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ORDER COMPARISONS ON CANONICAL ISOMORPHISMS

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Consider a nonnegative Hölder continuous 2-form $P(z)dxdy$ ($z = x + iy$) on a connected Riemann surface R. We denote by $P(R)$ the linear space of solutions u of the equation $\Delta u = Pu$ on R and by $PX(R)$ the subspace of $P(R)$ consisting of those u with a certain boundedness property X. We also use the standard notations $H(R)$ and $HX(R)$ for $P(R)$ and $PX(R)$ with $P \equiv 0$. As for X we take B to mean the finiteness of the supremum norm $\|u\| = \sup_R |u|$, D the finiteness of the Dirichlet integral $D(u) = \int_R du \wedge^* du$, E the finiteness of the energy integral $E(u) = \int_R (du \wedge^* du + u^2(z)P(z)dxdy)$, and their nontrivial combinations BD and BE. Let $Q(z)dxdy$ be another 2-form of the same kind. We say that $PX(R)$ is canonically isomorphic to $QX(R)$ if there exists a linear isomorphism T of $PX(R)$ onto $QX(R)$ such that u and Tu have the same ideal boundary values for every u in $PX(R)$ in the sense that $|u - Tu|$ is dominated by a potential on R, i.e. a nonnegative superharmonic function whose greatest harmonic minorant is zero. In the pioneering work [14] concerning canonical isomorphisms, Royden proved the following *order comparison theorem*: If there exists a constant $c \geq 1$ such that

$$c^{-1}P(z) \leq Q(z) \leq cP(z) \tag{1}$$

on hyperbolic R except possibly for a compact subset K of R, then $PB(R)$ and $QB(R)$ are canonically isomorphic. In this connection we wish to discuss the following two questions:

1°. *Is the condition* (1) *also sufficient for* $PX(R)$ *and* $QX(R)$ *to be canonically isomorphic for* $X = D$, E, BD, *and* BE?

2°. *In the affirmative case how large can we make the exceptional set* K *in* (1) *for* $X = B$, D, E, BD, *and* BE?

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We shall actually show that the answer to 1° is affirmative. To give the complete answer to 2° is probably very difficult. In this paper we shall considerably enlarge the class of exceptional sets as follows. Let W be an open subset of R with an analytic relative boundary ∂W and consider the relative class $HX(W;\partial W)=\{u\in HX(W)\cap C(R);\,u|R-W=0\}$ for $X=B,D,$ and BD. The Heins injection $\lambda_H: HX(W;\partial W)\to HX(R)$ is given by $\lambda_H u=\lim_{\Omega\to R}H_u^{\Omega}$. We say that a subset K of R is X-negligible ($X=B,D,$ and BD) if either R is parabolic or there exists a W with $K\subset R-W$ such that $\lambda_H: HX(W;\partial W)\to HX(R)$ is surjective. We will see that a subset K of hyperbolic R is B-negligible (BD-negligible, resp.) if and only if there exists a potential (Dirichlet finite potential, resp.) p such that $p\geq 1$ on K. Such a simple characterization for D-negligible sets is not available. We only know that if there exists a Dirichlet finite potential p harmonic outside a compact set of R such that $p\geq 1$ on K, then K is D-negligible. By these criterions we see that compact sets are trivial examples of our negligible sets. The purpose of this paper is to contribute to the solution of questions 1° and 2° as follows:

ORDER COMPARISON THEOREM. *Let R be a hyperbolic connected Riemann surface. If* (1) *is valid on R except possibly for a B-negligible subset of R, then $PB(R)$ and $QB(R)$ are canonically isomorphic; if* (1) *is valid on R except possibly for a BD-negligible subset of R, then $PBD(R)$ ($PBE(R)$, resp.) and $QBD(R)$ ($QBE(R)$, resp.) are canonically isomorphic; if* (1) *is valid on R except possibly for a D-negligible subset of R, then $PD(R)$ ($PE(R)$, resp.) and $QD(R)$ ($QE(R)$, resp.) are canonically isomorphic.*

We excluded parabolic R since in such a case $PX(R)=\{0\}$ for every $X=B,D,E,BD,BE$ and $P\not\equiv 0$, and $HX(R)=\{\text{constants}\}$. In nos. 1–4, we shall prove that the theorem is true if the exceptional set in (1) is empty. We consider canonical injections: $PX(R)\to QX(R)$ and reduction operators: $PX(R)\to HX(R)$ as preparations for considering canonical isomorphisms. The notion of quasipotential will be introduced and prove to be useful; at least it is a convenient terminolgy. In nos. 5–7, the surjectiveness of canonical extensions $PX(W;\partial W)\to PX(R)$ will be discussed. After these preparations, the proof of our order comparison theorem will be given in no. 8. Although our main concern in this paper is the order comparisons, we will append a sketch of the other kind of important criterion, the integral comparisons, for the existence of canonical isomorphisms in nos. 9–10. The first integral comparison theorem in no. 9 generalizes those thus far known. The second one in no. 10 completely characterizes the surjectiveness of reduction operators: $PX(R) \to HX(R)$ for $X = B, BD$, and BE. Methodologically the use of the compactification theory of Riemann surfaces would give more clearer geometric intuitive insight to the whole discussion in this paper. However to make the description as elementary as possible we will intentionally avoid its use even if it is preferable.

Canonical Injections

1. It will be convenient to include disconnected but separable surfaces in our considerations. Therefore we assume that our Riemann surface R is either connected or an open subset of a connected Riemann surface. By a *regular open set* we mean a finite union of closure disjoint regular regions in R. We use the notation Ω for regular open sets. The totality $\{\Omega\}$ of regular open sets Ω of R forms a directed net by inclusions converging to R. A 2-form $P(z)dxdy$ $(z = x + iy)$ on R is said to be Hölder continuous if, for each parametric disk (U, z), there exist constant $K = K(U, z) \in (0, \infty)$ and $\alpha = \alpha(U, z) \in (0, 1]$ such that $|P(z_1) - P(z_2)| \leq K|z_1 - z_2|^{\alpha}$ for every pair of points z_1 and z_2 in U. We say that $P(z)dxdy$ is nonnegative, $P(z)dxdy \geq 0$ or $P(z) \geq 0$ in notation, if, for each parametric disk (U, z), $P(z) \geq 0$ for every z in U. These are well defined since such properties are invariant under the change of local parameters. In particular the order $P(z)dxdy \geq Q(z)dxdy$ or $P(z) \geq Q(z)$ can be defined between two 2-forms by $(P(z) - Q(z))dxdy \geq 0$. The 2-form $P(z)dxdy$ is (identically) zero if, for each parametric disk (U, z), $P(z) \equiv 0$ in U. We simply denote this by $P \equiv 0$. Nonnegative Hölder continuous 2-forms on R will be denoted by $P(z)dxdy$, $Q(z)dxdy$, etc.

We denote by P_{φ}^{Ω} the solution of the Dirichlet problem of the equation $\Delta u = Pu$ on Ω with a continuous boundary values φ on the relative boundary $\partial\Omega$, i.e. $P_{\varphi}^{\Omega} \in P(\Omega) \cap C(\bar{\Omega})$ with $P_{\varphi}^{\Omega}|\partial\Omega = \varphi$, where $P(\Omega)$ is the linear space of C^2 solutions of $\Delta u = Pu$ on Ω. By the limiting process we can define P_{φ}^{Ω} even for upper- or lower- semicontinuous functions φ. We also use the standard notation H_{φ}^{Ω} and $H(\Omega)$ for P_{φ}^{Ω} and $P(\Omega)$ with $P \equiv 0$. Let $G_{\Omega}(z, \zeta)$ be the *harmonic* Green's function on Ω. If Ω is connected, then there is no question about its definition. If $\Omega = \bigcup_{i=1}^{n} \Omega_i$

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with Ω_i connected and $\bar{\Omega}_i \cap \bar{\Omega}_j = \phi$ $(i \neq j)$, then $G_\Omega(z,\zeta) = G_{\Omega_i}(z,\zeta)$ for $z, \zeta \in \Omega_i$ and $G_\Omega(z,\zeta) = 0$ for $z \in \Omega_i$ and $\zeta \in \Omega_j$ $(i \neq j)$. Then

$$P_\varphi^\Omega = H_\varphi^\Omega - \frac{1}{2\pi}\int_\Omega G(\cdot,\zeta)P_\varphi^\Omega(\zeta)P(\zeta)d\xi d\eta \,, \tag{2}$$

where $\zeta = \xi + i\eta$, and the *Dirichlet integral* $D_\Omega(P_\varphi^\Omega) = \int_\Omega dP_\varphi^\Omega \wedge * \, dP_\varphi^\Omega$ is given by

$$D_\Omega(P_\varphi^\Omega) = D_\Omega(H_\varphi^\Omega) + \frac{1}{2\pi}\int_{\Omega\times\Omega} G_\Omega(z,\zeta)P_\varphi^\Omega(z)P_\varphi^\Omega(\zeta)P(z)dxdyP(\zeta)d\xi d\eta \tag{3}$$

(cf. e.g. [9,10]). Since the *energy integral* $E_\Omega(u) = E_\Omega^P(u) = D_\Omega(u) + \int_\Omega u^2(z)P(z)dxdy$ is the variation whose Euler-Lagrange equation is $\Delta u = Pu$, we have the so-called energy principle (Dirichlet principle):

$$E_\Omega(P_\varphi^\Omega) = \min\left(E_\Omega(u)\,;\, u \in C(\bar{\Omega}) \cap C^w(\Omega), u|\partial\Omega = \varphi\right) , \tag{4}$$

where C^w is the class of weakly differentiable functions (cf. e.g. [1]). Another simple but important fact which will be used repeatedly is that $|u|$ *and* $u \cup 0 = \max(u, 0)$ *are subharmonic for every* $u \in P(R)$. Since $\Delta u = Pu \geq 0$ in $\{u > 0\}$, $u \cup 0 = u$ is subharmonic in $\{u > 0\}$, and so is $u \cup 0 = 0$ in $\{u < 0\}$. The submean value property is clearly valid at each point of $\{u = 0\}$ for $u \cup 0$. Therefore $u \cup 0$ and $(-u) \cup 0$ are subharmonic on R, and so is $|u| = u \cup 0 + (-u) \cup 0$. A *potential* p on R is a nonnegative superharmonic function whose greatest harmonic minorant is zero. If $\Omega \subset \Omega'$, then $0 \leq H_p^{\Omega'} \leq H_p^\Omega$ and $\lim_{\Omega\to R} H_p^\Omega$ is a nonnegative harmonic function on R dominated by p. Thus $\lim_{\Omega\to R} H_p^\Omega = 0$ and actually this is the defining property for a nonnegative superharmonic function p to be a potential. A function f on R will be referred to as a *quasipotential* if $|f|$ is majorated by a potential $p = p_f$. Clearly the class of quasipotentials (potentials, resp.) forms a linear (additive, resp.) space. Since $|P_f^\Omega| \leq P_{|f|}^\Omega \leq P_p^\Omega \leq H_p^\Omega$ with $p = p_f$, we have

$$\lim_{\Omega\to R} P_f^\Omega = \lim_{\Omega\to R} P_{|f|}^\Omega = 0 \tag{5}$$

for every upper- or lower- semicontinuous quasipotential f on R. The following fact will also be used repeatedly: *If* f *is a quasipotential such that* $|f|$ *is subharmonic, then* $f = 0$. This follows from (5) and the inequality $|f| \leq H_{|f|}^\Omega$.

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We denote by $PB(R)$ the subspace of $P(R)$ consisting of solutions u with the finite supremum norms: $\|u\| = \|u\|_R = \sup_R |u| < \infty$. The notation $PD(R)$ is used for the subspace consisting of solutions u with the finite Dirichlet integrals $D(u) = D_R(u) < \infty$. The subspace $PE(R)$ consists of solutions u such that the energy integrals $E(u) = E_R(u) = E_R^P(u) < \infty$. Similarly $PBD(R) = PB(R) \cap PD(R)$ and $PBE(R) = PB(R) \cap PE(R)$. Contrary to $PB(R)$ and $PD(R)$, the scale E for which every solution in $PE(R)$ is finite varys according to P. We use the standard notations $HX(R)$ for $PX(R)$ with $P \equiv 0$ $(X = B, D, E, BD, BE)$. In this case $E(u) = D(u)$ and thus e.g. $HE(R) = HD(R)$. We denote by $PX(R)^+$ the subset of $PX(R)$ consisting of nonnegative solutions. It is of fundamental importance that $PX(R)^+$ generates $PX(R)$ for $X = B, D, E, BD$, and BE (cf. e.g. [14], [7], [2]). We say that $PX(R)$ and $QX(R)$ $(X = B, D, E, BD, BE)$ are *canonically isomorphic* if there exists a linear isomorphism T of $PX(R)$ onto $QX(R)$ such that $u - Tu$ is a quasipotential for every $u \in PX(R)$. *The operator T is unique, order preserving, and isometric,* and will be referred to as the *canonical isomorphism.* In fact, if T' is another such operator, then $|Tu - T'u| \leq |u - Tu| + |u - T'u|$ and thus $|Tu - T'u|$ is a quasipotential. Since $|Tu - T'u|$ is subharmonic, $Tu = T'u$ for every $u \in PX(R)$. Suppose $u \in PX(R)^+$. Since $Tu \geq u - |u - Tu|$, $Q_u^\Omega \geq 0$, and $Q_{Tu}^\Omega = Tu$, we have $Tu \geq Q_u^\Omega - Q_{|u-Tu|}^\Omega \geq -Q_{|u-Tu|}^\Omega$ on Ω. On letting $\Omega \to R$, we conclude by (5) that $Tu \geq 0$, i.e. $Tu \in QX(R)^+$. For general $u \in PX(R)$, observe that $|Tu| \leq |u| + |u - Tu|$. Since $H_{|u|}^\Omega \leq \|u\|_{\partial\Omega} \leq \|u\|$, by the maximum principle for subharmonic functions we see that $|Tu| \leq H_{|Tu|}^\Omega \leq \|u\| + H_{|u-Tu|}^\Omega$ on Ω. Again by (5) we conclude that $\|Tu\| \leq \|u\|$. Similarly $|u| \leq |Tu| + |u - Tu|$ implies $|u| \leq H_{|u|}^\Omega \leq \|Tu\| + H_{|u-Tu|}^\Omega$ and then $\|u\| \leq \|Tu\|$. Therefore $\|Tu\| = \|u\| \leq \infty$.

2. To study the existence of canonical isomorphism, it is convenient to consider a *canonical injection* $T = T_{Q,P}$ of $PX(R)$ into $QX(R)$ $(X = B, D, E, BD, BE)$. It is a linear operator from $PX(R)$ into $QX(R)$ such that $u - Tu$ is quasipotential for every $u \in PX(R)$. It is actually *injective.* If $Tu = Tv$, then $|u - v| \leq |u - Tu| + |v - Tv|$ shows that the subharmonic function $|u - v|$ is a quasipotential and hence $u = v$. By exactly the same proof as in the last part of no. 2 we see that *the canonical injection is unique, order preserving, and isometric.* If T_{P_2,P_1} and T_{P_3,P_2} exist for X, then T_{P_3,P_1} exists for X and

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$$(6) \qquad T_{P_3,P_1} = T_{P_3,P_2} \circ T_{P_2,P_1} .$$

In fact, $T = T_{P_3,P_2} \circ T_{P_2,P_1}$ is a linear operator from $P_1X(R)$ into $P_3X(R)$.

$$\begin{array}{ccc} P_1X(R) & \xrightarrow{T_{P_3,P_1}} & P_3X(R) \\ {\scriptstyle T_{P_2,P_1}}\downarrow & \nearrow {\scriptstyle T_{P_3,P_2}} & \\ P_2X(R) & & \end{array}$$

Let $u \in P_1X(R)$ and set $v = T_{P_2,P_1}u \in P_2X(R)$. Since $|u - Tu| \le |u - T_{P_2,P_1}u| + |v - T_{P_3,P_2}v|$, $u - Tu$ is a quasipotential and thus T is the canonical injection T_{P_3,P_1}. To determine pairs (P, Q) such that $T_{Q,P}$ exists is very important but a difficult problem. For our present purpose we will only prove that pairs (P, Q) with $P \ge Q$ have this property, which is originally obtained by Royden [14] and in abstract setting by Loeb [4] for $X = B$ (cf. also Maeda [5], Glasner-Katz [2]). It is convenient to discuss first the existence of $T_P = T_{0,P}$, which is in particular referred to as the *reduction operator*. The term is employed by Singer [16] to suggest that the operator T_P reduces the study of the class $PX(R)$ to that of more manageable class $HX(R)$. In this context, it is also important to determine P such that T_P is surjective, i.e. $PX(R)$ is canonically isomorphic to $HX(R)$. For $X = B, BD$, and BE, a complete answer is known (cf. Appendix, no. 10). For $X = D$ and E, we only have partial informations (cf. Singer [17], [11]). Here we only prove that *the reduction operator T_P always exists uniquely for every P*. Let $u \in PX(R)$. If $X = D$ or E, then $D(u) < \infty$ and the harmonic decomposition of Royden-Brelot (cf. e.g. [1], [15]) assures that $\lim_{\Omega \to R} H^{\Omega}_u \in HD(R)$. If $X = B$, then let $u = u_1 - u_2$ with $u_i \in PB(R)^+$ $(i = 1, 2)$. Since $\{H^{\Omega}_{u_j}\}$ is increasing, $|H^{\Omega}_{u_j}| \le \|u_i\|$, and $H^{\Omega}_u = H^{\Omega}_{u_1} - H^{\Omega}_{u_2}$, we also conclude that $\lim_{\Omega \to R} H^{\Omega}_u \in HB(R)$. Therefore the linear operator T_P of $PX(R)$ into $HX(R)$ can be defined by

$$(7) \qquad T_P u = \lim_{\Omega \to R} H^{\Omega}_u$$

for every $u \in PX(R)$ $(X = B, D, E, BD, BE)$. To see that (7) is actually the reduction operator, let $u = u_1 - u_2$ with $u_i \in PX(R)^+$ $(i = 1, 2)$. Let h_i be an arbitrary harmonic function with $h_i \ge u_i$. By the subharmonicity of u_i, $u_i \le H^{\Omega}_{u_i} \le h_i$ and thus $T_P u_i \le h_i$. This means that $T_P u_i$ is the least harmonic majorant of u_i. Therefore $T_P u_i - u_i$ is a potential $(i = 1, 2)$ and $|u - T_P u| \le |u_1 - T_P u_1| + |u_2 - T_P u_2|$ and a fortiori $u - T_P u$ is a quasipotential.

Besides (7) the following representation of T_P is also useful. We define the *harmonic Green's function* $G_R(z,\zeta)$ on a Riemann surface R as follows. Let $R=\bigcup_n R_n$ be the decomposition into connected components R_n. Thus R_n is a connected Riemann surface. For $z,\zeta\in R_n$, let $G_R(z,\zeta)=G_{R_n}(z,\zeta)$ the usual harmonic Green's function on R_n if R_n is hyperbolic and $G_R(z,\zeta)=+\infty$ if R_n is parabolic. For $z\in G_n$ and $\zeta\in R_m$ $(n\neq m)$, we set $G_R(z,\zeta)=0$. Therefore

$$G_R(z,\zeta)=\lim_{\Omega\to R} G_\Omega(z,\zeta)$$

Let $u\in PX(R)$. Then

$$T_Pu=u+\frac{1}{2\pi}\int_R G_R(\cdot,\zeta)u(\zeta)P(\zeta)d\xi d\eta\ . \tag{8}$$

More precisely if $P\not\equiv 0$ on R_n and $PX(R_n)$ contains a nonzero function then $G_R(z,\zeta)\not\equiv\infty$ for z and ζ in R_n. Since $P_u^\Omega=u$, we see by (2) that

$$H_u^\Omega=u+\frac{1}{2\pi}\int_\Omega G_\Omega(\cdot,\zeta)u(\zeta)P(\zeta)d\xi d\eta\ .$$

If $u\in PX(R)^+$, then the integrand is increasing with respect to Ω and therefore the Lebesgue-Fatou theorem implies that

$$T_Pu=u+\frac{1}{2\pi}\int_R (\lim_{\Omega\to R} G^\Omega(\cdot,\zeta))u(\zeta)P(\zeta)d\xi d\eta\ .$$

Since $PX(R)^+$ generates $PX(R)$, we see the validity of (8). By (3) we also have

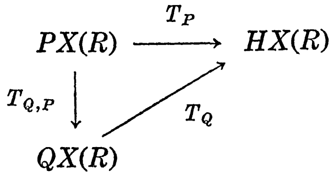
$$D(u)=D(T_Pu)+\frac{1}{2\pi}\int_{R\times R} G_R(z,\zeta)u(z)u(\zeta)P(z)dxdyP(\zeta)d\xi d\eta \tag{9}$$

for $u\in PX(R)$ such that the integral has definite meaning. This is the case for $u\in PX(R)^+$ for every X and for $u\in PX(R)$ $(X=D,E)$; and if $u\in PX(R)$ $(X=D,E)$, then each term in (9) is finite. By the energy principle (Dirichlet principle)

$$\langle u,u\rangle_R^P=\int_{R\times R} G_R(z,\zeta)u(z)u(\zeta)P(z)dxdyP(\zeta)d\xi d\eta\geq 0 \tag{10}$$

as soon as the integral is definite. The quantity is referred to as the *P-Green energy* and the relation (10) is known as that the Green kernel is of positive type in the Green potential theory.

If the canonical injection $T_{Q,P}$ of $PX(R)$ into $QX(R)$ exists, then, since $|T_{Q,P}u - u|$ and $|T_Q(T_{Q,P}u) - T_{Q,P}u|$ are quasipotentials, the inequality $|T_Q(T_{Q,P}u) - u| \leq |T_Q(T_{Q,P}u) - T_{Q,P}u| + |T_{Q,P}u - u|$ shows that $u - T_Q(T_{Q,P}u)$ is a quasipotential. Therefore, by the uniqueness of the reduction operator,



the linear operator $T_Q \circ T_{Q,P}$ from $PX(R)$ into $HX(R)$ must be the reduction operator T_P, i.e.

$$T_P = T_Q \circ T_{Q,P} . \tag{11}$$

We also have an analogue of (7): If $T_{Q,P}$ for X exists, then

$$T_{Q,P}u = \lim_{\Omega \to R} Q_u^\Omega \tag{12}$$

for every $u \in PX(R)$. In fact, $Q_u^\Omega = Q_{T_{Q,P}u}^\Omega + Q_{u - T_{Q,P}u}^\Omega$ and (5) imply the relation. From (11) it follows that *$T_{Q,P}$ from $PX(R)$ into $QX(R)$ exists if and only if*

$$T_P(PX(R)) \subset T_Q(QX(R)) , \tag{13}$$

and in this case $T_{Q,P} = T_Q^{-1} \circ T_P$. The necessity is clear by (11). Conversely, if (13) is valid, then the inverse T_Q^{-1} of T_Q from $T_Q(QX(R))$ onto $QX(R)$ can be defined on $T_P(PX(R))$. Let $u \in PX(R)$ and set $v = T_Pu$ and $w = T_Q^{-1}v \in QX(R)$. Then $|u - T_Q^{-1}(T_Pu)| = |u - w| \leq |w - v| + |u - v| = |w - T_Qw| + |u - T_Pu|$ shows that $u - T_Q^{-1}(T_Pu)$ is a quasipotential for every $u \in PX(R)$. Therefore the operator $T_Q^{-1} \circ T_P$ is the canonical injection from $PX(R)$ into $QX(R)$. From this we see that *if the canonical injections $T_{Q,P}$ and $T_{P,Q}$ exist for X, then they are canonical isomorphisms, i.e. $T_{P,Q} = T_{Q,P}^{-1}$ and $T_{Q,P} = T_{P,Q}^{-1}$.* In this case, we see by (13)

$$T_P(PX(R)) = T_Q(QX(R)) \tag{14}$$

and

$$T_{Q,P} = T_Q^{-1} \circ T_P , \qquad T_{P,Q} = T_P^{-1} \circ T_Q \tag{15}$$

are both surjective. Actually (14) *is a necessary and sufficient condition*

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for $PX(R)$ and $QX(R)$ to be canonically isomorphic. In particular, if T_P and T_Q are surjective, then $PX(R)$ and $QX(R)$ are canonically isomorphic.

3. We are ready to prove that *if $P \geq Q$, then the canonical injection $T_{Q,P}$ from $PX(R)$ into $QX(R)$ exists* ($X = B, D, E, BD, BE$). Since $P \geq Q$ implies $u = P_u^\Omega \leq Q_u^\Omega$ for $u \in PX(R)^+$, $u \leq Q_u^\Omega \leq H_u^\Omega \leq T_P u$ and $\{Q_u^\Omega\}$ is increasing. Therefore

$$Tu = \lim_{\Omega \to R} Q_u^\Omega$$

exists and $u - Tu$ is a quasipotential in view of $|u - Tu| = Tu - u \leq T_P u - u = |u - T_P u|$ for every $u \in PX(R)^+$. Since $PX(R)^+$ generates $PX(R)$, this is also true for every $u \in PX(R)$ and T is a linear operator from $PX(R)$ into $Q(R)$. We only have to show that $Tu \in QX(R)$ in order to conclude that $T = T_{Q,P}$. If $X = B$, then $|Q_u^\Omega| \leq \|u\|$ and $Tu \in QB(R)$. If $X = E$, then, by the energy principle and $Q \leq P$,

$$E_\Omega^Q(Q_u^\Omega) \leq E_\Omega^Q(u) \leq E_\Omega^P(u) \leq E_R^P(u) .$$

Therefore, since $dQ_u^\Omega \wedge^* dQ_u^\Omega \to dTu \wedge^* dTu$, the Fatou lemma yields $E_R^Q(Tu) \leq E_R^P(u)$ and $Tu \in QE(R)$. Finally let $u \in PD(R)$. We wish to show that $Tu \in QD(R)$. For this purpose we may assume that $u \in PD(R)^+$. Then, by $Q \leq P$, $0 \leq u = P_u^\Omega \leq Q_u^\Omega$. Thus $E_\Omega^Q(Q_u^\Omega) \leq E_\Omega^Q(u)$ implies that

$$\begin{aligned} D_\Omega(Q_u^\Omega) &\leq D_\Omega(u) - \int_\Omega (Q_u^\Omega(z))^2 Q(z)dxdy + \int_\Omega u^2(z)Q(z)dxdy \\ &\leq D_\Omega(u) - \int_\Omega u^2(z)Q(z)dxdy + \int_\Omega u^2(z)Q(z)dxdy = D_\Omega(u) . \end{aligned}$$

Hence $D_\Omega(Q_u^\Omega) \leq D_\Omega(u) \leq D_R(u)$ and the Fatou lemma yields $D_R(Tu) \leq D_R(u)$, i.e. $Tu \in QD(R)$. Next we prove that *$PX(R)$ and $(cP)X(R)$ are canonically isomorphic for every $c > 0$.* We only have to show this for $c \leq 1$. For, if $c > 1$, then $(cP)X(R)$ is canonically isomorphic to $(c^{-1}(cP))X(R) = PX(R)$ since $c^{-1} < 1$. Then, since $P \geq cP$, the canonical injection $T_{cP,P}$ exists and by (13),

$$T_P(PX(R)) \subset T_{cP}((cP)X(R)) \quad (\subset HX(R)) ,$$

and by (14) we only have to show that this inclusion is improper, i.e. any $h \in T_{cP}((cP)X(R))$ belongs to $T_P(PX(R))$. Since T_{cP} is order preserving and $(cP)X(R)^+$ generates $(cP)X(R)$, we may assume that $h = T_{cP}v \geq 0$ with $v \in (cP)X(R)^+$. For brevity set $Q = cP$. Observe that $0 \leq P_v^\Omega \leq Q_v^\Omega = v$, $P_v^\Omega = P_h^\Omega + P_{v-T_Q v}^\Omega$, and $\{P_v^\Omega\}$ is decreasing. Thus

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$$0 \leq u = \lim_{\Omega \to R} P_v^\Omega = \lim_{\Omega \to R} P_h^\Omega \leq v$$

exists and $u \in P(R)^+$. By (8)

$$h = v + \frac{1}{2\pi}\int_R G_R(\cdot\,, \zeta)v(\zeta)cP(\zeta)d\xi d\eta$$

and in particular

$$\int_R G_R(\cdot\,, \zeta)v(\zeta)P(\zeta)d\xi d\eta < \infty\ .$$

Therefore, since $G_\Omega(\cdot\,, \zeta)P_v^\Omega(\zeta) \leq G_R(\cdot\,, \zeta)v(\zeta)$ and $h = \lim_{\Omega \to R} H_v^\Omega$, by applying the Lebesgue dominated convergence theorem to

$$H_v^\Omega = P_v^\Omega + \frac{1}{2\pi}\int_R G_\Omega(\cdot\,, \zeta)P_v^\Omega(\zeta)P(\zeta)d\xi d\eta$$

as $\Omega \to R$, we obtain

$$h = u + \frac{1}{2\pi}\int_R G_R(\cdot\,, \zeta)u(\zeta)P(\zeta)d\xi d\eta\ .$$

If $X = B$, then $|P_v^\Omega| \leq \|v\|$ implies $\|u\| < \infty$ and a fortiori $u \in PB(R)^+$. Since $h - u$ is a potential, we must have $T_P u = h$. If $X = E$, then the energy principle implies that

$$E_\Omega^P(P_v^\Omega) \leq E_\Omega^P((cP)_v^\Omega) = E_\Omega^P(v) \leq c^{-1}E_\Omega^{cP}(v) \leq c^{-1}E_R^{cP}(v)\ .$$

The Fatou lemma yields $E_R^P(u) < \infty$ and $u \in PE(R)^+$. Similarly as above $h = T_P u$. Finally let $X = D$. By (9) and (10)

$$D(v) = D(h) + \frac{1}{2\pi}\int_{R\times R} G_R(z, \zeta)v(z)v(\zeta)cP(z)dxdycP(\zeta)d\xi d\eta$$

and we have $\langle v, v\rangle_R^{cP} < \infty$. Thus $\langle u, u\rangle_R^P = c^{-2}\langle u, u\rangle_R^{cP} \leq c^{-2}\langle v, v\rangle_R^{cP} < \infty$. Again by (9), $D(u) = D(h) + (1/2\pi)\langle u, u\rangle_R^P < \infty$. Therefore $u \in PD(R)$ and as before $h = T_P u$. Combining two main assertions in this no., we maintain:

PROPOSITION. *If there exists a constant $c \geq 1$ such that $c^{-1}P \leq Q \leq cP$ on R, then $PX(R)$ and $QX(R)$ are canonically isomorphic for $X = B, D, E, BD$, and BE.*

Proof. Since $Q \geq c^{-1}P$, $T_{c^{-1}P,Q}$ exists. In view of that $(c^{-1}P)X(R)$ and $PX(R)$ are canonically isomorphic, $T_{P,c^{-1}P}$ exists and therefore by (6),

$$\begin{array}{ccc} QX(R) & \xrightarrow{T_{P,Q}} & PX(R) \\ {\scriptstyle T_{c^{-1}P,Q}}\downarrow & \nearrow {\scriptstyle T_{P,c^{-1}P}} & \\ (c^{-1}P)X(R) & & \end{array} \qquad \begin{array}{ccc} PX(R) & \xrightarrow{T_{Q,P}} & QX(R) \\ {\scriptstyle T_{cP,P}}\downarrow & \nearrow {\scriptstyle T_{Q,cP}} & \\ (cP)X(R) & & \end{array}$$

$T_{P,Q} = T_{P,c^{-1}P} \circ T_{c^{-1}P,Q}$ exists. Similarly $T_{Q,P}$ exists (see the above diagrams) and therefore $PX(R)$ and $QX(R)$ are canonically isomorphic.

Q.E.D.

Canonical Extensions

4. For convenience we say that an open subset W of R is *normal* if each point z of the relative boundary ∂W of W posesses a parametric disk U at z such that $U \cap \partial W$ is a diameter of U. Hereafter we always use W for normal open subsets of R. Consider the linear spaces

$$P(W; \partial W) = \{u \in P(W) \cap C(R); u|R - W = 0\}$$

and

$$PX(W; \partial W) = \{u \in PX(W) \cap C(R); u|R - W = 0\}$$

for $X = B, D, E, BD$, and BE. A linear operator $\lambda = \lambda_P = \lambda_P^W$ from $PX(W; \partial W)$ into $PX(R)$ is said to be a *canonical extension* if $u - \lambda u$ is a quasipotential on R for every $u \in PX(W; \partial W)$. We write $H(W; \partial W)$, $HX(W; \partial W)$, and λ_H for $P(W; \partial W)$, $PX(W; \partial W)$, and λ_P with $P \equiv 0$. As $PX(R)$, $PX(W; \partial W)$ is generated by $PX(W; \partial W)^+$ (cf. e.g. [9]). Similarly as in nos. 1 and 2, we see that *the canonical extension is unique, injective, order preserving, and isometric.* We next prove that *the canonical extension λ_P always exists for $X = B, D, E, BD$, and BE and for any P.* First let $u \in PX(W; \partial W)^+$. Since u is subharmonic on R, $u \le P_u^\Omega \le P_u^{\Omega'} \le H_u^{\Omega'}$ for $\Omega \subset \Omega'$. If $X = B$, then $0 \le P_u^\Omega \le \|u\|$. If $X = E$, then $E_\Omega^P(P_u^\Omega) \le E_\Omega^P(u) \le E_R^P(u)$. If $X = D$, then $E_\Omega^P(P_u^\Omega) \le E_\Omega^P(u)$ implies that

$$D_\Omega(P_u^\Omega) \le D_\Omega(u) - \int_\Omega ((P_u^\Omega(z))^2 - u^2(z))P(z)dxdy$$

Since $(P_u^\Omega(z))^2 - u^2(z) \ge 0$, $D_\Omega(P_u^\Omega) \le D_\Omega(u) \le D_R(u)$. In the latter two cases, the harmonic decomposition of u yields the convergence of $\{H_u^\Omega\}$. Therefore

$$\lambda_P u = \lim_{\Omega \to R} P_u^\Omega \tag{16}$$

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exists and belongs to $PX(R)$ for every $u \in PX(W; \partial W)^+$ and hence for every $u \in PX(W; \partial W)$. Thus $\lambda = \lambda_P$ is a linear operator from $PX(W; \partial W)$ into $PX(R)$ $(X = B, D, E, BD, BE)$. Again let $u \in PX(W; \partial W)^+$ and $h \in H(R)$ with $u \leq h$. Then $u \leq P_u^\Omega \leq H_u^\Omega \leq H_h^\Omega = h$ implies that $\lambda_H u = \lim_{\Omega \to R} H_u^\Omega \leq h$ and thus $\lambda_H u$ is the least harmonic majorant of u. Therefore $\lambda_H u - u$ is a potential. Since $|u - \lambda_P u| = \lambda_P u - u \leq \lambda_H u - u$, $u - \lambda_P u$ is a quasipotential. For general $u \in PX(W; \partial W)$, let $u = u_1 - u_2$ with $u_i \in PX(W; \partial W)^+$ $(i = 1, 2)$. Then $|u - \lambda_P u| \leq |u_1 - \lambda_P u_1| + |u_2 - \lambda_P u_2|$ shows that $u - \lambda_P u$ is a quasipotential, i.e. λ_P is the canonical extension.

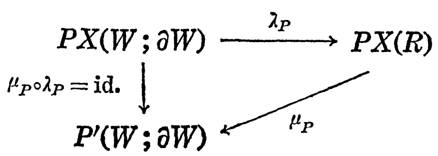
5. We denote by $P'(R)(P'(W; \partial W)$, resp.) the subspace of $P(R)$ $(P(W; \partial W)$, resp.) generated by $P(R)^+(P(W; \partial W)^+$, resp.). Let $\chi = \chi_W$ be the characteristic function of W, i.e. $\chi | W = 1$ and $\chi | R - W = 0$. We define an operator $\mu = \mu_P = \mu_P^W$, which will be referred to as the *canonical restriction*, by

$$\mu_P v = \lim_{\Omega \to R} P_{\chi v}^{W \cap \Omega} \tag{17}$$

on W and $\mu_P v = 0$ on $R - W$. We use μ_H for μ_P with $P \equiv 0$. This is a linear operator from $P'(R)$ into $P'(W; \partial W)$. To see that μ_P is well defined, let $v = v_1 - v_2$ with $v_i \in P(R)^+$ $(i = 1, 2)$. Since $0 \leq P_{\chi v_i}^{X \cap \Omega} \leq v_i$ and $P_{\chi v_i}^{W \cap \Omega} | \partial W = 0$, $\{P_{\chi v_i}^{W \cap \Omega}\}$ forms a decreasing net and thus $\lim_{\Omega \to R} P_{\chi v_i}^{W \cap \Omega}$ exists and belongs to $P(W; \partial W)^+$ if it is extended to R by setting zero on $R - W$. From $P_{\chi v}^{W \cap \Omega} = P_{\chi v_1}^{W \cap \Omega} - P_{\chi v_2}^{W \cap \Omega}$, the existence of (17) in $P'(W; \partial W)$ follows. Observe that $PX(R) \subset P'(R)$ and $PX(W; \partial W) \subset P'(W; \partial W)$. The merit of considering μ_P lies in the following relation:

$$(\mu_P \circ \lambda_P) u = u \tag{18}$$

for every $u \in PX(W; \partial W)$ $(X = B, D, E, BD, BE)$. To prove this, it is sufficient to assume that $u \in PX(W; \partial W)^+$. Since $P_u^{W \cap \Omega} = P_{\chi u}^{W \cap \Omega} = u$,



$P_{\chi \lambda_P u}^{W \cap \Omega} = u + P_{\chi(\lambda_P u - u)}^{W \cap \Omega}$. On the other hand, $0 \leq P_{\chi(\lambda_P u - u)}^{W \cap \Omega} \leq P_{\lambda_P u - u}^{\Omega} \to 0$ as $\Omega \to R$. Therefore $\mu_P(\lambda_P u) = u$. The relation (18) neither means that $\mu_P(PX(R)) \subset PX(W; \partial W)$ nor that μ_P is injective on $PX(R)$ unless λ_P is surjective. It only means that $\mu_P(\lambda_P(PX(W; \partial W)) = PX(W; \partial W)$ and μ_P

is injective on $\lambda_P(PX(W;\partial W))$. In this sense it is interesting to determine W such that $\lambda_P^W: PX(W;\partial W)\to PX(R)$ is surjective. Except for the case $X=B$ the problem seems to be very difficult (cf. [13]). Here we only give a sufficient condition.

6. In this no, we always assume that R is *connected.* To discuss the surjectiveness of λ_P, we introduce three kinds of negligible sets. A subset K of R is said to be *X-negligible* ($X=B$, D, and BD, resp.) if either R is parabolic or there exists a normal open subset W such that $K\subset R-W$ and $\lambda_H: HX(W;\partial W)\to HX(R)$ ($X=B$, D, and BD, resp.) is surjective. We shall try to restate the concept in a more intuitively understandable term. First

A subset K of hyperbolic R is B-negligible if and only if there exists a potential p on R such that $p\geq 1$ on K.

If $\lambda_H: HB(W;\partial W)\to HB(R)$ is surjective with $K\subset R-W$, then $h=\mu_H 1\in HB(W;\partial W)$ $0\leq h\leq 1$, and $\lambda_H h=1$. Since h is subharmonic, $p=1-h$ is superharmonic. On the other hand $p=1-h=|h-\lambda_H h|$ is a quasipotential and thus p is a potential. Clearly $p=1\geq 1$ on K. Conversely assume the existence of such a p. By multiplying p by a suitable constant and by choosing W suitably, we can assume that $p\geq 1$ on $R-W$. Let $h\in HB(R)^+$. Then clearly $\mu_H h\in HB(W;\partial W)$. Observe that $0\leq h-\chi_W h\leq \|h\|p$ implies $0\leq H_h^{W\cap\Omega}-H_{\chi_W h}^{W\cap\Omega}\leq\|h\|p$. On letting $\Omega\to R$, $0\leq h-\mu_H h\leq\|h\|p$ on W and trivially on $R-W$, and a fortiori $h-\mu_H h$ is a quasipotential on R. By $|h-\lambda_H\circ\mu_H h|\leq|\mu_H h-\lambda_H(\mu_H h)|+|h-\mu_H h|$ we see that the subharmonic function $|h-\lambda_H\circ\mu_H h|$ is a quasipotential and thus $\lambda_H\circ\mu_H h=h$. Since $HB(R)^+$ generates $HB(R)$, we conclude that $\lambda_H: HB(W;\partial W)\to HB(R)$ is surjective. If we use the term of the compactification theory (cf. e.g. [1], [15]) we can restate the above assertion as follows: K is B-negligible if and only if the closure of $R-K$ in the Wiener compactification of R is a neighborhood of the Wiener harmonic boundary. Next we prove

A subset K of hyperbolic R is BD-negligible if and only if there exists a Dirichlet finite potential p on R such that $p\geq 1$ on K.

If $\lambda_H: HBD(W;\partial W)\to HBD(R)$ is surjective, then $h=\mu_H 1$ belongs to $HBD(W;\partial W)$, $0\leq h\leq 1$, and $\lambda_H h=1$. As above, $p=1-h$ is a potential, $p=1\geq 1$ on K, and $D(p)=D(h)<\infty$. Conversely assume the existence of such a p on R. On replacing p by $p\cap 1=\min(p,1)$, we may assume that $0\leq p\leq 1$ on R and $p=1$ on K. Let $h\in HBD(R)^+$.

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By the above, $\mu_H h \in HB(W;\partial W)$ and $\lambda_H(\mu_H h) = h$. Since $HBD(R)^+$ generates $HBD(R)$, we only have to show that $D_W(\mu_H h) < \infty$. Since $D_R((1-p)h) < \infty$, by the harmonic decomposition of $(1-p)h$ (cf. e.g. [1], [15]) we see the existence of $k = \lim_{\Omega \to R} H^{W \cap \Omega}_{(1-p)h}$ in $HBD(W;\partial W)$. Observe that $0 \leq h-(1-p)h \leq \|h\|p$ implies $0 \leq H^{W \cap \Omega}_h - H^{W \cap \Omega}_{(1-p)h} \leq \|h\|p$ and a fortiori $0 \leq h - k \leq \|h\|p$. Thus $|k-\mu_H h| \leq |\mu_H h - \lambda_H(\mu_H h)| + \|h\|p$, i.e. $|k-\mu_H h|$ is a quasipotential. Since $|k-\mu_H h|$ is subharmonic, we conclude that $\mu_H h = k \in HBD(W;\partial W)$. Note that

$$D(\mu_H h) \leq D(ph) . \tag{19}$$

In terms of compactifications (cf. e.g. [1], [15]), we see that K is BD-negligible if and only if the closure of $R-K$ in the Royden compactification of R is a neighborhood of the Royden harmonic boundary. From these two characterizations and the first two trivial inclusions it follows that

$$\{\text{compact sets}\} \subset \{D\text{-negligible sets}\} \subset \{BD\text{-negligible sets}\} \subset \{B\text{-negligible sets}\} .$$

Although we do not give explicit examples here, it is not hard to see that the above inclusions are all strict. We are not successful in potential term characterization of D-negligible sets and only give the following sufficient condition:

A subset K of hyperbolic R is D-negligible if there exists a Dirichlet finite potential p on R harmonic outside a compact set such that $p \geq 1$ on K.

The set $K_{\varepsilon,\zeta} = \{z \in R; G_R(z,\zeta) \geq \varepsilon > 0\}$ for any ε is an example of a D-negligible set since $p = (G_R(\cdot,\zeta) \cap \varepsilon')/\varepsilon$ for large ε' is a potential as stated above. To prove the assertion suppose that $p \in H(R-\Omega_0)$. Since p is a potential $\sup_{R-\Omega_0} p = \sup_{\partial\Omega_0} p = c < \infty$. On replacing p by $p \cap c$, we may assume that p is bounded. On applying the regularization we can also assume that $p \in C^2(R) \cap H(R-\Omega_0)$. Let $h \in HD(R)^+$ and $h_n = h \wedge n$ = the greatest harmonic minorant of h and n. Then $D(h_n) \leq D(h)$, $h_n \uparrow h$, and $D(h_n - h) \to 0$ (cf. e.g. [15]). Let $p_\Omega = p - H^\Omega_p$ for $\Omega \supset \Omega_0$ and let $k \in HD(R)$. By the Green formula

$$\begin{aligned} D_\Omega(p_\Omega k) &= -\int_\Omega p_\Omega k d^* d(p_\Omega k) \\ &= -\int_{\Omega_0} p_\Omega k^2 d^* dp_\Omega - \frac{1}{2}\int_\Omega dp_\Omega^2 \wedge^* dk^2 \\ &= -\int_\Omega p_\Omega k^2 d^* dp + \int_\Omega p_\Omega^2 dk \wedge^* dk . \end{aligned}$$

By the Fatou lemma we conclude as $\Omega \to R$ that

(20) $$D(pk) \leq \|k\|^2_{\Omega_0} D(p) + \|p\|^2 D(k) .$$

By the second characterization above, $\mu_H h_n \in HD(W; \partial W)$. By (19) and (20) we have

$$D(\mu_H h_n - \mu_H h_{n+p}) \leq D(p) \|h_n - h_{n+p}\|^2_{\Omega_0} + \|p\|^2 D(h_n - h_{n+p}) .$$

Therefore $u = \lim_{n\to\infty} \mu_H h_n \in HD(W; \partial W)$ and $D(u - \mu_H h_n) \to 0$ as $n \to \infty$. On the other hand, $D(\lambda_H u - h_n) = D(\lambda_H u - \lambda_H \mu_H h_n) \leq D(u - \mu_H h_n)$ shows that $D(\lambda_H u - h) = 0$, i.e. $\lambda_H u + a = h$ with a constant a. Take $e \in HBD(W; \partial W)$ with $\lambda_H e = a$. Then $\lambda_H(u + e) = h$. Since $HD(R)^+$ generates $HD(R)$, we conclude that $\lambda_H : HD(W; \partial W) \to HD(R)$ is surjective.

7. In the definition of negligible sets we presupposed that R is connected. The connectedness is not an essential restriction because we only have to consider componentwise if R is not connected. If R is parabolic, then R itself is of degenerate character but does not quite match our definition in terms of the surjectiveness of λ_H. In this case $HX(R) =$ {constants} and $HX(W; \partial W) = \{0\}$ for $X = B, D, BD$ if $R - W \neq \phi$ (cf. e.g. [15]). From (8) in no. 2, it follows that $PX(R) = \{0\}$ and $PX(W; \partial W) = \{0\}$ for $X = B, D, E, BD, BE$ if R is parabolic and $P \not\equiv 0$. This unpleasant situation can be conventionally resolved if we includes nonnegative constants into the class of potentials when R is parabolic. However, instead of providing such an artificial convention, we would rather avoid parabolic surfaces. The role of negligible sets is clarified by the

PROPOSITION. *Let W be a normal open set in a hyperbolic connected Riemann surface R. If $R - W$ is B-negligible, then $\lambda_P : PB(W; \partial W) \to PB(R)$ is surjective for every P; if $R - W$ is D-negligible, then $\lambda_P : PX(W; \partial W) \to PX(R)$ $(X = BD, BE)$ is surjective for every P; if $R - W$ is D-negligible, then $\lambda_P : PX(W; \partial W) \to PX(R)$ $(X = D, E)$ is surjective for every P.*

Proof. Suppose that $R - W$ is B-negligible. Take a potential p on R such that $p \geq 1$ on $R - W$. Let $u \in PB(R)^+$. Clearly $\mu_P u \in PB(W; \partial W)$. Observe that $0 \leq u - \chi_W u \leq \|u\| p$ on R. Thus $0 \leq P_u^{W \cap \Omega} - P_{\chi_W u}^{W \cap \Omega} \leq \|u\| p$ and a fortiori $0 \leq u - \mu_P u \leq \|u\| p$, i.e. $u - \mu_P u$ is a quasipotential. By $|u - \lambda_P \circ \mu_P u| \leq |\mu_P u - \lambda_P(\mu_P u)| + |u - \mu_P u|$, we see that the subharmonic function $|u - \lambda_P \circ \mu_P u|$ is a quasipotential and therefore $\lambda_P \circ \mu_P u = u$. Since $PB(R)^+$ generates $PB(R)$, $\lambda_P : PB(W; \partial W) \to PB(R)$ is surjective.

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Next suppose $R-W$ is BD-negligible. As in no. 6, take a Dirichlet finite potential p such that $0 \leq p \leq 1$ and $p = 1$ on $R-W$. Let $u \in PBY(R)^+$ $(Y = D, E)$. Since $R-W$ is B-negligible, $\mu_P u \in PB(W; \partial W)$ and $\lambda_P \mu_P u = u$. Since $PBY(R)^+$ generates $PBY(R)$, we can conclude the surjectiveness of $\lambda_P : PBY(W; \partial W) \to PBY(R)$ if we show $Y(\mu_P u) < \infty$ $(Y = D, E)$. First let $u \in PBD(R)^+$, i.e. $D(u) < \infty$. Observe that

$$u = T_P u - \frac{1}{2\pi}\int_R G_R(\cdot, \zeta) u(\zeta) P(\zeta) d\xi d\eta ,$$

$P^{W \cap \Omega}_{\chi_W u} \downarrow \mu_P u$, $G_{W \cap \Omega}(\cdot, \zeta) P^{W \cap \Omega}_{\chi_W u} \leq G_R(\cdot, \zeta) u(\zeta)$, and $H^{W \cap \Omega}_{\chi_W u} \to \mu_H T_P u$ since $\chi_W u - \chi_W T_P u$ is a quasipotential. By the Lebesgue dominated convergence theorem,

$$P^{W \cap \Omega}_{\chi_W u} = H^{W \cap \Omega}_{\chi_W u} - \frac{1}{2\pi}\int_{W \cup \Omega} G_{W \cap \Omega}(\cdot, \zeta) P^{W \cap \Omega}_{\chi_W u}(\zeta) P(\zeta) d\xi d\eta$$

implies that

$$\mu_P u = \mu_H T_P u - \frac{1}{2\pi}\int_W G_W(\cdot, \zeta) \mu_P u(\zeta) P(\zeta) d\xi d\eta .$$

Since $D(u) = D(T_P u) + (1/2\pi)\langle u, u\rangle^P_R < \infty$, $T_P u \in HBD(R)$, and a fortiori the surjectiveness of λ_H implies that $D(\mu_H T_P u) < \infty$. Therefore, by $0 \leq \mu_P u \leq u$, we see that

$$\begin{aligned} D(\mu_P u) &= D(\mu_H T_P u) + \frac{1}{2\pi}\langle \mu_P u, \mu_P u\rangle^P_W \\ &\leq D(\mu_H T_P u) + \frac{1}{2\pi}\langle u, u\rangle^P_R < \infty . \end{aligned}$$

Next let $E^P(u) < \infty$. Since $D(u) < \infty$ by the above we have $D(\mu_P u) < \infty$. Therefore $0 \leq \mu_P u \leq u$ implies that

$$\begin{aligned} E^P(\mu_P u) &= D(\mu_P u) + \int_R (\mu_P u(\zeta))^2 P(\zeta) d\xi d\eta \\ &\leq D(\mu_P u) + \int_R u^2(\zeta) P(\zeta) d\xi d\eta \\ &\leq D(\mu_P u) + E^P(u) < \infty . \end{aligned}$$

Finally suppose that $R-W$ is D-negligible. Since $PX(R)^+$ generates $PX(R)$, we only have to show that $\mu_P u \in PX(W; \partial W)$ and $\lambda_P \mu_P u = u$ for every $u \in PX(R)^+$ $(X = D, E)$. By exactly the same proof as above we see that $\mu_P u \in PX(W; \partial W)$. In the above proof, $D(\mu_H T_P u) < \infty$ followed from the surjectiveness of $\lambda_H: HBD(W; \partial W) \to HBD(R)$. In the present case it follows from that of $\lambda_H: HD(W; \partial W) \to HD(R)$. Thus we only have to show that $\lambda_P \mu_P u = u$. Set $\mu_P u = \mu_H T_P u - p$, where $p = (1/2\pi) \cdot \int G_W(\cdot, \zeta)\mu_P u(\zeta)P(\zeta)d\xi d\eta$ is a quasipotential since it is dominated by $(1/2\pi)\int G_R(\cdot, \zeta)u(\zeta)P(\zeta)d\xi d\eta$, a potential. Observe that

$$P^{\Omega}_{\mu_P u} = H^{\Omega}_{\mu_P u} - \frac{1}{2\pi}\int_{\Omega} G_{\Omega}(\cdot, \zeta)P^{\Omega}_{\mu_P u}(\zeta)P(\zeta)d\xi d\eta$$
$$= H^{\Omega}_{\mu_H T_P u} - H^{\Omega}_p - \frac{1}{2\pi}\int_{\Omega} G_{\Omega}(\cdot, \zeta)P^{\Omega}_{\mu_P u}(\zeta)P(\zeta)d\xi d\eta \,.$$

Since $0 \le P^{\Omega}_{\mu_P u} \le u$ and $\lim_{\Omega \to R} H^{\Omega}_p = 0$, the Lebesgue dominated convergence theorem yields

$$\lambda_P \mu_P u = \lambda_H \mu_H T_P u - \frac{1}{2\pi}\int_R G_R(\cdot, \zeta)\lambda_P \mu_P u(\zeta)P(\zeta)d\xi d\eta \,.$$

i.e. $T_P \lambda_P \mu_P u = \lambda_H \mu_H T_P u$. Since $\lambda_H: HD(W; \partial W) \to HD(R)$ is surjective, $\lambda_H \mu_H T_P u = T_P u$ and thus $T_P \lambda_P \mu_P u = T_P u$. The injectiveness of T_P implies that $\lambda_P \mu_P u = u$. Q.E.D.

8. We now complete the proof of our order comparison theorem stated in the introduction. If R is parabolic, then $HX(R) = \{\text{constants}\}$ $(X = B, D, BD)$ (cf. e.g. [15]) and by (8) in no. 2 $PX(R) = \{0\}$ for $P \not\equiv 0$ and $X = B, D, E, BD$ and BE. Thus the comparison question is of interest only for the case R is *hyperbolic*. Suppose (1) is valid on R except for a B-negligible set K. Let W be a normal open subset of R such that $R - W \supset K$ and $R - W$ is B-negligible. Since (1) is valid on the whole W, a Riemann surface, Proposition 3 assures that there exists the canonical isomorphism $T^W_{Q,P}$ of $PB(W)$ onto $QB(W)$. By using (12), it is not hard to see that $T^W_{Q,P}$ may be considered as a linear isomorphism of $PB(W; \partial W)$ onto $QB(W; \partial W)$. By Proposition 7, $\lambda_Q \circ T^W_{Q,P} \circ \mu_P$ is a linear isomorphism of $PB(R)$ onto $QB(R)$. To see that $\lambda_Q \circ T^W_{Q,P} \circ \mu_P = T_{Q,P}$ is the canonical isomorphism of $PB(R)$ onto $QB(R)$, we have

$$\begin{array}{ccc} PB(W; \partial W) & \xrightarrow{\mu_P} & PB(R) \\ {\scriptstyle T^W_{Q,P}}\downarrow & & \downarrow{\scriptstyle T_{Q,P}} \\ QB(W; \partial W) & \xrightarrow{\lambda_Q} & QB(R) \end{array}$$

to show that $u - T^W_{Q,P}u$ is a quasipotential on R for every $u \in PB(W; \partial W)$. We may assume that $u \geq 0$. Since $u - T^W_{Q,P}u$ is a quasipotential on W, $T^W_P u = T^W_Q(T^W_{Q,P}u) = h \in HB(W; \partial W)$. Observe that

$$u = h - \frac{1}{2\pi}\int_W G_W(\cdot, \zeta)u(\zeta)P(\zeta)d\xi d\eta$$

and similarly

$$T^W_{Q,P}u = h - \frac{1}{2\pi}\int_W G_W(\cdot, \zeta)T^W_{Q,P}u(\zeta)Q(\zeta)d\xi d\eta .$$

Therefore by $u \leq \lambda_P u \equiv v$ and $T^W_{Q,P}u \leq \lambda_Q T^W_{Q,P}u \equiv w$,

$$\begin{aligned} |u - T^W_{Q,P}u| &\leq \frac{1}{2\pi}\int_W G_W(\cdot, \zeta)(v(\zeta)P(\zeta) + w(\zeta)Q(\zeta))d\xi d\eta \\ &\leq \frac{1}{2\pi}\int_R G(\cdot, \zeta)(v(\zeta)P(\zeta) + w(\zeta)Q(\zeta))d\xi d\eta \\ &= (T_P v - v) + (T_Q w - w) < \infty , \end{aligned}$$

i.e. $|u - T^W_{Q,P}u|$ is dominated by the potential $(T_P v - v) + (T_Q w - w)$. By the similar applications of Propositions 3 and 7 as above, the other part of our comparison theorem can be proven verbatimly.

Appendix: Integral Comparisons

9. The order comparison (1) is very handy in many practical applications (cf. e.g. [8], [12]). However it is very far from being necessary. In pursuing the complete condition for the existence of canonical isomorphisms it is indespensable to consider the so-called integral comparisons. We denote by $G^P_R(z, \zeta)$ the Green's function of the equation $\Delta u = Pu$ on R. Hence $G_R(z, \zeta) = G^P_R(z, \zeta)$ with $P \equiv 0$. Let W be a normal open subset. We say that (P, Q) satisfies the condition (B) on W if

$$\text{(B)} \qquad \begin{cases} \int_W G^P_W(\cdot, \zeta)|Q(\zeta) - P(\zeta)|d\xi d\eta < \infty , \\ \int_W G^Q_W(\cdot, \zeta)|P(\zeta) - Q(\zeta)|d\xi d\eta < \infty . \end{cases}$$

We say that (P, Q) satisfies the condition (D) on W if

$$\text{(D)} \qquad \begin{cases} \int_{W \times W} G^P_W(z, \zeta)|Q(z) - P(z)|\cdot|Q(\zeta) - P(\zeta)|dxdyd\xi d\eta < \infty , \\ \int_{W \times W} G^Q_W(z, \zeta)|P(z) - Q(z)|\cdot|P(\zeta) - Q(\zeta)|dxdyd\xi d\eta < \infty . \end{cases}$$

Finally we say that (P, Q) satisfies the condition (E) on W if

$$\text{(E)} \qquad \int_W |P(\zeta) - Q(\zeta)| d\xi d\eta < \infty .$$

In our former paper [8] we showed that if (P, Q) satisfies the condition (X) on R, then $PBX(R)$ and $QBX(R)$ are canonically isomorphic $(X = B, D, E, ; BB = B)$. If we use this in the proof in no. 8 instead of Proposition 3, then we obtain:

INTEGRAL COMPARISON THEOREM 1. *If $R - W$ is B-negligible and (P, Q) satisfies the condition (B) on W, then $PB(R)$ and $QB(R)$ are canonically isomorphic; If $R - W$ is BD-negligible and (P, Q) satisfies the condition* (D) ((E), *resp.*), *then $PBD(R)$ ($PBE(R)$, resp.) and $QBD(R)$ ($QBE(R)$, resp.) are canonically isomorphic.*

The prototypes of this theorem are found in [6], Maeda [5], Glasner-Katz [2], etc. The integral conditions for $PX(R)$ and $QX(R)$ $(X = D, E)$ to be canonically isomorphic are not known. To find them is one of very important open problem on canonical isomorphisms. It may be instructive to point out that even if $PBX(R)$ and $QBX(R)$ are canonically isomorphic, $PX(R)$ and $QX(R)$ need not be canonically isomorphic for $X = D, E$ (cf. Singer [17], [11]), although $PBX(R)$ $(X = D, E)$ are dense in $PX(R)$.

10. The above theorem applied to $(P, 0)$ takes the more precise form. First, since $\int_R G_R^P(\cdot, \zeta)P(\zeta)d\xi d\eta \leq 2\pi$ and $G_R^P(\cdot, \zeta) \leq G_R(\cdot, \zeta)$, the condition

$$(\text{B}_0) \qquad \int_W G_R(\cdot, \zeta)P(\zeta)d\xi d\eta < \infty$$

implies (B) for $(P, 0)$ on W, the condition

$$(\text{D}_0) \qquad \int_{W \times W} G_R(z, \zeta)P(z)P(\zeta)dxdyd\xi d\eta < \infty$$

implies (D) for $(P, 0)$ on W, and condition

$$(\text{E}_0) \qquad \int_W P(\zeta)d\xi d\eta < \infty$$

trivially implies (E) for $(P, 0)$ on W. Thus the condition (B_0) for B-negligible $R - W$ implies the surjectiveness of $T_P: PB(R) \to HB(R)$.

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The condition (X_0) $(X=D, E)$ for BD-negligible $R-W$ implies the surjectiveness of $T_P: PBX(R) \to HBX(R)$. Conversely assume $T_P: PX(R) \to HX(R)$ is surjective and let $e = T_P^{-1}1$ $(X = B, BD, BE)$. We can choose a normal open set W such that

$$\left\{z \in R\,;\, e(z) \geq \frac{1}{2}\right\} \subset W \subset \left\{z \in R\,;\, e(z) > \frac{1}{3}\right\}.$$

By using relations

$$1 = e + \frac{1}{2\pi}\int_R G_R(\cdot, \zeta)e(\zeta)P(\zeta)d\xi d\eta$$

and

$$D(e) = \frac{1}{2\pi}\int_{R\times R} G_R(z,\zeta)e(z)e(\zeta)P(z)dxdyP(\zeta)d\xi d\eta\,,$$

we derive (B_0) ((D_0), resp.) on W if $X = B$ (D, resp.). If $X = E$, then

$$E_R^P(e) = D(e) + \int_R e^2(\zeta)P(\zeta)d\xi d\eta < \infty$$

implies (E_0) on W. Since $p = 2(1-e) = \frac{1}{\pi}\int_R G_R(\cdot,\zeta)e(\zeta)P(\zeta)d\xi d\eta$ is a potential and $p \geq 1$ on $R-W$, $R-W$ is B-negligible. If $X = D$ or E, then $D(e) < \infty$ and a fortiori $D(p) < \infty$, i.e. $R-W$ is BD-negligible. Thus we have shown (cf. [6], Glasner-Nakai [3], Glasner-Katz [2])

INTEGRAL COMPARISON THEOREM 2. *The linear spaces $PB(R)$ and $HB(R)$ are canonically isomorphic if and only if there exists a normal open subset W on which* (B_0) *is valid such that $R-W$ is B-negligible; $PBD(R)$ ($PBE(R)$, resp.) and $HBD(R)$ ($HBD(R)$, resp.) are canonically isomorphic if and only if there exists a normal open subset W on which* (D_0) ((E_0), *resp.*) *is valid such that $R-W$ is BD-negligible.*

Again the condition when $T_P; PX(R) \to HX(R)$ $(X = D, E)$ is surjective has not yet been obtained. Examples are known that $T_P: PBX(R) \to HBX(R)$ is surjective but $T_P: PX(R) \to HX(R)$ is not for $X = D, E$ (Singer [17], [11]). To seek the (complete) condition for $T_P: PX(R) \to HX(R)$ $(X = D, E)$ to be surjective seems to be urgently important.

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