A NOTE ON SURFACE FILM DRIVEN CONVECTION

by BRIAN STRAUGHAN

(Received 7 November, 1989)

1. Introduction. In [**6**] McTaggart presented a nonlinear energy stability analysis of the problem of convection in the presence of a surface film overlying a non-shallow layer of fluid heated from below. In her work the film is regarded as a two-dimensional continuum and surface tension is then introduced naturally as a combination of a surface density and the derivative of a surface free energy. In fact, the model originated with work of Landau and Lifschitz [**4**] on the effect of adsorbed films on the motion of a liquid. The precise model she uses was developed from a continuum thermodynamic viewpoint by Lindsay and Straughan [**5**].

While McTaggart's [**6**] analysis is very useful and shows quantitatively how the effects of surface viscosity, surface thermal conductivity and surface tension play an important role on the onset of convection, she leaves open a fundamental problem. Her analysis encounters a cubic surface term which she formally includes in the Euler-Lagrange equations: however, to make progress she is forced to set the coefficient of this term equal to zero. It is the purpose of this note to rectify this situation and show how McTaggart's [**6**] results represent rigorous nonlinear stability bounds in the theory of film driven convection. We point out Joseph [**3**] also found difficulty in bounding a surface term when he applied energy theory to convection driven by interfacial tension between two fluid layers: the present analysis leads one to wonder whether thin film theory will overcome his difficulty too.

2. Nonlinear energy stability. For a fluid in the infinite region $z \in (0, d)$, with prescribed temperature $T_0 = T(0)$ and $T_d = T(d)$, $z = d$ being the film and gravity being in the negative z-direction, McTaggart [**6**] studies the stability of the stationary (conduction) solution

$$\mathbf{v} \equiv \mathbf{0}, \qquad T = T_0 - \beta z,$$

where $\beta = (T_0 - T_d)/d$ (>0) and the (hydrostatic) pressure $\bar{p}$ is quadratic in z. She denotes a perturbation to $(\mathbf{v}, T, \bar{p})$ by $(\mathbf{u}, \theta, p)$ and then non-dimensionalises the perturbation equations. For disturbances periodic in (x, y) the energy functional she employs is

$$E(t) = \tfrac{1}{2}(\|\mathbf{u}\|^2 + \Pr\|\theta\|^2) + \tfrac{1}{2}S_1(\|\mathbf{u}\|_s^2 + \Pr\|\theta\|_s^2) \tag{1}$$

where $\|\,.\,\|$ and $\|\,.\,\|_s$ denote the L^2 norms in Ω and Γ, respectively, Ω being the period cell of the disturbance in the non-dimensionalised layer $z \in (0, 1)$ and Γ being that part of the boundary of Ω which intersects $z = 1$. The non-dimensional parameters Pr and S_1 are the Prandtl number $\Pr = \nu/\kappa$ and a non-dimensional density ratio, $S_1 = \gamma/\rho d$, where γ, ρ are, respectively, surface and bulk fluid density.

She shows that the energy (1) satisfies the equation

$$\begin{aligned}\frac{dE}{dt} = -D(\mathbf{u}) - D(\theta) + 2R\langle\theta w\rangle - \frac{B}{R}(1 - A_4)[\theta_{;\alpha}u^\alpha] - A_3[\theta_{;\alpha}\theta_{;\alpha}]\\ - A_1[(d^\lambda_\lambda)^2] - A_2[d^{\alpha\beta}d_{\alpha\beta}] - A_5[\theta^2 u^\alpha_{;\alpha}]\end{aligned} \tag{2}$$

where $D(.)$ denotes the Dirichlet integral on Ω, e.g.

$$D(\theta) = \|\nabla\theta\|^2,$$

$\langle . \rangle$ and $[.]$ denote integration over Ω and Γ, respectively, α, β denote surface coordinates x, y, $d_{\alpha\beta} = \frac{1}{2}(u_{\alpha;\beta} + u_{\beta;\alpha})$, and $R, B, A_1 - A_5$ are non-dimensional parameters defined as follows:

$$R^2 = \frac{g\alpha\beta d^4}{\kappa\nu}, \qquad B = \frac{s\beta d^2}{\rho\nu\kappa}, \qquad A_1 = \nu_2/\mu d, \qquad A_2 = \nu_6/\mu d,$$

$$A_3 = -q_0/dk, \qquad A_4 = T_d g\alpha/c_v\beta, \qquad A_5 = sd/\rho\nu k,$$

where g, α, κ, ν, s, are gravity, thermal expansion coefficient, bulk thermal diffusivity, bulk kinematic viscosity and surface tension. The terms R^2 and B are the Rayleigh and Marangoni numbers, and A_1, A_2 are surface/bulk viscosity ratios. A_3 represents the surface/bulk thermal conductivity effect, and c_v is the specific heat at constant volume of the bulk fluid.

McTaggart's [**6**] analysis essentially disregards the cubic surface term in (2), i.e. $A_5[\theta^2 u^{\alpha}_{;\alpha}]$. However, since $A_5 = sd/\rho\nu k$, we should not do this as it clearly implies vanishing surface tension or very thin layer theory. Hence, we re-develop her theory leaving in the nonlinear term. Define

$$\mathcal{D} = D(\mathbf{u}) + D(\theta) + A_3[\theta_{;\alpha}\theta_{;\alpha}] + A_1[(d^{\mu}_{\mu})^2] + A_2[d^{\alpha\beta}d_{\alpha\beta}], \tag{3}$$

$$I = 2\langle \theta w \rangle - N(1 - A_4)[\theta_{;\alpha}u^{\alpha}], \tag{4}$$

$$\mathcal{N} = -A_5[\theta^2 u^{\alpha}_{;\alpha}], \tag{5}$$

where $N = B/R^2$. The only difference with (3)–(5) and the analysis in [**6**] is that we have left $\mathcal{N}$ defined separately instead of including it in the production term I.

Using (3)–(5) in (2) it is now easy to derive

$$\frac{dE}{dt} \leqq -\frac{\mathcal{D}}{R_E}(R_E - R) + \mathcal{N}, \tag{6}$$

where

$$R_E^{-1} = \max_{\mathcal{H}} (I/\mathcal{D}),$$

$\mathcal{H}$ being the space of admissible solutions. To handle the nonlinear term we use the Cauchy-Schwarz inequality to write:

$$\mathcal{N} \leqq A_5 \|\theta^2\|_s \|d^{\mu}_{\mu}\|_s.$$

To proceed from this we need to assume the average of θ over Γ is zero, *i.e.*

$$\int_{\Gamma} \theta \, dA = 0,$$

and then by use of inequality (9) we observe that

$$\mathcal{N} \leqq A_5 c \|\theta\|_s [\theta_{;\alpha}\theta_{;\alpha}]^{1/2} \|d^{\mu}_{\mu}\|_s,$$

https://doi.org/10.1017/S0017089500008181 Published online by Cambridge University Press

where a value for the constant c is given in (9). Using the definitions of $E(t)$ and $\mathscr{D}$, see (1), (3), we may then write

$$\mathcal{N} \leqq \frac{cA_5 2^{1/2}}{(\Pr S_1 A_1 A_3)^{1/2}} E^{1/2}\mathscr{D} \equiv c_1 E^{1/2}\mathscr{D}. \tag{7}$$

Hence, putting (7) into (6) we find

$$\frac{dE}{dt} \leqq -\mathscr{D}\left\{\left(\frac{R_E - R}{R_E}\right) - c_1 E^{1/2}\right\}.$$

From this inequality it is now a standard process, using Poincaré's inequality for the bulk terms and Wirtinger's inequality for the surface terms in $\mathscr{D}$, to show that provided

$$\text{(A)} \quad R < R_E \quad \text{and} \quad \text{(B)} \quad E^{1/2}(0) < \frac{R_E - R}{c_1 R_E}, \tag{8}$$

then $E \to 0$ at least exponentially as $t \to \infty$ and so there is global stability.

The eigenvalue problem for R_E is precisely the one studied by McTaggart [**6**] and so her results do yield rigorous nonlinear stability bounds provided the initial value of the energy is not too large, as determined by (8) (B).

3. Derivation of a surface inequality. Suppose $\theta(x, y)$ is periodic in (x, y), of x-period k and of y-period l. Define Γ to be the rectangle $(\hat{x}, \hat{x}+k) \times (\hat{y}, \hat{y}+l)$, $\hat{x}, \hat{y}$ fixed. We now show that provided $\int_\Gamma \theta \, dA = 0$,

$$\int_\Gamma \theta^4 \, dA \leqq \left[1 + 4\pi^{-2} + \frac{2(k+l)}{\pi(kl)^{1/2}}\right] \int_\Gamma \theta^2 \, dA \int_\Gamma \theta_{;\alpha}\theta_{;\alpha} \, dA. \tag{9}$$

The proof employs Joseph's method [**2**], p. 249. Since θ has x-period k, y-period l, we write

$$2\int_{\hat{x}}^{x} \theta(\xi, y)\theta_\xi(\xi, y)\, d\xi + \theta^2(\hat{x}, y) = \theta^2(x, y) = -2\int_{x}^{\hat{x}+k} \theta(\xi, y)\theta_\xi(\xi, y)\, d\xi + \theta^2(\hat{x}, y),$$

and

$$2\int_{\hat{y}}^{y} \theta(x, \eta)\theta_\eta(x, \eta)\, d\eta + \theta^2(x, \hat{y}) = \theta^2(x, y) = -2\int_{y}^{\hat{y}+l} \theta(x, \eta)\theta_\eta(x, \eta)\, d\eta + \theta^2(x, \hat{y}).$$

From these expressions it is easily seen that

$$\theta^2(x, y) \leqq \int_{\hat{x}}^{\hat{x}+k} |\theta(\xi, y)|\,|\theta_\xi(\xi, y)|\, d\xi + \theta^2(\hat{x}, y), \tag{10}$$

$$\theta^2(x, y) \leqq \int_{\hat{y}}^{\hat{y}+l} |\theta(x, \eta)|\,|\theta_\eta(x, \eta)|\, d\eta + \theta^2(x, \hat{y}). \tag{11}$$

We now multiply (10) and (11) together and integrate over Γ twice to find, with the help of the Cauchy-Schwarz inequality:

$$kl\int_\Gamma \theta^4 \, dA \leqq kl\int_\Gamma \theta^2 \, dA \int_\Gamma \theta_{;\alpha}\theta_{;\alpha}\, dA + (k+l)\left\{\int_\Gamma \theta^2 \, dA\right\}^{3/2}\left\{\int_\Gamma \theta_{;\alpha}\theta_{;\alpha}\, dA\right\}^{1/2} + \left\{\int_\Gamma \theta^2 \, dA\right\}^2. \tag{12}$$

https://doi.org/10.1017/S0017089500008181 Published online by Cambridge University Press

We now make use of the Wirtinger inequality, see Hardy *et al.* [**1**], p. 184, which shows

$$\int_\Gamma \theta^2\,dA \leqq \frac{4kl}{\pi^2}\int_\Gamma \theta_{;\alpha}\theta_{;\alpha}\,dA.$$

Putting this into (12) and dividing by kl we arrive at (9).

REFERENCES

1. G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities.* (Cambridge University Press, 1934).

2. D. D. Joseph, *Stability of fluid motions I.* (Springer-Verlag 1976).

3. D. D. Joseph, Two fluids heated from below. In, "*Energy stability and convection: Proceedings of the Capri workshop*", pp. 364–382. Pitman Research Notes in Mathematics, vol. 168. (Longman, 1988).

4. L. D. Landau, and E. M. Lifschitz, *Fluid mechanics.* (Pergamon Press, 1959).

5. K. A. Lindsay, and B. Straughan, A thermodynamic viscous interface theory and associated stability problems. *Arch. Rational Mech. Anal.* **71** (1979), 307–326.

6. C. L. McTaggart, On the stabilizing effect of surface films in Bénard convection. *Physico Chemical Hydrodynamics* **5** (1984), 321–331.

Department of Mathematics,
University Gardens,
Glasgow G12 8QW,
Scotland

https://doi.org/10.1017/S0017089500008181 Published online by Cambridge University Press