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UNIQUENESS OF FREE ACTIONS ON S^3 RESPECTING A KNOT

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In this paper we consider free actions of finite cyclic groups on the pair (S^3, K), where K is a knot in S^3. That is, we look at periodic diffeomorphisms f of (S^3, K) such that f^n is fixed point free, for all n less than the order of f. Note that such actions are always orientation preserving. We will show that if K is a non-trivial prime knot then, up to conjugacy, (S^3, K) has at most one free finite cyclic group action of a given order. In addition, if all of the companions of K are prime, then all of the free periodic diffeomorphisms of (S^3, K) are conjugate to elements of one cyclic group which acts freely on (S^3, K). More specifically, we prove the following two theorems.

THEOREM 1. *Let K be a non-trivial prime knot. If f and g are free periodic diffeomorphisms of (S^3, K) of the same order, then f is conjugate to a power of g.*

THEOREM 2. *Let K be a non-trivial prime knot, other than a torus knot, all of whose companions are prime. Then there is a cyclic group G which acts freely on (S^3, K) such that any free periodic diffeomorphism of (S^3, K) is conjugate to an element of G.*

After completing this work we were told by M. Sakuma that he independently obtained the same results for free symmetries of a knot [**10**]. In addition, he can extend these results to the case of rotations of a knot around a fixed axis by using Thurston's recent result on the geometrization of irreducible 3-dimensional orbifolds with singular locus of dimension at least one (cf. [**10**]). In this paper, we use only Thurston's Hyperbolization Theorem for Haken manifolds [**13**], but not the geometrization of orbifolds. We would like to thank M. Sakuma for valuable comments while comparing our proofs.

Our basic tools will be the theory of companionship of knots developed by Schubert [**11**], and the work of Jaco-Shalen [**3**] and Johannson [**4**] on characteristic decompositions. In particular, let K be a non-trivial knot in S^3 with exterior $E(K)$. There is a family τ of characteristic tori in $E(K)$, which is unique up to isotopy, such that:

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1) The tori in τ are disjoint, non-parallel, and essential (i.e., incompressible and non-boundary-parallel).

2) Each closed component of $E(K) - \tau$ is either Seifert fibered or both atoroidal and anannular (i.e., contains no essential torus or annulus).

3) The number $c(K)$ of tori in τ is minimal with respect to the above.

Let T be some torus in the characteristic family τ, and let V and Y be the components of $S^3 - T$. Then V is a solid torus and Y is the exterior of a knot K', which is said to be a companion of K. We shall refer to the component X_0 of $E(K) - \tau$ which contains $\partial E(K)$ as the *first component*. The components of $E(K) - X_0$ are knot complements $\{Y_i\}$. By an innermost disk argument we can find a meridional disk D_1 for some solid torus say $S^3 - Y_1$, such that D_1 is disjoint from all the other Y_i. By gluing a neighborhood of D_1 onto Y_1 we can find a ball B_1 which contains Y_1 but is disjoint from all the other Y_i. Repeating this argument in $S^3 - B_1$, we find a ball B_2 containing Y_2 but disjoint from B_1 and all the other Y_i. Continuing this process we obtain disjoint balls $\{B_i\}$ such that each Y_i is contained in a B_i. Let M be the manifold obtained from X_0 by replacing each of the knot complements Y_i, by a solid torus W_i, in such a way that a meridian of W_i is now where a longitude of Y_i was and a longitude of W_i is now where a meridian of Y_i was. Then M is the exterior $E(K_0)$ of a new knot K_0 in S^3. Let L_1 be the link which is made up of the cores of all the solid tori $\{W_i\}$. The disks D_i show that each component of L_1 is unknotted and that L_1 is in fact a trivial link. Now let $L = K_0 \cup L_1$, and observe that X_0 is just the exterior $E(L)$ of the link L. Since K_0 is homotopically non-trivial in $S^3 - Y_i$ for all i, every 2-sphere in $E(L)$ bounds a ball. By construction X_0 is either atoroidal and anannular, or X_0 is a Seifert fibered space. In the former case, by Thurston's Hyperbolization Theorem [**13**], X_0 has a complete hyperbolic structure of finite volume. In this case, we shall refer to L as a *hyperbolic link*. We shall need the following definitions for the case when X_0 is Seifert fibered.

Definition. A *cable space* is a manifold obtained from a solid torus $S^1 \times D^2$, by removing an open tubular neighborhood of a simple closed curve K that lies on a torus $S^1 \times J$, where J is a simple closed curve in Int D^2 and K is non-contractible in $S^1 \times D^2$.

Definition. A *composing space* is a manifold homeomorphic to $S^1 \times W$, where W is a disk with $n \geqq 2$ holes.

Cable spaces, composing spaces and torus knot complements all are Seifert fibered manifolds with planar orbit surfaces. It was observed by Jaco-Shalen [**3**] that these are the only Seifert fibered manifolds with incompressible boundary which are contained in a knot complement. They also observed that the first component of a knot complement is a composing space if and only if the knot is composite.

We prove Theorems 1 and 2 by induction on the number of characteristic tori in the family. In Section 1 we begin the induction by proving Theorem 1 for knots with no companions. We complete the proof of Theorem 1 in Section 2. Then in Section 3, we prove Theorem 2 and conclude with an example illustrating why the theorems fail for composite knots, and another example showing how Theorem 2 can fail for a knot with a composite companion.

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1. First step of the induction for theorem 1. Let K be a knot in S^3, whose exterior $E(K)$ is atoroidal. By Thurston's Hyperbolization Theorem [**13**], either $E(K)$ has a complete hyperbolic structure of finite volume or K is a torus knot.

Let a "keyring link" be a link $L = K_0 \cup L_1$ where K_0 is unknotted and L_1 bounds a collection D of embedded disjoint disks in S^3, each of which meets K_0 in exactly one point. A keyring link is not hyperbolic since its complement is a solid torus, $S^1 \times S^1 \times I$, or a composing space. Such a link is illustrated in figure 1.

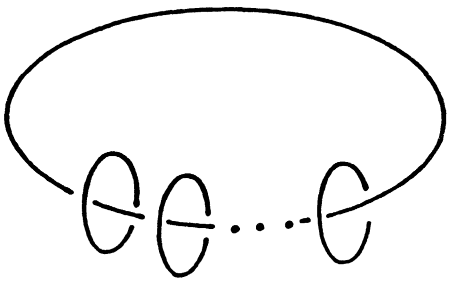

Figure 1 "Keyring link"

LEMMA 1.1. *Let $L = K_0 \cup L_1$ be a link such that L_1 is empty or is a trivial link and every 2-sphere in $E(L)$ bounds a ball. If there is a periodic diffeomorphism of (S^3, L_1) with fixed point set K_0 then L is a keyring link.*

Proof. Let h be a periodic diffeomorphism of (S^3, L_1) with fixed point set K_0. By the solution to the Smith Conjecture [**8**], K_0 cannot be knotted. Since L_1 is trivial, by the Equivariant Dehn's Lemma [**7**] there exists an invariant collection $\{D_i\}$ of embedded disjoint disks in S^3 where L_1 is the union of the boundaries of the D_i. Let D be some disk in $\{D_i\}$. If $D \cap K_0 = \emptyset$, then the boundary of a regular neighborhood of D is a 2-sphere which does not bound a ball in $E(L)$. Thus, K_0 intersects D in $n > 0$ points. Since K_0 is fixed pointwise by h we must have $h(D) = D$. Now $h|D$ is a periodic diffeomorphism of a disk with n fixed points. So $n = 1$. Thus L is a keyring link.

PROPOSITION 1.2. *Let $L = K_0 \cup L_1$ be a hyperbolic link such that L_1 is empty or a trivial link. Let f and g be free periodic diffeomorphisms of (S^3, K_0, L_1) of order p and q respectively. If $p = q$, then f is conjugate to a power of g by a diffeomorphism which is isotopic to the identity on (S^3, K_0, L_1). Otherwise, there is a free periodic diffeomorphism h of (S^3, K_0, L_1) of order $m = \operatorname{lcm}(p, q)$ such that $f^q = h^q$ and g^p is conjugate to h^p by a diffeomorphism which is isotopic to the identity on (S^3, K_0, L_1).*

Proof. We begin by showing that there is a hyperbolic structure for $E(L)$ such that $f|E(L)$ is an isometry and $g|E(L)$ is conjugate to an isometry $\tilde{g}$. Let $\langle f \rangle$ denote the action generated by f, and let $M = E(L)/\langle f \rangle$ be the orbit space of $E(L)$ under this free action. Then M has incompressible boundary, and is atoroidal and anannular. So, by Thurston's Hyperbolization Theorem [**13**], M has a complete hyperbolic structure of finite volume. We lift this structure to obtain a metric $\langle\ ,\ \rangle_f$ which gives $E(L)$ a complete hyperbolic structure of finite volume, and such that f is an isometry under this metric. Similarly we obtain a hyperbolic metric $\langle\ ,\ \rangle_g$ under which g is an isometry. By Mostow's Rigidity Theorem [**9**] together with Waldhausen [**15**], there is an isometry isotopic to the identity which takes $E(L)$ with $\langle\ ,\ \rangle_f$ to $E(L)$ with $\langle\ ,\ \rangle_g$. Thus g is conjugate to an isometry $\tilde{g}$ of $E(L)$ with $\langle\ ,\ \rangle_f$, by a diffeomorphism of $E(L)$ which is isotopic to the identity.

The group of orientation preserving isometries of $E(L)$ extends to a finite action of (S^3, L). By Lemma 1.1, the finite subgroup generated by f and $\tilde{g}$ is identified with its restriction to K_0. Hence together, f and $\tilde{g}$ generate a free cyclic action of (S^3, K_0, L_1) of order $m = \operatorname{lcm}(p, q)$. Also, if $p = q$ then f is just a power of $\tilde{g}$. Otherwise, pick $h = f\tilde{g}$.

Next we treat the case where K is a torus knot. Recall from the introduction that a cable space and a torus knot complement are Seifert fibered spaces

LEMMA 1.3. *Let M be either a cable space or the complement of a torus knot. Let f be a free orientation preserving periodic diffeomorphism of M which respects each boundary component. Then there is a Seifert fibration of M such that f leaves each fiber invariant.*

Proof. Let $N = M/\langle f \rangle$ be the orbit space of M under the action of f. Then N is orientable, irreducible, and has non-empty incompressible boundary. Hence by Theorem II.6.3 of [**3**], N is Seifert fibered. By lifting this Seifert fibration to M, we obtain an $\langle f \rangle$-invariant Seifert fibration of M.

The periodic diffeomorphism f induces a periodic diffeomorphism of the base space of the $\langle f \rangle$-invariant Seifert fibration of M. If M is a cable space then the Seifert fibration of M has one singular fiber and has base space an annulus. If M is the complement of a torus knot then there are two singular fibers with distinct multiplicities, and the base space is a disk. In either case, f must respect each singular fiber. Now, because f is free, it must preserve the orientation of each singular fiber. Therefore, again since f is orientation preserving, the induced periodic diffeomorphism on the base must be orientation preserving. But this induced map must fix each singular point and respect each boundary component, therefore it is the identity map. Thus f leaves each fiber of M invariant.

PROPOSITION 1.4. *Let L be a link whose exterior $E(L)$ is a torus knot complement or a cable space. Let f and g be free periodic diffeomorphisms of (S^3, L), of order p and q respectively, which respect each component of $\partial E(L)$. If $p = q$, then f is conjugate to a power of g by a diffeomorphism which is isotopic to the identity on (S^3, L). Otherwise, there is a free periodic diffeomorphism h of (S^3, L) of order $m = \text{lcm}(p, q)$, such that $f^q = h^q$ and g^p is conjugate to h^p by a diffeomorphism which is isotopic to the identity on (S^3, L).*

Proof. By Lemma 1.3, there is a Seifert fibration of $E(L)$ with respect to which f leaves each fiber invariant; and there is also a Seifert fibration with respect to which g leaves each fiber invariant. By [**14**] the Seifert fibration of $E(L)$ is unique up to isotopy. Hence we can conjugate g, by a diffeomorphism isotopic to the identity, to get $\tilde{g}$ which leaves each fiber of the first fibration invariant. We can extend the fibration of $E(L)$ to a fibration of S^3 where L consists of fibers. So we can extend $\tilde{g}$ to (S^3, L), still leaving each fiber invariant. Since f and $\tilde{g}$ are free, they preserve the orientation of each fiber and thus can be embedded in the S^1-action generating the Seifert fibration of (S^3, L). So after further conjugation we can arrange that f and $\tilde{g}$ commute on each fiber. Now by restriction to a generic fiber, we see that f and $\tilde{g}$ generate a free cyclic action of (S^3, L) of order $m = \text{lcm}(p, q)$. Also, if $p = q$ then f is just a power of $\tilde{g}$. Otherwise pick $h = f\tilde{g}$.

COROLLARY 1.5. *Let K be a knot whose exterior $E(K)$ is atoroidal. Let f and g be free periodic diffeomorphisms of (S^3, K) of the same order. Then $f|E(K)$ is conjugate to a power of $g|E(K)$ by a diffeomorphism of $E(K)$ which is isotopic to the identity.*

Proof. Since f and g are free actions of (S^3, K), they are orientation preserving by Smith Theory [**12**]. By Thurston [**13**], since $E(K)$ is atoroidal, K is either hyperbolic or is a torus knot. Thus we apply either Proposition 1.2 or Proposition 1.4.

2. Completion of the proof of theorem 1. The proof will be done by induction on the number $c(K)$ of tori in the characteristic family τ. The case of $c(K) = 0$ follows from Corollary 1.5. Now we work with knots with $c(K) > 0$. We begin by setting up the inductive step. Recall from the introduction that the component of $E(K) - \tau$ containing $\partial E(K)$ is called the "first component." Further, let $W = S^3 - E(K)$. Then we can attach solid tori $V_1, V_2, \ldots, V_n$ to the components of $\partial(X_0 \cup W)$ to obtain the complement of a link $L = K_0 \cup L_1$, where K_0 now denotes the core of W and L_1 is a trivial link.

LEMMA 2.1. *Let K be a non-trivial knot with exterior $E(K)$. Let f be a free periodic diffeomorphism of (S^3, K) which respects the characteristic family τ. Then*

1) *the first component X_0 is respected by f*

2) *$f|(X_0 \cup W)$ can be extended to a free periodic diffeomorphism $\tilde{f}$ of (S^3, K_0, L_1)*

3) *at most one component of $\partial X_0 - \partial E(K)$ is respected by f.*

Proof. 1) Since $f(\tau) = \tau$ and $f(\partial E(K)) = \partial E(K)$, we must have $f(X_0) = X_0$.

2) We define the map $\tilde{f}$ by extending $f|(X_0 \cup W)$ radially within the solid tori V_i. Then it is not hard to show that $\tilde{f}$ is a map of (S^3, K_0, L_1) which is orientation preserving and of finite order. Suppose that $\tilde{f}$ is not free. Since $f|(X_0 \cup W)$ is free, then $\tilde{f}$ must fix pointwise the core of some solid torus V_i. Now $\tilde{f}$ leaves a meridian m of V_i invariant. Let Y_i be the component of $E(K) - X_0$ corresponding to V_i. A meridian of V_i is a longitude of Y_i; so f leaves a longitude of Y_i invariant. Now by **[2]**, $f|Y_i$ cannot be free. Hence $\tilde{f}$ is a free periodic diffeomorphism of (S^3, K_0, L_1).

3) Suppose there are at least two components T_1 and T_2 of $\partial X_0 \cup \partial E(K)$ which are respected by f. These components are also respected by the map $\tilde{f}$, defined above. Since L_1 is a trivial link there is precisely one essential 2-sphere in $S^3 - (T_1 \cup T_2)$. Thus using the equivariant sphere Theorem (**[7]**, **[6]**), we obtain a 2-sphere S which is equivariant under $\tilde{f}$ and which separates T_1 and T_2. Now S bounds balls B_1 and B_2, in S^3, which contain T_1 and T_2 respectively. Since $\tilde{f}$ is of finite order and $\tilde{f}(T_i) = T_i$ we must have $\tilde{f}(B_i) = B_i$. So by the Brouwer fixed point theorem $\tilde{f}$ must fix a point of each B_i. But, as seen above, $\tilde{f}$ is fixed point free. Therefore f respects at most one component of $\partial X_0 - \partial E(K)$.

If f does respect some component T_1 of $\partial X_0 - \partial E(K)$, then f will respect the component Y of $E(K) - X_0$ which is bounded by T_1. It follows from **[11]** that Y is the exterior of a knot K_1 which is said to be a *companion* of K. Since $c(K_1) < c(K)$ we would like to apply the inductive hypothesis to K_1, but Theorem 1 only concerns prime knots and K_1 may be composite. So, first we must study free actions on composite knots. Recall from the introduction that the first component of a composite knot is a composing space $S^1 \times W$, where W is a disk with $r \geqq 2$ holes.

LEMMA 2.2. *Let K be a composite knot with exterior $E(K)$ and first component X_0. Suppose f is a free periodic diffeomorphism of (S^3, K) which respects X_0. Then any product structure for X_0 is isotopic to a product structure $S^1 \times W$ such that $f|X_0 = f_1 \times f_2$ where f_1 and f_2 are orientation preserving periodic diffeomorphisms of S^1 and W respectively, and both have the same order as f.*

Proof. We begin with some observations about any free periodic diffeomorphism f of (S^3, K). By Smith Theory [**12**], f is orientation preserving. Let p be the order of f. Now suppose there were some meridian m such that $f^n(m) = m$, for some $n < p$. Then m would bound some meridional disk D such that $f^n(D) = D$. But now, by the Brouwer Fixed Point Theorem, f^n would have to fix a point of D, and hence f would not have been free. Therefore, if m is any meridian for K and $n < p$ then $f^n(m) \neq m$. Also, it follows from [**2**] that if λ is any longitude for K and $n < p$ then $f^n(\lambda) \neq \lambda$.

Let $\langle f \rangle$ denote the action generated by f, and let $N = X_0/\langle f \rangle$; then as in the proof of Lemma 1.3, N is Seifert fibered. Lift the fibration of N to get an $\langle f \rangle$-invariant Seifert fibration of X_0. It follows from [**14**] that the Seifert fibration induced by this product structure is isotopic to the $\langle f \rangle$-invariant fibration. Embed X_0 in a solid torus V with $\partial E(K) = \partial V$, in such a way that the product structure and the $\langle f \rangle$-invariant fibration of X_0 are extended to V. Since $f(\partial E(K)) = \partial E(K)$, we can extend $f|X_0$ to an orientation preserving, fiber preserving, periodic diffeomorphism $\underline{f}$ of V. A meridian of V is a longitude of K. So, by our initial remarks, if m is a meridian for V and $n < p$ then $\underline{f}^n(m) \neq m$. Thus $\underline{f}$ is free.

Pick a meridional disk D such that

$$\underline{f}^n(D) \cap D = \emptyset \text{ for all } n < p.$$

Since the $\langle \underline{f} \rangle$-invariant fibration of V is isotopic to the Seifert fibration induced by the product structure, we can actually pick D so that each fiber of V will meet D in precisely one point. Let $W = D \cap X_0$. Then $f^n(W) \cap W = \emptyset$ for all $n < p$, and W meets each fiber of X_0 in precisely one point. Now X_0 has a product structure $S^1 \times W$ which is isotopic to the original product structure, and f is a product action with respect to this structure. Thus $f = f_1 \times f_2$ where f_1 and f_2 are periodic diffeomorphisms of S^1 and W respectively. Since f preserves the orientation of both S^3 and K, both f_1 and f_2 are orientation preserving. Since $f^n(W) \cap W = \emptyset$ for all $n < p$, the order of f_1 is p. Also, by our preliminary remarks, if m is a meridian for K then $f^n(m) \neq m$, for all $n < p$. So the order of f_2 is also p.

LEMMA 2.3. *Let K be a composite knot with exterior $E(K)$ and first component X_0. Suppose f and g are free periodic diffeomorphisms of (S^3, K) of the same order which respect X_0. Then there exists a diffeomorphism $\tilde{g}$ of*

X_0 which is conjugate to $g|X_0$ by a diffeomorphism which is isotopic to the identity on $\partial E(K)$ and such that f and $\tilde{g}$ induce the same permutation on the components of ∂X_0.

Proof. Let p be the order of f and g. Suppose there is some collection $\{T_1, T_2, \ldots, T_r\}$ of components of $\partial X_0 - \partial E(K)$ such that $f(T_i) = T_{i+1}$, for $i < r$, and $f(T_r) = T_1$. Then for all $i \leqq r$ we have $f^r(T_i) = T_i$. But by Lemma 2.1, at most one component of $\partial X_0 - \partial E(K)$ is respected by f^n, for any $n < p$. Therefore, either f rotates all the components of $\partial X_0 - \partial E(K)$ in cycles of order p, or f leaves one component invariant and rotates the others in cycles of order p. Similarly for g. Since the order of both f and g is p, it follows that f leaves one component invariant if and only if g leaves one component invariant. Both f and g leave $\partial E(K)$ invariant. So the permutations of the components of ∂X_0 induced by f and g are conjugate by a permutation which fixes $\partial E(K)$. Now it follows from Lemma 2.2 that we can conjugate g by a diffeomorphism isotopic to the identity on $\partial E(K)$, to get a diffeomorphism $\tilde{g}$ which will induce the same permutation as f on the components of ∂X_0.

LEMMA 2.4. *Let K be a composite knot with exterior $E(K)$ and first component X_0. Suppose f and g are free periodic diffeomorphisms of (S^3, K) which respect X_0, and which induce the same permutation on the components of ∂X_0. If $f|\partial E(K)$ is conjugate to $g|\partial E(K)$ by a diffeomorphism of $\partial E(K)$ which is isotopic to the identity, then $f|X_0$ is conjugate to $g|X_0$ by a diffeomorphism of X_0 which is isotopic to the identity on ∂X_0.*

Proof. By Lemma 2.2, X_0 has a product structure such that f is a product action, and another product structure such that g is a product action; and these structures are isotopic. So after conjugating g by a diffeomorphism isotopic to the identity, we can assume that both f and g are product actions under the same $S^1 \times W$ structure. So by Lemma 2.2, $f|X_0 = f_1 \times f_2$ and $g|X_0 = g_1 \times g_2$, where f_1, f_2, g_1, g_2, are orientation preserving periodic diffeomorphisms of the same order. So f_1 must be conjugate to g_1^t, for some $t < p$, by a diffeomorphism of S^1 which is isotopic to the identity. Also, since f and g induce the same permutation on the components of ∂X_0 and W is planar, f_2 must be conjugate to g_2^u, for some $u < p$, by a diffeomorphism which is isotopic to the identity on ∂W. So $f_1 \times f_2$ is conjugate to $g_1^t \times g_2^u$ by a diffeomorphism of X_0 which is isotopic to the identity on ∂X_0. However, by hypothesis, $f|\partial E(K)$ is conjugate to $g|\partial E(K)$ by a diffeomorphism of $\partial E(K)$ which is isotopic to the identity. Now

$$f|\partial E(K) = (f_1 \times f_2)\,|\partial E(K) \quad \text{and} \quad g|\partial E(K) = (g_1 \times g_2)\,|\partial E(K).$$

Thus $(f_1 \times f_2)\,|\partial E(K)$ is conjugate to $(g_1 \times g_2)\,|\partial E(K)$. So $(g_1^t \times g_2^u)\,|\,\partial E(K)$ is conjugate to $(g_1 \times g_2)\,|\partial E(K)$ by a diffeomorphism of $\partial E(K)$ which is isotopic to the identity. Thus $t = 1$ and $u = 1$, and $f|X_0$ is conjugate to $g|X_0$ by a diffeomorphism which is isotopic to the identity on ∂X_0.

Before we can prove Theorem 1 we need one final lemma to tell us how to glue together the conjugacy maps we obtain on different components of $E(K) - \tau$.

LEMMA 2.5. *Let $X \cup Y$ be a 3-manifold with $X \cap Y = T$, a torus. Let f and g be free periodic diffeomorphisms of $X \cup Y$ respecting X and Y. Let q and r be positive integers less than the order of g. Suppose $f|X$ is conjugate to $g^q|X$ by a diffeomorphism Φ of X which is isotopic to the identity on T; and $f|Y$ is conjugate to $g^r|Y$ by a diffeomorphism Ψ of Y which is isotopic to the identity on T. Then $q = r$ and f is conjugate to g^q by a diffeomorphism Γ such that $\Gamma|X = \Phi$.*

Proof. On T, we have $f = \Phi g^q \Phi^{-1}$ and $f = \Psi g^r \Psi^{-1}$, so

$$g^r = \Psi^{-1}\Phi g^q \Phi^{-1}\Psi.$$

Thus $q = r$.

Pick a collar neighborhood $T \times I$ of $T \times \{0\}$ in Y, such that $f|T \times I$ is a product action. On T, the map $\Phi\Psi^{-1}$ commutes with f, and $(\Phi\Psi^{-1})|T$ is isotopic to the identity. Therefore, there is an induced diffeomorphism on $T/\langle f \rangle$ which is isotopic to the identity. We can lift this isotopy of $T/\langle f \rangle$ to get an isotopy $H:T \times I \to T \times I$ such that H_1 is the identity,

$$H_0 = (\Phi\Psi^{-1})|T \times \{0\},$$

and H commutes with f on $T \times I$. We shall define the diffeomorphism Γ as follows. First define $\Theta: Y \to Y$ by

$$\Theta|T \times I = H \quad \text{and} \quad \Theta|(Y - (T \times I)) = \text{the identity}.$$

Then define $\Gamma|Y = \Theta\Psi$.

Now we finally prove Theorem 1′, which immediately implies Theorem 1.

THEOREM 1′. *Let K be a non-trivial prime knot with exterior $E(K)$. If f and g are free periodic diffeomorphisms of (S^3, K) respecting $E(K)$ which are of the same order, then f is conjugate to a power of g by a diffeomorphism which is isotopic to the identity on $\partial E(K)$.*

Proof. We argue by induction on the number $c(K)$ of tori in a characteristic family for $E(K)$. If $c(K) = 0$ then $E(K)$ is atoroidal. So the theorem follows from Corollary 1.5. Assume the theorem is true for any non-trivial prime knot K', with $c(K') < n$. Let K be a non-trivial prime knot with $c(K) = n > 0$. By **[5]**, there is an $\langle f \rangle$-invariant characteristic family τ, and a $\langle g \rangle$-invariant characteristic family. These characteristic families are isotopic by **[4]** **[3]**. So, after conjugating g by a diffeomorphism which is isotopic to the identity, we shall assume without loss of generality, that τ is invariant under both f and g. Thus

$$f(\tau) = \tau = g(\tau) \quad \text{and} \quad f(\partial E(K)) = \partial E(K) = g(\partial E(K)),$$

so $f(X_0) = X_0 = g(X_0)$.

Let $W = S^3 - E(K)$. By Lemma 2.1 part 2), we can extend $f|(X_0 \cup W)$ and $g|(X_0 \cup W)$ to free periodic diffeomorphisms $\tilde{f}$ and $\tilde{g}$ of (S^3, K_0, L_1). Since K is prime, X_0 is not a composing space, hence it is either a cable space, or it is atoroidal and anannular. In the latter case, by [**13**] $L = K_0 \cup L_1$ is a hyperbolic link. So by Proposition 1.2 or 1.4 applied to $\tilde{f}$ and $\tilde{g}$, there is an integer q such that $\tilde{f}$ is conjugate to $\tilde{g}^q$ by a diffeomorphism which is isotopic to the identity on (S^3, K_0, L_1). Hence $f|X_0$ is conjugate to $g^q|X_0$ by a diffeomorphism Φ of X_0 which is isotopic to the identity. In particular, f and g^q induce the same permutation of the components of ∂X_0.

Let $Y_1, \ldots, Y_m$ be the components of $E(K) - X_0$ which are moved by f and g^q. For all i less than the order p of f, by Lemma 2.1 part 3), f^i respects at most one component of $E(K) - X_0$. Therefore $Y_1, \ldots, Y_m$ are moved by f^i, for all $i < p$. Let

$$Z = X_0 \cup Y_1 \cup \ldots \cup Y_m.$$

Now the orbit space of Z under the action of f is homeomorphic to the orbit space of Z under the action of g^q. Thus the conjugacy map Φ of X_0 can be extended to a conjugacy map of Z, which we will still call Φ. If there was no component of $E(K) - X_0$ which f and g^q left invariant then $E(K) = Z$ and so we are done.

Suppose there is some component Y of $E(K) - X_0$ which f and g^q leave invariant. By [**11**], Y is the exterior $E(K')$ of a non-trivial knot K'. Since g^q leaves Y invariant, and p and q are relatively prime, g also must leave Y invariant. So $f|E(K')$ and $g|E(K')$ are free orientation preserving periodic diffeomorphisms of order p. Using radial extension, $f|E(K')$ and $g|E(K')$ can be extended to periodic diffeomorphisms f' and g' of (S^3, K'). Now since $f|\partial E(K')$ and $g|\partial E(K')$ are free, f' and g' preserve the orientation of K'. Thus if any power of f' and g' fixed any point of K' they would fix every point of K' and hence contradict the Smith Conjecture [**8**]. Thus, in fact, f' and g' are free.

We consider the situations when K' is prime and when K' is composite separately. Suppose K' is prime. Then since $c(K') < c(K)$, we can apply the inductive hypothesis to K'. Thus f' is conjugate to $(g')^r$ by a diffeomorphism Φ' which is isotopic to the identity on $\partial E(K')$. Now by Lemma 2.5, we can glue the conjugacy maps on $E(K')$ and X_0 together. Also be Lemma 2.5 we can conclude that $q = r$. Thus f is conjugate to g^q by a diffeomorphism which is isotopic to the identity on $\partial E(K)$, so we are done.

Now suppose K' was composite. Let X_1 be the first component of

$E(K') - \tau$. By Lemma 2.3, there is a diffeomorphism Ψ of $E(K')$ which is isotopic to the identity on $\partial E(K')$ and such that f' and $\Psi g' \Psi^{-1}$ induce the same permutation on the components of ∂X_1. Now start the proof from the beginning using $h = \Psi g' \Psi^{-1}$ instead of g. Thus $f|Z$ is conjugate to $h^q|Z$ by a diffeomorphism Φ of Z which is isotopic to the identity. Now f and h induce the same permutation of the components of ∂X_1. So by Lemma 2.4, $f|X_1$ is conjugate to $h^q|X_1$ by a diffeomorphism Φ' of X_1 which is isotopic to the identity on ∂X_1. Glue Φ and Φ' together using Lemma 2.5. Thus $f|(Z \cup X_1)$ is conjugate to $h^q|(Z \cup X_1)$ by a diffeomorphism which is isotopic to the identity on $\partial(Z \cup X_1)$. Now repeat the above arguments for the action of f and h on the components of $E(K_1) - X_1$. Eventually we conclude that f is conjugate to h^q by a diffeomorphism γ which is isotopic to the identity on $\partial E(K)$. Thus f is conjugate to g^q by $\Psi\gamma$ which is isotopic to the identity on $\partial E(K)$.

It is important to observe that the conjugacy map we constructed above is not necessarily isotopic to the identity on all of $E(K)$. This is because Dehn twists may occur when we glue the conjugacy maps together along the tori in τ.

THEOREM 2.6. *Let K be a composite knot with exterior $E(K)$. Suppose f and g are free periodic diffeomorphisms of (S^3, K) of the same order such that $f|\partial E(K)$ is conjugate to $g^q|\partial E(K)$ by a diffeomorphism of $\partial E(K)$ which is isotopic to the identity. Then f is conjugate to g^q by a diffeomorphism which is isotopic to the identity on $\partial E(K)$.*

Proof. This is just the second half of the proof of Theorem 1.

3. Theorem 2. As in Theorem 1, we will proceed by induction on the number of characteristic tori. We begin with a preliminary lemma.

LEMMA 3.1. *Let T^2 be a torus with free orientation preserving periodic diffeomorphisms h_1 and h_2, both of order m. Let p and q be relatively prime integers such that $m = pq$. Suppose h_1^q is conjugate to h_2^q, and h_1^p is conjugate to h_2^p, in each case by a diffeomorphism which is isotopic to the identity. Then h_1 is conjugate to h_2 by a diffeomorphism which is isotopic to the identity.*

Proof. For $i = 1$ and $i = 2$, the map h_i is conjugate by a diffeomorphism isotopic to the identity to a map of the form

$$g_i(\alpha, \beta) = (\alpha + 2\pi r_i/a_i, \beta + 2\pi s_i/b_i)$$

where $(r_i, a_i) = (s_i, b_i) = 1$ and $\operatorname{lcm}(a_i, b_i) = m$. By hypothesis, h_1^q is conjugate to h_2^q, and h_1^p is conjugate to h_2^p. Thus

$$qr_1/a_1 \equiv qr_2/a_2 \pmod 1, \quad qs_1/b_1 \equiv qs_2/b_2 \pmod 1,$$

$$pr_1/a_1 \equiv pr_2/a_2 \pmod 1, \quad \text{and} \quad ps_1/b_1 \equiv ps_2/b_2 \pmod 1.$$

Since p and q are relatively prime,

$$r_1/a_1 \equiv r_2/a_2 \pmod 1, \quad \text{and} \quad s_1/b_1 \equiv s_2/b_2 \pmod 1.$$

Hence h_1 is conjugate to h_2 by a diffeomorphism which is isotopic to the identity.

The proof of Proposition 3.2 is very similar to that of Theorem 1.

PROPOSITION 3.2. *Let K be a non-trivial prime knot all of whose companions are prime. Let f and g be free periodic diffeomorphisms of $E(K)$ of order p and q respectively, with $(p, q) = 1$. Then there is a free periodic diffeomorphism h of $E(K)$ of order $m = pq$, such that f^q and g^p are conjugate to h^q and h^p respectively, by diffeomorphisms which are isotopic to the identity on $\partial E(K)$.*

Proof. We proceed by induction on $c(K)$. If $c(K) = 0$, then K is a torus knot or a hyperbolic knot. So we apply Proposition 1.2 or 1.4. Suppose the proposition is true for any knot K' satisfying the hypotheses and such that $c(K) = n > 0$. As in the proof of Theorem 1, we can assume, using [**5**], that τ is a characteristic family for $E(K)$ which is invariant under both f and g. Let X_0 be the first component of $E(K) - \tau$. Also as in the proof of Theorem 1 we apply Proposition 1.2 or 1.4 to get a periodic diffeomorphism h_1 of X_0 such that the order of h_1 is $m = pq$, and $h_1^q = f^q|X_0$, and h_1^p is conjugate to $g^p|X_0$ by a diffeomorphism of X_0 which is isotopic to the identity.

By Lemma 2.1, h_1 respects at most one component of $E(K) - X_0$. Let $Y_1, \dots, Y_m$ be the components of $E(K) - X_0$ which are moved by h_1. Let

$$Z = X_0 \cup Y_1 \cup \dots \cup Y_m.$$

Again as in Theorem 1, we can extend h_1 so that $h_1^q = f^q|Z$, and h_1^p is conjugate to $g^p|Z$. If there was no component of $E(K) - X_0$ which h_1 left invariant, then we are done.

Suppose Y is a component of $E(K) - X_0$ which is respected by h_1. Again Y is the exterior $E(K')$ of a knot K', satisfying the hypotheses of the proposition. Since $c(K') < c(K)$ we can apply the inductive hypothesis to get a periodic diffeomorphism h_2 of $E(K')$ such that h_2^q and h_2^p are conjugate to $f^q|E(K')$ and $g^p|E(K')$, respectively, by diffeomorphisms which are isotopic to the identity on $\partial E(K')$. By Lemma 3.1, $h_1|\partial E(K')$ is conjugate to $h_2|\partial E(K')$ by a diffeomorphism of $\partial E(K')$ which is isotopic to the identity. So conjugate $h_2|E(K')$ to get a diffeomorphism $\widetilde{h}_2$ of $E(K')$ such that

$$\widetilde{h}_2|\partial E(K') = h_1|\partial E(K').$$

Now define h on $E(K)$ by $h|E(K') = \widetilde{h}_2$ and $h|Z = h_1|Z$. Thus h^q is conjugate to f^q and h^p is conjugate to g^p, both by diffeomorphisms isotopic to the identity on $\partial E(K)$.

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THEOREM 2. *Let K be a non-trivial prime knot, other than a torus knot, all of whose companions are prime. Then there is a cyclic group G which acts freely on* (S^3, K) *such that any free periodic diffeomorphism of* (S^3, K) *is conjugate to an element of G.*

Proof. Since K is not a torus knot, by [**1**], the orders of periodic diffeomorphisms of (S^3, K) are bounded. Let f be a free periodic diffeomorphism of (S^3, K) of order p, such that p is a maximum. Let g be any other free periodic diffeomorphism of (S^3, K). Raise g to an appropriate power to obtain a free periodic diffeomorphism g' which has order q^n, where q is a prime. Suppose $r = \gcd(p, q^n) < q^n$, and let $s = p/r$. Let $f' = f^r$. Then f' is a free periodic diffeomorphism of (S^3, K) with order s, and s is relatively prime to q^n. Now by Proposition 3.2, there is a free periodic diffeomorphism h of (S^3, K) of order $m = sq^n$. Since $r < q^n$, it follows that $m > p$. But p was chosen to be a maximum, hence $r = q^n$. Thus the order of g actually divides p. So by Theorem 1, g is conjugate to some power of f. Thus, in fact, $\langle f \rangle$ is the free action G.

If K is a torus knot then (S^3, K) has an S^1-action which induces the Seifert fibration of $E(K)$. By Lemma 1.3 and [**14**], any free periodic diffeomorphism of (S^3, K) is conjugate to one which embeds in this S^1-action by a diffeomorphism which is isotopic to the identity.

In figure 2 is an example to illustrate why our theorems fail for composite knots. This knot has three non-conjugate free $\mathbf{Z}_4$ actions determined by whether the knot is twisted meridianally by $\pi/2$, π, or $3\pi/2$ as it is rotated longitudinally by $\pi/2$. The induced action on the composing space leaves the outer boundary component invariant, and rotates the other four components.

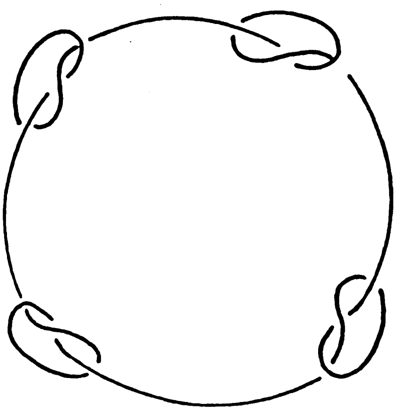

Figure 2

We conclude by describing an example of how Theorem 2 can fail for a prime knot with a composite companion. Let J be a knot with a free $\mathbf{Z}_2$-action. Let K_1 be the connected sum of three copies of J. Now, K_1 has a free $\mathbf{Z}_3$-action which rotates the copies of J, as well as a free $\mathbf{Z}_2$-action which leaves one copy of J invariant and switches the other two. Observe that these two free actions do not embed in a cyclic action. Let K be a (5, 7)-cable on K_1. Then K is a prime knot which has a free $\mathbf{Z}_2$-action and a free $\mathbf{Z}_3$-action, but these two actions do not embed in a cyclic action.

REFERENCES

1. E. Flapan, *Infinitely periodic knots*, Can. J. Math. *37* (1985), 17-28.
2. R. Hartley, *Knots with free period*, Can. J. Math. *37* (1981), 91-102.
3. W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Memoirs AMS (1979).
4. K. Johannson, *Homotopy equivalences of 3-manifolds with boundaries*, Lecture Notes in Mathematics *761* (1979).
5. W. Meeks and P. Scott, *Finite group actions on 3-manifolds*, preprint.
6. W. Meeks, L. Simon and S.-T. Yau, *Embedded minimal surfaces, exotic spheres and manifolds with positive ricci curvature*, Ann. of Math. *116* (1982), 621-659.
7. W. Meeks and S.-T. Yau, *Topology of three dimensional manifolds and the embedding problems in minimal surface theory*, Ann. of Math. *112* (1980), 441-485.
8. J. Morgan and H. Bass, *The Smith conjecture* (Academic Press, 1984).
9. G. Mostow, *Strong rigidity for locally symmetric spaces*, Ann. of Math. Study *78* (Princeton Univ. Press, 1973).
10. M. Sakuma, *Uniqueness of symmetries of knots*, preprint.
11. H. Schubert, *Knoten und Vollringe*, Acta Math *90* (1953), 131-286.
12. P. A. Smith, *Transformations of finite period II*, Ann. of Math. *40* (1939), 690-711.
13. W. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. *6* (1982), 357-381.
14. F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten*, Invent. Math. *3* (1967), 308-333, and *4* (1967), 87-117.
15. ——— *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. *87* (1968), 56-88.

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