On a function which is self-reciprocal in the Hankel transform

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It is well-known that if $\nu \geqq -\frac{1}{2}$ and

$$\phi(x) = \int_0^\infty (xt)^{\frac{1}{2}} J_\nu(xt)\, \phi(t) dt, \tag{1}$$

then $\phi(x)$ is said to be self-reciprocal in the Hankel transform and may be described as R_ν. If $\nu = \pm\frac{1}{2}$, (1) reduces to the Fourier sine or cosine transform. Functions of these two classes may be described as R_s and R_c.

In a list of pairs of reciprocal functions, G. A. Campbell[1] gives the example that

$$(a^2 + x^2)^{-\frac{1}{4}} K_{\frac{1}{4}}\{a\sqrt{(a^2 + x^2)}\} \tag{2}$$

is R_c, but there does not seem to be any explicit reference to this function in the literature. It suggests that there is a corresponding function which is R_ν. By considering the result[2] that, if $a > 0$, $b > 0$, $\mu > -1$,

$$\int_0^\infty J_\mu(bt) \frac{K_\nu\{a\sqrt{(t^2+z^2)}\}}{(t^2+z^2)^{\frac{1}{2}\nu}} t^{\mu+1} dt = \frac{b^\mu}{a^\nu}\left\{\frac{\sqrt{(a^2+b^2)}}{z}\right\}^{\nu-\mu-1} K_{\nu-\mu-1}\{z\sqrt{(a^2+b^2)}\},$$

on putting $z = a$, $b = x$, $\nu = \frac{1}{2}(\mu+1)$ it easily follows, since $K_\nu(t) = K_{-\nu}(t)$, that

$$x^{\frac{1}{2}+\mu}(x^2+a^2)^{\frac{1}{4}(-\mu-1)} K_{\frac{1}{2}(\mu+1)}\{a\sqrt{(x^2+a^2)}\} \tag{3}$$

is R_μ.

Since this appears to be a new example it is interesting to see where it fits into the general theory. Actually it is easy to show that the self-reciprocal character of the function (3) is a consequence of Hardy and Titchmarsh's[3] Theorem 7, with

$$\phi(s) = \int_0^\infty x^{s+\mu-\frac{1}{2}} (x^2+a^2)^{-\frac{1}{4}(\mu+1)} K_{\frac{1}{2}(\mu+1)}\{a\sqrt{(x^2+a^2)}\} dx.$$

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[1] *Bell System Technical Journal*, 7 (1928).

[2] Watson, *Bessel Functions*, § 13·47 (2), 416.

[3] *Quarterly Journal of Math.*, (Oxford Series) 1 (1930), 208.

The explicit formula

$$\phi(s) = 2^{\frac{1}{2}s+\frac{1}{2}\mu-\frac{3}{4}}\,\Gamma\left(\tfrac{1}{2}s+\tfrac{1}{2}\mu+\tfrac{1}{4}\right) K_{-\frac{1}{2}(s-\frac{1}{2})}(a^2)$$

is a consequence of the well-known integral due to Sonine.[1]

It is also of some interest that the only known parallel to the function (3) is the self-reciprocal function

$$\begin{aligned} F(x) &= x^{\frac{1}{2}-\mu}(x^2-b^2)^{\frac{1}{2}(\mu-1)} J_{\frac{1}{2}(\mu-1)}\{b\sqrt{(x^2-b^2)}\} \qquad (x > b > 0) \\ &= 0 \qquad (0 < x < b) \end{aligned}$$

considered by Hardy and Titchmarsh.[2]

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[1] See Watson, *Bessel Functions* §13·47 (6), 417.

[2] *Loc, cit.*, 211.

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