Seventh Meeting, *9th June* 1905.

Mr J. W. Butters in the Chair.

Note on the determination of the axes of a conic.

By R. F. Davis, M.A.

Let α, β, γ be the known trilinear coordinates (actual lengths of perpendiculars) of the centre of a conic inscribed in the triangle of reference ABC.

Since the product of the perpendiculars from the foci upon any tangent is equal to the square of the semi-minor axis ($=\rho^2$), it follows that if x, y, z be the coordinates of one focus then ρ^2/x, ρ^2/y, ρ^2/z are the coordinates of the other focus.

Also, since the centre bisects the join of the foci,

$$x + \rho^2/x = 2\alpha,$$

$$y + \rho^2/y = 2\beta,$$

$$z + \rho^2/z = 2\gamma.$$

Therefore $$x = \alpha + \sqrt{\alpha^2 - \rho^2},$$

$$y = \beta + \sqrt{\beta^2 - \rho^2},$$

$$z = \gamma + \sqrt{\gamma^2 - \rho^2},$$

But $$ax + by + cz = 2\Delta = a\alpha + b\beta + c\gamma\ ;$$

hence $$a\sqrt{\alpha^2 - \rho^2} + b\sqrt{\beta^2 - \rho^2} + c\sqrt{\gamma^2 - \rho^2} = 0.$$

Thus, given the centre of an inscribed conic the semi-axes are determined by the above equation.

It may be noted that if $\alpha = \beta = \gamma = r$ (radius of inscribed circle), the equation becomes $(a+b+c)\sqrt{r^2 - \rho^2} = 0$, and there is only one value of ρ^2.

The equation cleared of radicals is

$$L\rho^4 - 2M\rho^2 + N = 0,$$

where $L = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$

$$= (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

$$= 16\Delta^2,$$

$$M = a^2\alpha^2(b^2+c^2-a^2) + b^2\beta^2(c^2+a^2-b^2) + c^2\gamma^2(a^2+b^2-c^2)$$

$$= 2Rabc[\alpha^2\sin 2A + \beta^2\sin 2B + \gamma^2\sin 2C],$$

$$N = (a\alpha + b\beta + c\gamma)(b\beta + c\gamma - a\alpha)(c\gamma + a\alpha - b\beta)(a\alpha + b\beta - c\gamma).$$

It is to be noticed that

$$\rho_1^2 + \rho_2^2 = (\alpha^2\sin 2A + \beta^2\sin 2B + \gamma^2\sin 2C)/2\sin A\sin B\sin C$$

= the square of the tangent from α, β, γ to the polar circle of ABC.

Thus the director circle of any conic inscribed in the triangle ABC cuts orthogonally the polar circle of the same triangle. A particular case of this is that the centre of a rectangular hyperbola inscribed in the triangle ABC lies on the polar circle.

The director circle of a variable conic inscribed in a given quadrilateral cuts orthogonally the polar circles of the four triangles formed by three out of four sides (themselves coaxal), and therefore belongs to the conjugate coaxal system (Gaskin's Theorem).

It will be seen, for example, that the axes of Steiner's ellipse can be easily found from the above equation.

Putting $a\alpha = b\beta = c\gamma = 2\Delta/3$, we get

$$\rho_1^2 + \rho_2^2 = (a^2 + b^2 + c^2)/18 \quad \text{and} \quad \rho_1^2\rho_2^2 = \Delta^2/27.$$