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EVERY COUNTABLE REGULAR SPACE WITHOUT ISOLATED POINTS IS CONNECTIFIABLE

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ABSTRACT. It is proved that every countable regular space without isolated points can be embedded densely in a connected Hausdorff space with a dispersion point.

In [5] S. Watson poses the question of whether there exists a countable regular space without isolated points which is not connectifiable. (A Hausdorff space X is called *connectifiable* if X can be embedded densely in a connected Hausdorff space). We answer this problem in the negative by proving that every countable regular space X without isolated points can be embedded densely in a connected Hausdorff space with a dispersion point. The embedding of the space X is obtained by considering a continuous one-to-one function of X onto a dense subspace of a specific Hausdorff connected space $Y^{\#}$ with a dispersion point.

A space X is called *Urysohn* if for every two distinct points x, y of X there exist open neighbourhoods V, U of the points x, y such that $\bar{V} \cap \bar{U} = \emptyset$. A point x of a connected space X is called a *dispersion point* if $X \setminus \{x\}$ is totally disconnected.

We will use the following construction due to P. Roy [3]: Let C_n, $n = 1, 2, \ldots$, be a countable collection of disjoint dense subsets of the rationals Q such that $\bigcup_{n=1}^{\infty} C_n = Q$. On the set $Y = \{(r, n) : r \in C_n, n = 1, 2, \ldots\} \cup \{\infty\}$ we define the following topology: Basic open neighbourhoods of every point of the form $(r, 2n)$ are the sets $V_\epsilon(r, 2n) = \{(t, 2n) : |t - r| < \epsilon\}$. Basic open neighbourhoods of every point of the form $(r, 2n-1)$, $n = 2, 3, \ldots$ are the sets $V_\epsilon(r, 2n-1) = \{(t, m) : |t - r| < \epsilon, m = 2n-2, 2n-1, 2n\}$. For $n = 1$, basic open neighbourhoods of the points $(r, 1)$, are the sets $V_\epsilon(r, 1) = \{(t, m) : |t - r| < \epsilon, m = 1, 2\}$. A basis of open neighbourhoods of the point ∞ is the collection of sets $V_n(\infty) = \{\infty\} \cup \{(r, k) \in Y : k \geq 2n\}$. The space Y is countable connected Urysohn with dispersion point ∞.

It is obvious that the subspace $D = \{(r, n) \in Y : n = 2, 4, \ldots\}$ is regular first countable without isolated points and a dense subspace of Y.

LEMMA. *The space $Y \setminus C_1$ can be embedded densely in an uncountable Hausdorff connected space $Y^{\#}$ having the point ∞ as a dispersion point.*

PROOF. We first observe that $Y \setminus C_1$ is a countable connected Urysohn space having the point ∞ as a dispersion point.

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Let I be the set of irrationals. On the set

$$Y^{\#} = \{(r,2n) : r \in I \cup C_{2n}, n = 1,2,\ldots\} \cup \{(r,2n-1) : r \in C_{2n-1}, n = 2,3,\ldots\} \cup \{\infty\}$$

we define a topology in a similar manner as for the set Y above, that is, basic open neighbourhoods of every point of the form $(r,2n)$ are the sets

$$V_\epsilon(r,2n) = \{(t,2n) : |t-r| < \epsilon\}.$$

Basic open neighbourhoods of every point of the form $(r,2n-1)$ are the sets

$$V_\epsilon(r,2n-1) = \{(t,m) : |t-r| < \epsilon, m = 2n-2, 2n-1, 2n\}.$$

A basis of open neighbourhoods of the point ∞ is the collection of sets

$$V_n(\infty) = \{\infty\} \cup \{(r,k) \in Y^{\#} : k \geq 2n\}.$$

It is obvious that $Y^{\#}$ is an uncountable connected Hausdorff space containing $Y \setminus C_1$ as a dense subspace. But $Y^{\#}$ is not a Urysohn space because for every pair of points of the form $(r,2n-2)$, $(r,2n)$, $r \in I$, $n = 2,3,\ldots$ it holds that

$$\overline{O_\epsilon(r,2n-2)} \cap \overline{O_\epsilon(r,2n)} \neq \emptyset, \quad \text{for every } \epsilon > 0.$$

We prove that the point ∞ is a dispersion point of $Y^{\#}$. Let M be a connected subset of $Y^{\#} \setminus \{\infty\}$, and let R be the space of real numbers. We consider the function f of $Y^{\#} \setminus \{\infty\}$ onto the totally disconnected space $R \setminus C_1$, defined by $f\big((r,n)\big) = r$. Since f is continuous, it follows that if $r \in Q \setminus C_1$, then M is the singleton $\{(r,n)\}$, and if $r \in I$, then $M = \bigcup_{n=1}^{\infty}\{(r,2n)\}$, which is not connected.

Observe that the countable subspace $D = \{(r,n) \in Y^{\#} : r \in C_n, n = 2,4,\ldots\}$ remains a regular first countable dense subspace of $Y^{\#}$ not having isolated points.

PROPOSITION. *Every countable regular space without isolated points can be embedded as an open dense subspace in a connected Hausdorff space with a dispersion point.*

PROOF. Let (X,τ) be a countable regular space without isolated points. By [1, Proposition 5.1] there exists a weaker regular first countable topology σ on X which has no isolated points. Hence by the Sierpinski's theorem [4], there exists a homeomorphism f of (X,σ) onto the dense subspace D of the space $Y^{\#}$ of the Lemma. Obviously the function f of (X,τ) onto D is continuous one-to-one.

We consider the set $Z = X \cup (Y^{\#} \setminus D)$ on which we define the following topology: Every open set in (X,τ) is open. For every point of the form $(r,2n)$, a basis of open neighbourhoods is the collection of sets

$$O_\epsilon(r,2n) = \{(t,2n) : |t-r| < \epsilon, t \in I\} \cup f^{-1}\big(\{(t,2n) : |t-r| < \epsilon, t \in C_{2n}\}\big).$$

For every point of the form $(r,2n-1)$ a basis of open neighbourhoods is the collection of sets

$$\begin{aligned} O_\epsilon(r,2n-1) = \{(t,2n-1) : |t-r| < \epsilon\} & \\ \cup \{(t,m) : |t-r| < \epsilon, t \in I, m = 2n-2, 2n\} & \\ \cup f^{-1}\big(\{(t,m) : |t-r| < \epsilon, t \in C_m, m = 2n-2, 2n\}\big). & \end{aligned}$$

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For the point ∞ a basis of open neighbourhoods is the collection of sets

$$O_n(\infty) = \{\infty\} \cup \{(r,k) \in Y^\# \setminus D : k \geq 2n\} \cup f^{-1}\Big(\{(r,k) \in D : k \geq 2n\}\Big).$$

We will prove that Z is the required space. That it is Hausdorff containing X as an open dense subspace is obvious by the definition of topology on Z.

We prove that Z is connected. Suppose not, and let A, B be disjoint open-and-closed subsets such that $A \cup B = Z$. Let $\infty \in A$. Then there exists an open neighbourhood $O_n(\infty)$ of the point ∞ such that

$$\overline{O_n(\infty)} = O_n(\infty) \cup \{(r, 2n-1) : r \in C_{2n-1}\} \subseteq A.$$

No point of the set $\{(r, 2n-2) : r \in I\}$ belongs to B. For if $(r, 2n-2)$ is such a point, then there exists an open neighbourhood $O_\epsilon(r, 2n-2)$ such that $\overline{O_\epsilon(r, 2n-2)} \subseteq B$. But this is impossible because

$$\overline{O_\epsilon(r, 2n-2)} \cap \{(r, 2n-1) : r \in C_{2n-1}\} \neq \emptyset.$$

Therefore $\{(r, 2n-2) : r \in I\} \subseteq A$, which implies that $f^{-1}\Big(\{(r, 2n-2) : r \in C_{2n-2}\}\Big) \subseteq A$, from which it follows that $\{(r, 2n-3) : r \in C_{2n-3}\} \subseteq A$. Continuing in this manner we conclude that $B \subseteq X$. But then $f(B)$ is a countable subset of $D \subseteq Y^\#$, not having isolated points, and hence it has a limit point $(t, 2k)$, $t \in I$, which is impossible because, by the definition of topology on Z, the point $(t, 2k)$ must then be a limit point of B.

It remains to prove that ∞ is a dispersion point of Z. Let M be a connected subset of $Z \setminus \{\infty\}$. We consider the function F of $Z \setminus \{\infty\}$ onto $Y^\# \setminus \{\infty\}$ defined by $F(x) = f(x)$, for every $x \in X$, and $F\big((r,n)\big) = (r,n)$, for every $(r,n) \in Y^\# \setminus D \cup \{\infty\}$. Since F is continuous and one-to-one, it follows that $F(M)$ is a connected subset of the totally disconnected space $Y^\# \setminus \{\infty\}$, and hence $F(M)$ is a singleton, which implies that M is a singleton.

REMARK. Since the space $Y^\#$ is not countable, the question which arises from the Watson's problem is the following: Can every countable regular space without isolated points be embedded densely in a countable connected Urysohn almost regular space with a dispersion point, or in a countable connected, locally connected Urysohn, almost regular space? (A space is called *almost regular* if it contains a dense subset at every point of which the space is regular, see [2]).

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