Pi = \otimes_v \Pi_v$ be an irreducible generic representation contained in $\theta_{GE_7}(\pi)$. Such Π always exists in the case when θ_{GE_7} is cuspidal. Let us denote the Langlands parameters of π_v and Π_v by $t_{v,u}$ and t_χ, respectively.

In this section our goal is to prove the following theorem:

THEOREM 10. *For π_v, Π_v as above one has $r(t_{v,\mu}) = t_\chi$.*

Proof. Let us denote the unramified representation of Spin_7 whose Langlands parameter is $r(t_{v,\mu})$ by $\Pi_{r(v,\mu)}$. Then it is enough to prove that $\Pi_\chi = \Pi_{r(v,\mu)}$. The proof of Theorem 10 follows from Propositions 10.1–10.6 below. First we show how these propositions imply Theorem 10 and then prove them.

Recall that any element in T_{Spin_7} can be written uniquely in the form $t = (s_\alpha, s_\beta, s_\gamma)$ as explained in Proposition 3.2. Denote $T_{\mathrm{Spin}_7}(F) \ni t_a = (a, a^2, a)$ for any $a \in F^*$, and

$$Z_G(F)\backslash T_G(F) \ni t(b, c, \lambda) = \mathrm{diag}(1, \lambda)\mathrm{diag}(b, c, b^{-1}\lambda, c^{-1}\lambda).$$

PROPOSITION 10.1. *One has*

$$K_\chi(t_a) = \int_{\Delta_a} K_{v,\mu}(t(b, 1, \lambda))\tilde{\sigma}(\lambda^{-1})\mathrm{d}^*b\mathrm{d}^*\lambda,$$

where

$$T_G(F) \supset \Delta_a = \{t(b, c, \lambda) : |b|, |\lambda^{-1}| \leqslant 1, |c| = 1, |b\lambda^{-1}| \geqslant |a|\}.$$

Define

$$J_\chi(s) = \int_{|a| \leqslant 1} K_\chi(t_a)|a|^s \,\mathrm{d}a. \tag{10.7}$$

By the Casselman–Shalika formula $K_\chi(t_a)$ is of polynomial growth in $|a|$ and hence $J_\chi(s)$ converges for $\mathrm{Re}\, s \gg 0$. Using Proposition 10.1 we obtain

PROPOSITION 10.2. *One has*

$$(1 - q^{-s})J_\chi(s) = Q_{v,\mu}(q^{-s})(1 - q^{-s})^2 L(\Pi_{r(v,\mu)}, \omega_2, s)$$

where $Q_{v,\mu}(q^{-s})$ is a polynomial in q^{-s} whose coefficients are rational functions in μ_i, v_i and $\tilde{\sigma}$.

PROPOSITION 10.3. *One has*

$$J_\chi(s) = P_\chi(q^{-s})(1 - q^{-s})^2 L(\Pi_\chi, \omega_2, s)$$

where $P_\chi(q^{-s})$ is a polynomial in q^{-s} whose coefficients are rational functions in χ_i.

From Propositions 10.2 and 10.3 we have

PROPOSITION 10.4.

(a) $(1-q^{-s})P_\chi(q^{-s}) = Q_{\nu,\mu}(q^{-s})$.

(b) $L(\Pi_\chi, \omega_2, s) = L(\Pi_{r(\nu,\mu)}, \omega_2, s)$.

Using the result of Proposition 10.4 (a) we obtain

PROPOSITION 10.5. $\mathrm{tr}[2,0,0](t_\chi) = \mathrm{tr}[2,0,0](r(t_\nu, t_\mu))$.

PROPOSITION 10.6. *Any unramified representation* Π_χ *of* Spin_7 *is uniquely determined by the second fundamental L-function* $L(\Pi_\chi, \omega_2, s)$ *and by* $\mathrm{tr}[2,0,0](t_\chi)$.

Summing up the results of Propositions 10.4 (b), 10.5 and 10.6 we conclude that $\Pi_{r(\nu,\mu)} = \Pi_\chi$ as required. Now functoriality on the level of unramified representations is proved and this finishes the proof of Theorem 10. It remains to prove Propositions 10.1–10.6.

Proof of Proposition 10.1. Let $W_{\nu,\mu}$ be the unique K-fixed function in the space of the Whittaker model of π normalized such that $W_{\nu,\mu}(1) = 1$. In Section 8 we obtained a formula for the Whittaker model of $\theta_{GE_7}(\pi)$. By (8.6), the local lifting θ_{GE_7} is well-defined. Denote by W_χ the image of $W_{\nu,\mu}$. Then

$$W_\chi(g) = \int_{Z(F)N(F)\backslash G(F)} \int_{F^3} f(w[1]x_{\alpha_1}(r)wn(1)x_\alpha(r_1)x_\gamma(r_2)(g_1,g_2)g)\times$$
$$\times\, \psi(r)\psi(r_1+r_2)W_{\nu,\mu}(g_1,g_2)\,\mathrm{d}r\,\mathrm{d}r_i\mathrm{d}g_1\,\mathrm{d}g_2. \tag{10.8}$$

Here N stands for N_G, Z stands for Z_G and so on. It is easy to see that W_χ is also a K-fixed vector, but not necessarily normalized. Using the Iwasawa decomposition for G we get

$$W_\chi(g) = \int_{Z(F)\backslash T(F)} \int_{F^3} f(w[1]wx_{1122221}(r)n(1)x_\alpha(r_1)x_\gamma(r_2)t(b,c,\lambda)g)\times$$
$$\times\, \psi(r)\psi(r_1+r_2)W_{\nu,\mu}t(b,c,\lambda)\delta_B^{-1}t(b,c,\lambda)\,\mathrm{d}r\,\mathrm{d}r_i\mathrm{d}t\,(b,c,\lambda) \tag{10.9}$$

Plugging $g = t_a$ and conjugating $t(b,c,\lambda)t_a$ to the left we get

$$W_\chi(t_a) = \int_{Z(F)\backslash T(F)} \delta_P^{3/17}(w[1]wt(b,c,\lambda)t_aw^{-1}w[1])\times$$
$$\times\, K_{\nu,\mu}t(b,c,\lambda)\delta_B^{-1/2}t(b,c,\lambda)|c|^{-1}|bc\lambda|^{-1}|a|^{-1}St(b,c,\lambda,t_a)\,\mathrm{d}t(b,c,\lambda), \tag{10.10}$$

where

$$St(b, c, \lambda, t_a) = \int_{F^3} f(w[1]wx_{1122221}(r)n(b, c, \lambda)x_{0112221}(r_1)x_{0111000}(r_2))\times$$

$$\times \psi\left(\frac{\lambda}{bc}ar\right)\psi(r_1)\psi\left(\frac{r_2}{c}\right) \mathrm{d}r\, \mathrm{d}r_1\, \mathrm{d}r_2. \tag{10.11}$$

The two following lemmas can be proved by direct computations:

LEMMA 10.1.1. *One has*

$$St(b, c, \lambda, t_a) = \begin{cases} (1 - q^{-3/17})f(1), & t(b, c, \lambda) \in \Delta_a, \\ 0, & \text{otherwise.} \end{cases}$$

Consider $t = t(b, c, \lambda) \in \Delta_a$.

LEMMA 10.1.2.

(a) $\delta_P^{3/17}(w[1]wt(b, c, \lambda)t_a w^{-1}w[1]) = |b/\lambda|^3|a|^3$,
(b) $\delta_B t(b, c, \lambda) = |b/\lambda|^4$,
(c) $\delta_B(t_a) = |a|^8$,
(d) $\sigma(w[1]wt(b, c, \lambda)w^{-1}w[1]) = \tilde{\sigma}(\lambda^{-1}) = \sigma t(b, c, \lambda)$.

Recall that for $t(b, c, \lambda) \in A_G$,

$$W_{\nu,\mu}t(b, c, \lambda) = K_{\nu,\mu}t(b, c, \lambda)\delta_B^{1/2}t(b, c, \lambda) = K_{\nu,\mu}t(b, c, \lambda)\left|\frac{b}{\lambda}\right|^2.$$

Summing all this we have

$$W_\chi(t_a) = (1 - q^{-3/17})f(1) \int_{\Delta_a} K_{\nu,\mu}t(b, c, \lambda)\tilde{\sigma}(\lambda^{-1})|a|^4\, \mathrm{d}t(b, c, \lambda)$$

Note that $W_\chi(1) = (1 - q^{-3/17})f(1)$. Recall that K_χ is defined using the Whittaker function normalized to be 1 at the identity. So

$$K_\chi(t_a) = \int_{\Delta_a} K_{\nu,\mu}t(b, c, \lambda)\sigma t(b, c, \lambda)\, \mathrm{d}t(b, c, \lambda)$$

and Proposition 10.1 is proved. □

Proof of Proposition 10.2. We write $\tilde{\sigma}$ for $\tilde{\sigma}(p)$. Using Proposition 10.1 we get

$$\begin{aligned} J_\chi(s) &= \int_{|a| \leqslant 1} \int_{\Delta_a} K_{\nu,\mu}t(b, c, \lambda)\sigma t(b, c, \lambda)|a|^s\, \mathrm{d}t(b, c, \lambda)\, \mathrm{d}a \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{k=0}^{n-l} K_{\nu,\mu}\big(t(p^k, 1, p^{-l})\big)\tilde{\sigma}^l q^{-ns}. \end{aligned}$$

Changing the order of summation is equal to

$$\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} K_{\nu,\mu}(t(p^k,1,p^{-l}))\tilde{\sigma}^l q^{-ns}q^{-ls}q^{-ks}$$
$$= \frac{1}{1-q^{-s}}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} K_{\nu,\mu}\big(t(p^k,1,p^{-l})\big)\tilde{\sigma}^l q^{-ls}q^{-ks}.$$

Applying the second version of the Casselman–Shalika formula to $K_{\nu,\mu}$ and computing the sum above one has:

$$(1-q^{-s})J_\chi(s)$$
$$= C(\mu_1,\mu_2)\Bigg(\frac{1-\mu_1{}^{-1}\mu_2}{(1-\mu_1 q^{-s})(1-\mu_2 q^{-s})(1-\tilde{\sigma}\mu_3{}^{-1}q^{-s})(1-\nu^{-1}\tilde{\sigma}\mu_3{}^{-1}q^{-s})}-$$
$$-\frac{\mu_2{}^{-1}(1-\mu_1{}^{-1}\mu_2{}^{-1})}{(1-\mu_1 q^{-s})(1-\mu_2{}^{-1}q^{-s})(1-\tilde{\sigma}(\mu_2\mu_3)^{-1}q^{-s})(1-\nu^{-1}\tilde{\sigma}(\mu_2\mu_3)^{-1}q^{-s})}+$$
$$+\frac{\mu_1{}^{-2}(1-\mu_1{}^{-1}\mu_2{}^{-1})}{(1-\mu_1{}^{-1}q^{-s})(1-\mu_2 q^{-s})(1-\tilde{\sigma}(\mu_1\mu_3)^{-1}q^{-s})(1-\nu^{-1}\tilde{\sigma}(\mu_1\mu_3)^{-1}q^{-s})}-$$
$$-\frac{\mu_1{}^{-2}\mu_2{}^{-2}(1-\mu_1{}^{-1}\mu_2)}{(1-\mu_1{}^{-1}q^{-s})(1-\mu_2{}^{-1}q^{-s})(1-\tilde{\sigma}(\mu_1\mu_2\mu_3)^{-1}q^{-s})(1-\nu^{-1}\tilde{\sigma}(\mu_1\mu_2\mu_3)^{-1}q^{-s})}\Bigg).$$

Recall that $\tilde{\sigma} = (\nu\mu_1\mu_2\mu_3{}^2)^{1/2}$. Since the nontrivial eigenvalues of $\omega_2(r(t_{\nu,\mu}))$ are as in the denominator of J_χ, the result follows. □

Proof of Proposition 10.3. We use the second version of the Casselman–Shalika formula. Since $t_a \in A_G$ for $|a| \leq 1$ one has

$$W_\chi(t_a) = C(H)\sum_{w\in W(H)} \mathrm{d}_w(\chi)(w\chi)(t_a)\delta_B^{1/2}(t_a)$$

where $d_w(\chi)$ are rational functions in χ_i. Note that the Weyl elements w_α and w_γ centralize t_a, hence

$$W_\chi(t_a) = \sum_{w\in W(H)/\{w_\alpha,w_\gamma\}} \mathrm{d}_w(\chi)(w\chi)(t_a)\delta_B^{1/2}(t_a).$$

So

$$\int_{|a|\leq 1} K_\chi(t_a)|a|^s \mathrm{d}a = \sum_{n=0}^{\infty}\sum_{w\in W/\{w_\alpha,w_\gamma\}} \mathrm{d}_w(\chi)(w\chi)(t_p)^n q^{-ns}$$
$$= \sum_{w\in W/\{w_\alpha,w_\gamma\}} \frac{\mathrm{d}_w(\chi)}{1-(w\chi)(t_p)q^{-s}}.$$

We have $\sharp(W/\{w_\alpha, w_\gamma\}) = 12$. Since ω_2 is a 14-dimensional representation, $\omega_2(t_\chi)$ has 14 eigenvalues and two of them are the identity. It is not hard to check that the non-trivial eigenvalues of $\omega_2(t_\chi)$ coincide with the set $(w\chi)(t_p)$. Hence

$$\begin{aligned} J_\chi(s) &= \frac{P_\chi(q^{-s})}{\prod_{w\in W/\{w_\alpha,w_\gamma\}}(1-(w\chi)(t_p)q^{-s})} \\ &= P_\chi(q^{-s})(1-q^{-s})^2 L(\Pi, \omega_2, s) \end{aligned}$$

as required. □

Proof of Proposition 10.4. Note that $1/((1-q^{-s})^2 L(\Pi_\chi, \omega_2, s))$ is a polynomial in q^{-s}. Since

$$\frac{(1-q^{-s})P_\chi(q^{-s})}{1/(1-q^{-s})^2 L(\Pi_\chi, \omega_2, s)} = \frac{Q_{v,\mu}(q^{-s})}{1/(1-q^{-s})^2 L(\Pi_{r(v,\mu)}, \omega_2, s)}$$

and both sides are irreducible quotients of polynomials in q^{-s}, one concludes

$$(1-x)P_\chi(x) = Q_{v,\mu}(x) \quad \text{and} \quad L(\Pi_\chi, \omega_2, s) = L(\Pi_{r(v,\mu)}, \omega_2, s)$$

as required. □

Proof of Proposition 10.5. The element of the torus t_a is in A_{Spin_7} iff $|a| \leqslant 1$ is integer. It is easy to see that $\rho^*(t_a) = [0, n, 0]$, where $|a| = q^{-n}$. Then from the Casselman–Shalika formula it follows that

$$J_\chi(s) = \sum_{n=0}^{\infty} \mathrm{tr}\,[0, n, 0](t_\chi) q^{-ns}.$$

From [GiRa] we have

$$L(\Pi_\chi, \omega_2, s) = \frac{1}{(1-x)(1-x^2)(1-x^3)} \sum_{n,m=0}^{\infty} \mathrm{tr}[m, n, m](t_\chi) x^{n+3m}(1-x^{n+1}),$$

where $x = q^{-s}$. So using Proposition 10.3 the equality

$$\begin{aligned} &\frac{(1-x)^2 P_\chi(x)}{(1-x)(1-x^2)(1-x^3)} \left(\sum_{n,m=0} \mathrm{tr}[m, n, m](t_\chi) x^{n+3m}(1-x^{n+1}) \right) \\ &\quad = \sum_{k=0}^{\infty} \mathrm{tr}[0, k, 0](t_\chi) x^k \end{aligned}$$

implies

$$\begin{aligned} &P_\chi(x) \left(\sum_{m,n=0}^{\infty} tr[m, n, m](t_\chi) x^{n+3m}(1-x^{n+1}) \right) \\ &\quad = (1+x)(1-x^3) \sum_{k=0}^{\infty} \mathrm{tr}[0, k, 0](t_\chi) x^k. \end{aligned}$$

By comparing coefficients of x we deduce

$$P_\chi(x) = 1 + 2x + (2 - tr[0,1,0](t_\chi))x^2 + \\ + (2 - \mathrm{tr}[0,1,0](t_\chi) + \mathrm{tr}[2,0,0](t_\chi))x^3 + \cdots$$

On the other hand, from Proposition 10.2 we obtain

$$Q_{\nu,\mu}(x) = 1 + x - \mathrm{tr}[0,1,0](r(t_{\nu,\mu}))x^2 + \mathrm{tr}[2,0,0](r(t_{\nu,\mu}))x^3 + \cdots$$

Since by Proposition 10.4, $Q_{\nu,\mu}(x) = (1-x)P_\chi(x)$, comparing coefficients of x^2 and x^3 one has $tr[2,0,0](t_\chi) = tr[2,0,0](r(t_{\nu,\mu}))$ as required. □

Proof of Proposition 10.6. This is a purely combinatorial problem. Our aim is to recover the conjugacy class of t_χ from the set of 14 eigenvalues of $\omega_2(t_\chi)$ and the number $\mathrm{tr}[2,0,0](t_\chi)$.

Consider a representative of that conjugacy class $t_\Pi = t_\chi = \mathrm{diag}(a_1, a_2, a_3, a_3{}^{-1}, a_2{}^{-1}, a_1{}^{-1}) \in PSp_6(\mathbb{C})$ determined by the numbers a_1, a_2, a_3. We define the following equivalence relation.

$$(a_1, a_2, a_3) \simeq (a_1', a_2', a_3') \Leftrightarrow (a_1', a_2', a_3') = \pm(a_{s(1)}^{\pm 1}, a_{s(2)}^{\pm 1}, a_{s(3)}^{\pm 1})$$

for some permutation s of $\{1,2,3\}$. Obviously two triples are equivalent if they define the same conjugacy class in $PSp_6(\mathbb{C})$.

Let us define the numbers $b_i = a_i + a_i{}^{-1}$ for $i = 1,2,3$. We define

$$(b_1, b_2, b_3) \simeq (b_1', b_2', b_3') \Leftrightarrow (b_1', b_2', b_3') = \pm(b_{s(1)}, b_{s(2)}, b_{s(3)})$$

for some permutation s of $\{1,2,3\}$. Obviously, the equivalence class of the set $\{b_i\}$ determines uniquely the equivalence class of the set $\{a_i\}$ and, hence, uniquely determines the representation Π. So to detemine the conjugacy class of Π it suffices to determine the set $\{b_1, b_2, b_3\}$ up to sign.

On the other side, the information we are given consists of a set of 14 numbers that are eigenvalues of $\omega_2(t_\Pi)$ and the number $\mathrm{tr}[2,0,0](t_\Pi)$. Among the 14 eigenvalues there is eigenvalue 1 with a multiplicity 2. Consider the remaining 12 eigenvalues. The inverse of an eigenvalue is also an eigenvalue. Hence, we have six pairs of eigenvalues. Denote them by $(p_i, p_i{}^{-1})$ for $i = 1, \ldots, 6$. Recall that every $p_i = a_k{}^{\pm 1} a_l{}^{\pm 1}$ for some $k \neq l$. Define six numbers $r_i = p_i + p_i{}^{-1}$ for $i = 1, \ldots, 6$.

Denote by $P_i(x_1, .., x_n)$ the ith standard symmetric polynomial. These polynomials $\{P_i : i = 1, \ldots, n\}$ generate the ring of symmetric polynomials and that the values of $P_i(a_1, \ldots, a_n) : i = 1, \ldots, n$ completely determine the numbers $a_1, \ldots, a_n$ up to permutation.

One can express $\sigma_i = P_i(r_1, \ldots, r_6)$ as symmetric polynomials of b_i as follows.

$$\sigma_1 = \sum\nolimits_i r_i = b_1b_2 + b_1b_3 + b_2b_3,$$
$$\sigma_2 = \sum\nolimits_{i,j} r_ir_j = 2({b_1}^2 + {b_2}^2 + {b_3}^2) + b_1b_2b_3(b_1 + b_2 + b_3) - 12,$$
$$\sigma_3 = \sum\nolimits_{i,j,k} r_ir_jr_k = ({b_1}^2 + {b_2}^2 - 4)(b_1 + b_2)b_3 + \\ + ({b_2}^2 + {b_3}^2 - 4)(b_2 + b_3)b_1 + ({b_1}^2 + {b_3}^2 - 4)(b_1 + b_3)b_2 + (b_1b_2b_3)^2.$$

Any symmetric polynomial is a polynomial of standard symmetric polynomials. Hence we can express σ_i as polynomials of $s_j = P_j(b_1, b_2, b_3)$:

$$\sigma_1 = s_2,$$
$$\sigma_2 = 2{s_1}^2 - 4s_2 + s_1s_3 - 12,$$
$$\sigma_3 = -8s_2 + {s_1}^2 s - 2 - 2{s_2}^2 + s_1s_3 + {s_3}^2.$$

For the group PSp_6 the representation $[2, 0, 0]$ is a symmetric square representation. Hence

$$tr[2,0,0](t_\chi) = \sum_{i,j} {a_i}^2 + {a_i}^{-2} + \sum p_j + {p_j}^{-1} + 3 \\ = \sum {b_i}^2 - 6 + b_1b_2 + b_2b_3 + b_1b_3 + 3 = {s_1}^2 - s_2 - 3.$$

We are given the numbers σ_i and $tr[2, 0, 0](t_\chi)$. We claim that if the system of these four equations has any solution in s_1, s_2, s_3, then there are exactly two solutions and if (s_1, s_2, s_3) is one solution, then another is of the form $(-s_1, s_2, -s_3)$. These two solutions determine the numbers b_i up to permutation and multiplication of all of them on -1 simultaneously. Thus we can recover from the set of eigenvalues of $\omega_2(t_\Pi)$ the equivalence class of $\{b_i\}$, hence the equivalence class of $\{a_i\}$, hence the representation Π. It remains to prove our claim. After simplifying the system looks as follows:

$$s_2 = \sigma_1 \qquad s_1s_3 = A_1 - 2{s_1}^2,$$
$${s_3}^2 = A_2 + A_3{s_1}^2 \qquad {s_1}^2 - s_2 + 3 = tr[2,0,0](t_\chi).$$

Here all A_i are polynomial functions of σ_i so they are also given.

From the Equations (1), (4) and (3) of the last system we get numbers $s_2, {s_1}^2$ and ${s_3}^2$. And from Equation (2) we see that s_1s_3 is fixed. This proves our claim and Proposition 10.6. □

As we noted above, Propositions 10.1–10.6 give the proof of Theorem 10. □

Acknowledgements

I wish to thank David Ginzburg for suggesting the problem considered in this paper and for his constant guidance. I also wish to thank the referee for his valuable remarks.

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