ON THE ARENS SEMI-REGULARITY OF WEIGHTED GROUP ALGEBRAS

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Abstract. In this paper we prove that the weighted group algebra $L_1(G, w)$ is semi-regular if and only if G is either abelian or discrete.

1. Introduction. A number of Banach algebras commonly occurring in functional and harmonic analysis are not Arens regular. In fact, the group algebra $L_1(G)$ of a locally compact Hausdorff group is regular if and only if G is finite, and the closure of the algebra of finite rank operators on a Banach space X is regular if and only if X is reflexive (see [**13**], [**14**]). In 1984, Grosser [**5**] introduced a notion for Banach algebras with a bounded approximate identity which on the one hand is weaker than Arens regularity but on the other is adequate to characterise some of the nicer non-regular Banach algebras. These algebras are called Arens semi-regular Banach algebras. For members of this class the Arens products behave in a reasonable way although in general they do not coincide (see [**5**, Theorem 3]). It turns out that every commutative Banach algebra with a bounded approximate identity is Arens semi-regular [**5**, Theorem 2], so that, in particular, $L_1(G)$, for an infinite commutative locally compact group, is semi-regular but not Arens regular. The Arens semi-regularity of the algebra of compact operators on a Banach space has been studied in detail by Grosser in [**6**].

In [**5**] Grosser states that the group algebra $L_1(G)$, where G is a locally compact Hausdorff group, is semi-regular if and only if G is either abelian or discrete. This observation is based on results established by Losert and Rindler in [**8**]. The proof of [**8**, Theorem 1], however, assumes that, if G is a non-discrete locally compact Hausdorff group, then there exists a closed subgroup H such that the quotient space topology on G/H is metrisable and non-discrete. Whilst this result is true for a compact group (see, for example, 8.20 of [**7**, p. 80]), it is not clear whether or not the result is true in general. There appears therefore to be a gap in Losert and Rindler's arguments. In this paper we prove by arguments which do not involve the above result that the weighted group algebra $L_1(G, w)$ is semi-regular if and only if G is abelian or discrete.

2. Preliminaries. Throughout we assume that A is a Banach algebra with a bounded two-sided approximate identity (abbreviated to b.a.i.). For the definitions of the two Arens products on the bidual A^{**} of A, and a survey of the results on Arens regularity up to 1979, we refer the reader to [**2**].

A mapping $S: A \to A$ (resp. $T: A \to A$) is said to be a *left* (*right*) *multiplier* on A if and only if $S(ab) = (Sa)b$ ($T(ab) = aTb$). Every left and right multiplier on a Banach algebra with a b.a.i. is linear and continuous, and, if $M_l(A)$ (resp. $M_r(A)$) denotes the set of all left (right) multipliers on A, then $M_l(A)$ and $M_r(A)$ are Banach algebras with identity under the usual algebraic operations for operators and the norm given by the operator norm. An ordered pair (S, T) of mappings on A is said to be a *double multiplier* if and only if $aSb = (Ta)b$ for all a, $b \in A$. If (S, T) is a double multiplier on A, then $S \in M_l(A)$ and

$T \in M_r(A)$, and the set $M(A)$ of all double multipliers on A is a Banach algebra with identity under the algebraic operations

$$(S_1, T_1) + (S_2, T_2) = (S_1 + S_2, T_1 + T_2),\ \alpha(S, T) = (\alpha S, \alpha T)(\alpha \in \mathbf{C}),$$
$$(S_1, T_1)(S_2, T_2) = (S_1 S_2, T_2 T_1),$$

and the norm

$$\|(S, T)\| = \max(\|S\|, \|T\|).$$

For details of the above results, see, for example, §3 of [**10**].

An element E of A^{**} is said to be a *mixed unit* if it is a right identity with respect to the first Arens product and a left identity with respect to the second. It is clear that every $\sigma(A^{**}, A^*)$-cluster point of a b.a.i. is a mixed unit. On the other hand, if A^{**} has a mixed unit, it follows from the fact that the canonical image of the unit ball in A is $\sigma(A^{**}, A^*)$-dense in the unit ball of A^{**} and Proposition 4 of [**1**, p. 58] that A has a b.a.i. If we denote by $\cdot$ (resp. $*$) the product which leads to the definition of the first (second) Arens product on A^{**}, we define A^*A and AA^* by

$$A^*A = \{f \cdot a : f \in A^*, a \in A\} \text{ and } AA^* = \{a * f : a \in A, f \in A^*\}.$$

It is easy to show that, if E is a mixed unit in A^{**}, then $E + (A^*A + AA^*)^\perp$ is the set of all mixed units, where, for any subset $\mathcal{F}$ of A^*,

$$\mathcal{F}^\perp = \{F \in A^{**}\langle f, F\rangle = 0 \text{ for all } f \in \mathcal{F}\}.$$

It follows that a mixed unit in A^{**} is unique if and only if $(A^*A + AA^*)^\perp = \{0\}$.

The notion of Arens semi-regularity was introduced by Grosser in [**5**], where he defines a Banach algebra with a b.a.i. to be *Arens semi-regular* if and only if $S^{**}E = T^{**}E$ for all $(S, T) \in M(A)$ and mixed units E in A^{**}.

The elementary properties of Arens semi-regular algebras may be found in §3 of [**5**]. In particular, Grosser proves [**5**, Proposition 1] that, for any mixed unit E in A^{**} and $(S, T) \in M(A)$, $S^{**}E - T^{**}E \in (A^*A + AA^*)^\perp$. It follows that, if $(A^*A + AA^*)^\perp = \{0\}$, then A is semi-regular. This observation motivates the following definition.

DEFINITION 1. If a Banach algebra A with a b.a.i. is such that A^{**} has a unique mixed unit, then we say that A is *semi-regular of the first kind.*

We now give some results on semi-regular algebras of the first kind.

THEOREM 2. *If the bidual of a Banach algebra A has an identity with respect to either the first or second Arens product, then $(A^*A + AA^*)^\perp = \{0\}$, and so A is semi-regular of the first kind.*

Proof. Suppose that E is an identity in A^{**} with respect to the first Arens product, and let F be any element of $(A^*A + AA^*)^\perp$. Then, for any $a \in A$, $\hat{a} \cdot F = 0$, where $\hat{a}$ denotes the canonical image of a in A^{**}. Let $\{a_\alpha : \alpha \in I\}$ be a bounded net in A such that $E = \sigma(A^{**}, A^*)\text{-}\lim_\alpha \hat{a}_\alpha$. Then

$$F = E \cdot F = \sigma(A^{**}, A^*)\text{-}\lim_\alpha \hat{a}_\alpha \cdot F = 0;$$

that is, $(A^*A+AA^*)^{\perp}=\{0\}$, as required. (The proof for the second Arens product is almost identical.) □

COROLLARY 3. *A Banach algebra A with an identity is semi-regular of the first kind.*

Proof. If e is an identity for A, then $\hat{e}$ is an identity for A^{**} with respect to both Arens products. □

COROLLARY 4 ([**5**, Theorem 1]). *If A is an Arens regular Banach algebra with a b.a.i., then A is semi-regular of the first kind.*

Proof. Since A is regular, then A^{**} has an identity with respect to either the first or second Arens product. □

It follows from Corollary 4 that a B^*-algebra is semi-regular of the first kind and so has a unique mixed unit (cf. [**5**, p. 48]). The Banach algebra $M(G)$ of bounded regular Borel measures on a locally compact group G has an identity, and so is semi-regular of the first kind by Corollary 3. However, $M(G)$ is Arens regular if and only if G is finite [**2**]. The following provides us with an example of a semi-regular Banach algebra which is not of the first kind.

EXAMPLE. We recall from remarks made in the introductory paragraph that the group algebra $L_1(G)$ of an infinite compact commutative Hausdorff group is semi-regular but not Arens regular. However, in this case, $L_1(G)$ is not semi-regular of the first kind, as we now show. We denote $L_1(G)$ by A. If $\{e_\alpha:\alpha\in I\}$ is a b.a.i. in A and A_* is the set $\{f\in A^*:e_\alpha * f\rightarrow f\}$, then it is straightforward to show that A_* is a closed subspace of A^* and it follows from the factorization theorem (see, for example, Theorem 10 of [**1**, p. 61]) that $A_*=AA^*$. Recently, Ülger [**11**] has proved that wap $A=AA^*=A^*A$, where wap A is the set of all weakly almost periodic functionals on A. Since A is not Arens regular, wap A is a proper closed subspace of A^* [**2**, Theorem 2]. It follows that (A^*A+AA^*) is a proper closed subspace of A^* and so by the Hahn-Banach theorem there exists a non-zero element $F\in A^{**}$ such that $\langle A^*A+AA^*, F\rangle=0$. Thus A is semi-regular but not of the first kind.

For the sake of completeness, we give an improvement of a result due to Grosser [**5**, Corollary on p. 48] which characterizes the semi-regularity of a weakly completely continuous (abbreviated to w.c.c.) Banach algebra in terms of its regularity. (A Banach algebra is said to be w.c.c. if and only if, for each $a\in A$, the mappings $x\rightarrow ax$ and $x\rightarrow xa$ are weakly completely continuous on A.)

THEOREM 5. *Let A be a w.c.c. Banach algebra with a b.a.i. Then A is regular if and only if it is semi-regular of the first kind.*

Proof. If A is regular, then by Corollary 4 it is semi-regular of the first kind.

Conversely, suppose that A is semi-regular of the first kind. By [**5**, Lemma 4], $AA^*=A^*=A^*A$. Let $f\in A^*$. Then there exist elements $a\in A$, $g\in A^*$ such that $f=g\cdot a$. Thus, for any $F, G\in A^{**}$, we have

$$\langle f, F\cdot G\rangle=\langle g\cdot a, F\cdot G\rangle=\langle G\cdot(g\cdot a), F\rangle=\langle G\cdot g, \hat{a} * F\rangle=\langle g, (\hat{a} * F)\cdot G\rangle.$$

Since A is w.c.c., the canonical image of A is a two-sided ideal of $(A^{**}, \cdot)$ or $(A^{**}, *)$ by [**12**, p. 443], and so

$$\langle f, F \cdot G\rangle = \langle g, \hat{a} * F * G\rangle = \langle g \cdot a, F * G\rangle = \langle f, F * G\rangle,$$

which implies that $F \cdot G = F * G$; that is, A is regular, as required. □

Let G denote a locally compact Hausdorff group with identity e and left Haar measure dx. Following §7.1 of [**9**, p. 83] a real-valued function w on G is said to be a *weight function* if it has the following properties:

(i) $w(x) \geq 1$ for all $x \in G$,

(ii) w is measurable and locally bounded, and

(iii) $w(xy) \leq w(x)w(y)$ for all $x, y \in G$.

With multiplication by convolution, the functions f in $L_1(G)$ such that $fw \in L_1(G)$ form a subalgebra of $L_1(G)$ which is a Banach algebra under the norm

$$\|f\|_{w,1} = \int_G |f(x)|\, w(x)\, dx.$$

This algebra is denoted by $L_1(G, w)$ and is called the *weighted group algebra*. If $C_{00}(G)$ is the space of continuous real-valued functions on G with compact support, then $C_{00}(G) \subseteq L_1(G, w)$, and, since $C_{00}(G)$ is dense in $L_1(G)$, it follows that $C_{00}(G)$ is $\|\cdot\|_{w,1}$-dense in $L_1(G, w)$. The dual space of $L_1(G, w)$ is $L_\infty(G, w)$, the space of all complex-valued measurable functions ϕ on G such that

$$\|\phi\|_{w,\infty} = \text{ess.} \sup_{x \in G} \frac{|\phi(x)|}{w(x)} < \infty.$$

See §7.3 of [**9**, p. 84].

The group algebra $L_1(G)$ has a b.a.i. which consists of functions u_α in $C_{00}(G)$ with the properties that $\|u_\alpha\|_1 = 1$ and the support of each u_α is contained in some compact neighbourhood, S say, of e. (See, for example, §5.6 of [**9**, p. 77]). It is straightforward to show that the net $\{u_\alpha : \alpha \in\}$ is also a b.a.i. for $L_1(G, w)$.

For the measure theoretic concepts required, we follow the presentation given by Reiter in [**9**, p. 46 et seq.] and refer the reader to this reference for details; we merely note one or two results which we use in the proof of our main theorem.

For a bounded measure μ on G and $f \in L_1(G)$, the convolution products $\mu * f$ and $f * \mu$ are defined respectively by

$$(\mu * f)(x) = \int_G f(y^{-1}x)\, d\mu(y)$$

and

$$(f * \mu)(x) = \int_G f(xy^{-1})\Delta(y^{-1})\, d\mu(y),$$

where $\Delta(.)$ is the modular function of G. It follows immediately from the above definitions that, for the Dirac measure δ_a concentrated at $a \in G$, $(\delta_a * f)(x) = f(a^{-1}x)$

and $(f * \delta_a)(x) = f(xa^{-1})\Delta(a^{-1})$. If G is discrete, then, for any $a \in G$, the characteristic function $\chi_{\{a\}}$ is in $C_{00}(G)$ and determines the measure δ_a. In this case each $\delta_a \in L_1(G, w)$ and, in particular, δ_e is an identity for $L_1(G, w)$.

Let $M(w)$ consist of those measures $\mu \in M(G)$ for which the upper integral $\int_G^* w(x)\, d|\mu|(x)$ is finite. It is straightforward to show that, for any $a \in G$,

$$\int_G^* w(x)\, d(\delta_a(x)) = w(a),$$

and so, since $w(a)$ is finite, $\delta_a \in M(w)$.

The multiplier algebras of $L_1(G, w)$ have been characterized by Gaudry in [**3**]. It follows from [**3**, Theorem 4] that the double multiplier algebra of $L_1(G, w)$ is algebraically and topologically isomorphic to $M(w)$, the correspondence $(S, T) \leftrightarrow \mu$ being such that $Sf = \mu * f$ and $Tf = f * \mu$.

3. The main result. We open this section with a result on locally compact Hausdorff groups which are not extremely disconnected. (Extremally disconnected means that the closure of every open set is open.) A locally compact extremally disconnected group is discrete [**4**, p. 322], and so, if the locally compact group G is not discrete, then it is not extremely disconnected.

LEMMA 6. *Let G be a locally compact Hausdorff group. Then the following are equivalent.*

(a) *G is not extremally disconnected.*

(b) *For each non-empty open subset U of G, there exists a non-empty open subset V of U such that $\bar{V} \cap \overline{(U \setminus V)}^0 \neq \varnothing$.*

Proof. Suppose that (b) holds for any open subset of G and that G is extremally disconnected. With $U = G$, the set $\bar{V}$ is clopen and so $\overline{(G \setminus V)}^0 = \overline{(G \setminus \bar{V})} = G \setminus \bar{V}$, which implies that $\overline{(G \setminus V)}^0 \cap \bar{V} = \varnothing$, contradicting (b). Thus (b)$\Rightarrow$(a).

On the other hand, suppose that G is not extremally disconnected. Then there exists an open subset W of G such that $\bar{W}$ is not open. Thus $\bar{W}$ contains an element x such that $\bar{W}$ is not a neighbourhood of x. Let U be any open non-empty subset of G and suppose that $y \in U$. Let $W_1 = Wx^{-1}y$. Then $y \in \bar{W}_1$ and $\bar{W}_1$ is not a neighbourhood of y. If $V = W_1 \cap U$, then V is open and non-empty since $y \in \bar{W}_1$. Now $y \in \bar{V} \cap \overline{(U \setminus V)}^0$; clearly $y \in \bar{V}$, and, since U is a neighbourhood of y and $\bar{W}_1$ is not, every open neighbourhood of y meets $U \setminus \bar{W}_1$, implying that $y \in \overline{(U \setminus V)}^0$. Thus (a)$\Rightarrow$(b).

THEOREM 7. *Let G be a locally compact Hausdorff group. Then $L_1(G, w)$ is semi-regular if and only if G is either abelian or discrete.*

Proof. Suppose that G is abelian. Then the algebra $L_1(G, w)$ is commutative and so is semi-regular by [**5**, Theorem 2]. If G is discrete, then δ_e is an identity for $L_1(G, w)$, and so by Corollary 3 is semi-regular of the first kind.

Conversely, suppose that $L_1(G, w)$ is semi-regular but that G is neither discrete nor abelian. Then there exist x, $y \in G$ such that $xyx^{-1} \neq y$. Since G is Hausdorff there exist non-empty open subsets U_1, U_2 of G such that $xyx^{-1} \in U_1$, $y \in U_2$, and $U_1 \cap U_2 = \varnothing$. By continuity there exists an open set U_3, containing y, such that $xU_3x^{-1} \subseteq U_1$. Let

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$U = U_3 \cap U_2$. Then $y \in U$, $xUx^{-1} \subseteq U_1$, and so $xUx^{-1} \cap U = \emptyset$. Since G is locally compact we may assume, without loss of generality, that U is relatively compact.

Since G is non-discrete, corresponding to U, there exists by Lemma 6 an open set V such that $V \subseteq U$ and $\bar{V} \cap \overline{(U \setminus V)^0} = \emptyset$.

Let ϵ be any positive number. By Lemma 1(ii) of **[8]** there exists a net $\{v_\alpha\}$ in $L_1(G)$, with $\operatorname{supp} v_\alpha \subseteq U$, $\|v_\alpha\|_1 = \epsilon$, $\int_V v_\alpha\,dx = \epsilon/2$, and $\lim_\alpha \|f * v_\alpha\|_1 = \lim_\alpha \|v_\alpha * f\|_1 = 0$ for all $f \in L_1(G)$. Since w is locally bounded and U is relatively compact, each $v_\alpha \in L_1(G, w)$. If $\{u_\alpha\}$ is the b.a.i. introduced in §2, then $\{u_\alpha + v_\alpha\}$ is a b.a.i. for $L_1(G, w)$; for, if g is any element of $C_{00}(G)$, with $K = \operatorname{supp} g$, then

$$\|g * (u_\alpha + v_\alpha) - g\|_{1,w} \le \|g * u_\alpha - g\|_{1,w} + \|g * v_\alpha\|_{1,w}.$$

The support of $g * v_\alpha$ is contained in $K\bar{U}$, and so

$$\|g * v_\alpha\|_{1,w} \le \sup_{x \in K\bar{U}} w(x) \|g * v_\alpha\|_1 \to 0.$$

Since $C_{00}(G)$ is $\|\cdot\|_{1,w}$-dense in $L_1(G, w)$, $\{u_\alpha + v_\alpha\}$ is a right a.i. for $L_1(G, w)$. Similarly we can show that $\{u_\alpha + v_\alpha\}$ is a left a.i. The net $\{u_\alpha + v_\alpha\}$ is $\|\cdot\|_{1,w}$-bounded since

$$\|u_\alpha + v_\alpha\|_{1,w} \le \sup_{x \in S} w(x) + \Big(\sup_{x \in \bar{U}} w(x)\Big)\epsilon$$

for all α.

With x as defined earlier, we show that

$$\langle v_\alpha * \delta_x - \delta_x * v_\alpha, C_{Vx} \rangle = \epsilon/3,$$

where C_{Vx} denotes the characteristic function of the set Vx. We first note that

$$(v_\alpha * \delta_x - \delta_x * v_\alpha)(t) = \Delta(x^{-1})(v_\alpha - \delta_x * v_\alpha * \delta_{x^{-1}})(tx^{-1}),$$

and that

$$(\delta_x * v_\alpha * \delta_{x^{-1}})(tx^{-1}) = v_\alpha(x^{-1}t).$$

The support of v_α is contained in U and so the support of $\delta_x * v_\alpha * \delta_{x^{-1}}$ is contained in xUx^{-1}. Now $xUx^{-1} \cap U = \emptyset$ and so $\operatorname{supp}(\delta_x * v_\alpha * \delta_{x^{-1}}) \cap V = \emptyset$. Thus

$$\begin{aligned}\langle v_\alpha * \delta_x - \delta_x * v_\alpha, C_{Vx} \rangle &= \Delta(x^{-1}) \int_G (v_\alpha - \delta_x * v_\alpha * \delta_{x^{-1}})(tx^{-1}) C_{Vx}(t)\,dt \\ &= \Delta(x^{-1}) \int_G (v_\alpha - \delta_x * v_\alpha * \delta_{x^{-1}})(tx^{-1}) C_V(tx^{-1})\,dt \\ &= \Delta(x^{-1})\Delta(x) \int_G (v_\alpha - \delta_x * v_\alpha * \delta_{x^{-1}})(t) C_V(t)\,dt \\ &= \int_V v_\alpha(t)\,dt = \epsilon/2,\end{aligned}$$

as required.

Finally, since

$$\langle \delta_x * (u_\alpha + v_\alpha) - (u_\alpha + v_\alpha) * \delta_x, C_{Vx} \rangle = \langle \delta_x * u_\alpha - u_\alpha * \delta_x, C_{Vx} \rangle + \langle \delta_x * v_\alpha - v_\alpha * \delta_x, C_{Vx} \rangle$$

and $C_{Vx} \in L_\infty(G, w)$, it follows that $(\delta_x * (u_\alpha + v_\alpha) - (u_\alpha + v_\alpha) * \delta_x)$ and $(\delta_x * u_\alpha - u_\alpha * \delta_x)$ cannot both converge weakly to 0. This contradicts the semi-regularity of $L_1(G, w)$; for, if (S, T) is the double multiplier on $L_1(G, w)$ corresponding to the measure δ_x and E_1 (resp. E_2) is a cluster point of $\{u_\alpha\}$ (resp. $\{u_\alpha + v_\alpha\}$), then $S^{**}E_1 = T^{**}E_1$ and $S^{**}E_2 = T^{**}E_2$ imply that both

$$(\delta_x * u_\alpha - v_\alpha * \delta_x)$$

and

$$(\delta_x * (u_\alpha + v_\alpha) - (u_\alpha + v_\alpha) * \delta_n)$$

converge weakly to 0, contrary to the above. The contradiction proves that G must be either abelian or discrete. □

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