TENSOR PRODUCTS OF MAXIMAL ABELIAN SUBALGBERAS OF C*-ALGEBRAS

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Abstract. It is shown that if C_1 and C_2 are maximal abelian self-adjoint subalgebras (masas) of C*-algebras A_1 and A_2, respectively, then the completion $C_1 \otimes C_2$ of the algebraic tensor product $C_1 \odot C_2$ of C_1 and C_2 in any C*-tensor product $A_1 \otimes_\beta A_2$ is maximal abelian provided that C_1 has the extension property of Kadison and Singer and C_2 contains an approximate identity for A_2. Examples are given to show that this result can fail if the conditions on the two masas do not both hold. This gives an answer to a long-standing question, but leaves open some other interesting problems, one of which turns out to have a potentially intriguing implication for the Kadison-Singer extension problem.

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1. Introduction. If A_1 and A_2 are C*-algebras with centres Z_1 and Z_2, respectively, it was shown by Richard Haydon and the author [**6**] that the centre of the minimal C*-tensor product $A_1 \otimes_{min} A_2$ is just the closure of the algebraic tensor product $Z_1 \odot Z_2$ in $A_1 \otimes_{min} A_2$. This closure is naturally isomorphic to $Z_1 \otimes_{min} Z_2$. The result was also shown to follow from a more general slice map result [**12**, Theorem 4] and the analogous result for arbitrary C*-tensor products was subsequently established by Archbold using the Dixmier approximation property [**3**]. Batty later gave a neat alternative proof of Archbold's general result [**5**].

Analogous questions arise for maximal abelian self-adjoint C*-subalgebras (or masas) of C*-algebras. If C_1 and C_2 are masas of A_1 and A_2, respectively, then the closure of the algebraic tensor product $C_1 \odot C_2$ in any C*-completion $A_1 \otimes_\beta A_2$ of $A_1 \odot A_2$ is naturally isomorphic to $C_1 \otimes_{min} C_2$ since the algebras in the tensor product are abelian. The slice map result [**12**, Theorem 4] implies that $C_1 \otimes_{min} C_2$ is again a masa in $A_1 \otimes_{min} A_2$, and it is natural to ask whether $C_1 \otimes_{min} C_2$ is a masa in $A_1 \otimes_\beta A_2$ for any C*-norm β on $A_1 \odot A_2$ other than the minimal norm when $A_1 \odot A_2$ has more than one C*-norm.

Although this question was originally raised in [**12**], up to now little progress seems to have been made. In this paper we give a solution to the problem. There are two main results. The first, a positive one, states that if one of the masas C_1, C_2 possesses the Kadison-Singer extension property and the other contains an approximate identity for the ambient algebra, then the question has a positive answer. The second, a negative answer to the general question, is a pair of example of masas C_1 and C_2 of C*-algebras A_1 and A_2, respectively, such that $C_1 \otimes_{min} C_2$ is not maximal abelian in $A_1 \otimes_{max} A_2$ in each case. Connections with the Kadison-Singer extension problem [**9**] of whether

$\ell^\infty(\mathbb{N})$ has the extension property relative to $B(\ell^2(\mathbb{N}))$ will be discussed in the final section.

2. Masas with the extension property. A masa C of a C*-algebra A is said to have the *extension property* (see [**9**], [**2**], [**4**]) if

(i) any pure state (i.e. character) of C has a unique pure state extension to A and (when A is non-unital).

(ii) no pure state of A annihilates C.

Condition (ii) is well-known to be equivalent to the condition (see [**1**, Proof of Lemma 2.32])

(ii)′ C contains an increasing approximate identity for A.

It is a straightforward consequence of the Krein-Milman theorem that any pure state of a masa C with the extension property has a unique state extension to A. An alternative characterisation [**4**] of the extension property for unital A states that, if $\mathcal{U}(C)$ denotes the unitary group of C, then for any $x \in A$ the intersection $C \cap \overline{\text{co}}\{uxu^* : u \in \mathcal{U}(C)\}$ of C with the closed convex hull $\overline{\text{co}}\{uxu^* : u \in \mathcal{U}(C)\}$ contains exactly one point.

EXAMPLES 1. In the reduced C*-algebra $C_r^*(\mathbb{F}_2)$ of the free group on two generators, with u and v the canonical unitary generators, the abelian C*-subalgebras generated by u and v, respectively, are masas with the extension property [**4**, Example (i)]. Moreover $C_r^*(\mathbb{F}_2)$ is not nuclear [**11**]; (see also [**14**]).

2. No non-atomic masa of $B(\ell^2(\mathbb{N}))$ has the extension property [**9**].

In what follows the minimal C*-tensor product of two C*-algebras will be denoted by $A \otimes B$ when at least one of A and B is abelian. The following well-known factorisation result for states on a tensor product will be required in the proof of Theorem 2.

LEMMA 1 [**5**, Lemma 3]. *Let φ be a state on $A_1 \otimes_\beta A_2$ such that the restriction φ_1 of φ to A_1 is a pure state of A_1. Then $\varphi = \varphi_1 \otimes \varphi_2$ for some state φ_2 on A_2.*

In the proof of the following result, which is analogous to that of [**5**, Theorem 4], the unitization $\tilde{A}$ of A will be taken to be the subalgebra $A + \mathbb{C}.1$ of the multiplier algebra $M(A)$, so that $\tilde{A} = A$ if A is itself unital. If β is a C*-norm on $A_1 \odot A_2$, $\tilde{\beta}$ will denote the unique C*-norm on $\tilde{A}_1 \odot \tilde{A}_2$ extending β; (see [**8**]).

THEOREM 2. *Let A_1 and A_2 be C*-algebras with A_2 unital and let C be a masa of A_1 with the extension property. Then in any C*-completion $A_1 \otimes_\beta A_2$,*

$$(C \otimes 1)^c = C \otimes A_2,$$

where $(C \otimes 1)^c = \{x \in A_1 \otimes_\beta A_2 : x(c \otimes 1) = (c \otimes 1)x \text{ for all } c \in C\}$.

Proof. 1. Assume first that A_1 is unital. If $\Phi : A_1 \otimes_\beta A_2 \to A_1 \otimes_{min} A_2$ is the canonical homomorphism, its restriction to $C \otimes A_2$ is an isomorphism and $C \otimes A_2$ can be identified with its image in $A_1 \otimes_{min} A_2$ under Φ. With this identification $\Phi|_{C_1 \otimes A_2}$ is just the identity map. If $x \in (C \otimes 1)^c$, then $[x, C \otimes 1] = 0$, which implies that $[\Phi(x), C \otimes 1] = 0$. By [**12**, Theorem 4], $\Phi(x) \in C \otimes A_2$. Replacing x by $(x - \Phi(x))^*(x - \Phi(x))$, it is sufficient to show that if $x \geq 0$ and $\Phi(x) = 0$ then $x = 0$.

With these assumptions $\|x\| \in \mathrm{Sp}(x)$, $\|x\|1 - x$ is singular and the closed left ideal I of $D = (C \otimes 1)^c$ generated by $\|x\|1 - x$ is proper. Let

$$J = \{c \in C : c \otimes 1 \in I\}.$$

Since $I(C \otimes 1) \subseteq I$, J is a proper closed two-sided ideal of C and since $J \otimes 1 = (C \otimes 1) \cap I$, there are canonical isometric isomorphisms

$$C/J \cong (C \otimes 1)/(J \otimes 1) \cong (C \otimes 1 + I)/I,$$

by [**10**, 1.17.6]. Let χ be a character of C that annihilates J. Via these isomorphisms, χ corresponds to a linear functional φ on $(C \otimes 1 + I)/I$ such that $\|\varphi\| = 1$ and $\varphi(1 + I) = 1$. By the Hahn-Banach theorem φ extends to a linear functional of norm 1 on D/I which, when composed with the quotient map, gives a state $\bar{\varphi}$ on D such that $\bar{\varphi}(c \otimes 1) = \chi(c)$ for $c \in C$. Let ψ be an extension of $\bar{\varphi}$ to a state on $A_1 \otimes_\beta A_2$. Letting ψ_1 be the restriction of ψ to A_1, so that $\psi_1(a) = \psi(a \otimes 1)$, ψ_1 is a pure state since C has the extension property. By Lemma 1, $\psi = \psi_1 \otimes \psi_2$ for some state ψ_2 on A_2. Now $\psi(\|x\|1 - x) = 0$, since $\psi|_I = 0$, and $\psi(x) = (\psi_1 \otimes \psi_2)(\Phi(x)) = 0$. Thus $\|x\| = \psi(\|x\|1 - x) = 0$, which implies that $x = 0$, as required.

2. If A_1 is not unital, let $\tilde{A}_1$ be the unitisation of A_1 and let $\tilde{C} = C + \mathbb{C}1$. Then $\tilde{C}$ is a masa in $\tilde{A}_1$. To see that $\tilde{C}$ has the extension property in $\tilde{A}_1$, let f be a pure state of $\tilde{C}$ and let $\bar{f}$ be a pure state extension of f to $\tilde{A}_1$. If f is the unique pure state annihilating C, then $g = \bar{f}|_{A_1} = 0$, since otherwise g would be a pure state of A_1 that annihilated C. In this case $\bar{f}$ is the unique pure state of $\tilde{A}_1$ that annihilates A_1. If $f|_C \neq 0$, then $f|_C$ is a pure state of C and $\bar{f}|_{A_1}$ is a pure state extension of $f|_C$ and so uniquely determined by f. Since $\bar{f}$ is uniquely determined by its restriction to A_1, it follows that $\bar{f}$ is uniquely determined by f.

For $x \in A_1 \otimes_\beta A_2$, if $x \in (C \otimes 1)^c$ then $x \in (\tilde{C} \otimes 1)^c \subseteq \tilde{A}_1 \otimes_{\tilde{\beta}} A_2$, which implies by part 1 that $x \in \tilde{C} \otimes A_2$. Let φ be the state on $\tilde{A}_1$ that annihilates A_1. Then $\chi = \varphi|_{\tilde{C}}$ is the character of $\tilde{C}$ that annihilates C and the kernel of the map $\chi \otimes id_{A_2}$ is just $C \otimes A_2$. Since $0 = (\varphi \otimes id_{A_2})(x) = (\chi \otimes id_{A_2})(x)$, it follows that $x \in C \otimes A_2$, as required. □

COROLLARY 3. *Let A_1 and A_2 be C*-algebras and let C_1 and C_2 be masas of A_1 and A_2, respectively, such that C_1 has the extension property and C_2 contains an approximate identity for A_2 if A_2 is not unital. Then $C_1 \otimes C_2$ is a masa of $A_1 \otimes_\beta A_2$ for any C*-norm β on $A_1 \odot A_2$. Moreover $C_1 \otimes C_2$ has the extension property if and only if C_2 does.*

Proof. Assume that C_1 has the extension property. If A_2 is unital, it is immediate from Theorem 2 that $(C_1 \otimes C_2)^c \subseteq (C_1 \otimes 1)^c = C_1 \otimes A_2$. If A_2 is non-unital and $\{e_\lambda\}$ is an approximate identity of A_2 in C_2, for $x \in (C_1 \otimes C_2)^c$, $[x, c \otimes e_\lambda] = 0$ for any $c \in C_1$ and any λ. Since $x(c \otimes 1) = \lim_\lambda x(c \otimes e_\lambda)$ and $(c \otimes 1)x = \lim_\lambda (c \otimes e_\lambda)x$, it follows that, as an element of $A_1 \otimes_{\tilde{\beta}} \tilde{A}_2$, x lies in $(C_1 \otimes 1)^c$, which equals $C_1 \otimes_{\tilde{\beta}} \tilde{A}_2$, the closure of $C_1 \odot \tilde{A}_2$ in $A_1 \otimes_{\tilde{\beta}} \tilde{A}_2$, by Theorem 2. Since C_1 is abelian, $C_1 \otimes_{\tilde{\beta}} \tilde{A}_2$ identifies naturally with $C_1 \otimes \tilde{A}_2$ and so $x \in (C_1 \otimes \tilde{A}_2) \cap (A_1 \otimes_\beta A_2)$. Let Ψ denote the canonical homomorphism from $A_1 \otimes_{\tilde{\beta}} \tilde{A}_2$ to $A_1 \otimes_{\min} \tilde{A}_2$. Then $\Psi|_{C_1 \otimes \tilde{A}_2}$ is the identity map on $C_1 \otimes \tilde{A}_2$, since $C_1 \otimes_{\tilde{\beta}} \tilde{A}_2$ and $C_1 \otimes \tilde{A}_2$ are naturally identified, and $\Psi(x)$ is in $(C_1 \otimes \tilde{A}_2) \cap (A_1 \otimes_{min} A_2)$, which equals $C_1 \otimes A_2$ by [**12**, Corollary 5] (or a simple slice map argument), since C_1 is abelian. Thus $x \in C_1 \otimes_\beta A_2$. If X is the spectrum of C_1, there is a natural isomorphism $C_1 \otimes A_2 \cong C_0(X, A_2)$. Via this isomorphism $C_1 \otimes C_2$ identifies with $C_0(X, C_2)$ and x identifies with a function $\bar{x}$ in $C_0(X, A_2)$. For $\chi \in X$

and $c \in C_2$, there is a function $f \in C_0(X, C_2)$ such that $f(\chi) = c$. Since $x \in (C_1 \otimes C_2)^c$, $[\overline{x}(\chi), c] = [\overline{x}, f](\chi) = 0$, which implies that $\overline{x}(\chi) \in C_2$ for $\chi \in X$ and so $\overline{x} \in C_0(X, C_2)$. Thus $x \in C_1 \otimes C_2$, as required.

Now assume that C_1 and C_2 have the extension property and let φ be a character of $C_1 \otimes C_2$. Then $\varphi = \varphi_1 \otimes \varphi_2$ for suitable characters φ_1 and φ_2 of C_1 and C_2, respectively. Let ψ_i be the unique pure state extension of φ_i to A_i for $i = 1, 2$ and let ψ be the pure state $\psi_1 \otimes \psi_2$ of $A_1 \otimes_\beta A_2$. If $\overline{\varphi}$ is any pure state extension of φ to $A \otimes_\beta A_2$, and $\overline{\varphi}_1$ and $\overline{\varphi}_2$ are the restrictions of $\overline{\varphi}$ to A_1 and A_2, respectively, then $\overline{\varphi}_i|_{C_i} = \varphi_i$, which implies that $\overline{\varphi}_i = \psi_i$ for $i = 1, 2$. By Lemma 1, $\overline{\varphi} = \psi_1 \otimes \psi_2 = \psi$ and condition (i) in the definition of the extension property holds. If A_1 or A_2 is non-unital, it is an immediate consequence of condition (ii)$'$ in the definition of the extension property that condition (ii)$'$ holds for $C_1 \otimes C_2$ relative to $A_1 \otimes_\beta A_2$. Thus $C_1 \otimes C_2$ has the extension property. Conversely, if $C_1 \otimes C_2$ has the extension property, it is a simple exercise using similar methods to show that C_2 has the extension property. $\square$

Note. 1. An alternative proof of Theorem 2 when A_1 is unital can be given using the characterisation of the extension property in terms of unitary conjugates given in the paragraph following condition (ii)$'$ at the beginning of this section.

2. If both C_1 and C_2 have the extension property a more direct proof of the first part of Corollary 3 can be given as follows. If C is an abelian C*-subalgebra of $A_1 \otimes_\beta A_2$ containing $C_1 \otimes C_2$, let $\varphi = \varphi_1 \otimes \varphi_2$ be a character of $C_1 \otimes C_2$. If $\overline{\varphi}$ is a character of C extending φ, then $\overline{\varphi}$ extends to a pure state ψ of $A_1 \otimes_\beta A_2$ which equals $\psi_1 \otimes \psi_2$, where ψ_i is the unique state extension of φ_i to A_i for $i = 1, 2$, by the argument of the second paragraph of the above proof. Thus $\overline{\varphi} = (\psi_1 \otimes \psi_2)|_C$, which means that φ has a unique character extension to C. Moreover no pure state of $A_1 \otimes_\beta A_2$ and hence no character of C has a restriction to $C_1 \otimes C_2$ equal to 0, since $C_1 \otimes C_2$ contains an approximate identity for $A_1 \otimes_\beta A_2$. It follows by the Stone-Weierstrass theorem that $C = C_1 \otimes C_2$.

3. Masas with tensor products which are not maximal abelian.

3.1. Let $A = C_r^*(\mathbb{F}_2) + K(\ell^2(\mathbb{F}_2))$ in $B(\ell^2(\mathbb{F}_2))$, where $K(\ell^2(\mathbb{F}_2))$ denotes the compact linear operators on $\ell^2(\mathbb{F}_2)$. Then A is a C*-algebra, $K(\ell^2(\mathbb{F}_2))$ is an ideal of A and $A/K(\ell^2(\mathbb{F}_2)) \cong C_r^*(\mathbb{F}_2)$. Let q be the canonical quotient map from A onto $C_r^*(\mathbb{F}_2)$, and let λ and ρ denote the representations of $C_r^*(\mathbb{F}_2)$ corresponding to the left- and right-regular representations of $\mathbb{F}_2$ on $\ell^2(\mathbb{F}_2)$, respectively. Then $\{\lambda \circ q, \rho \circ q\}$ is a commuting pair of representations of the pair $\{A, A\}$ on $\ell^2(\mathbb{F}_2)$ with corresponding representation π_r of $A \odot A$ given by

$$\pi_r\left(\sum a_i \otimes b_i\right) = \sum \lambda(q(a_i))\rho(q(b_i)).$$

A C*-norm $\| \ \|_\alpha$ on $A \odot A$ is defined by

$$\|x\|_\alpha = \max\{\|x\|_{min}, \|\pi_r(x)\|\} \quad (x \in A \odot A).$$

Let $\{\xi_g : g \in \mathbb{F}_2\}$ be the canonical orthonormal basis of $\ell^2(\mathbb{F}_2)$, for each $g \in \mathbb{F}_2$ let e_g be the projection onto the one dimensional subspace $\mathbb{C}\xi_g$ and let C be the abelian C*-algebra generated by $\{e_g : g \in \mathbb{F}_2\} \cup \{1\}$. Then $C \subset K(\ell^2(\mathbb{F}_2)) + \mathbb{C}.1 \subset A$.

PROPOSITION 4. *The algebra C is maximal abelian in A, but $C \otimes C$ is not maximal abelian in $A \otimes_\beta A$ for any C*-norm $\| \ \|_\beta$ satisfying $\|x\|_\beta \geq \|x\|_\alpha$ on $x \in A \odot A$, and in particular if $\| \ \|_\beta = \| \ \|_{max}$.*

Proof. 1. To see that C is maximal abelian in A, let L be the closure of C in the weak operator topology. Then L is a maximal abelian $*$-subalgebra of $B(\ell^2(\mathbb{F}_2))$ isomorphic to $\ell^\infty(\mathbb{N})$ and the canonical projection σ from $B(\ell^2(\mathbb{F}_2))$ onto L is given by

$$\sigma(x) = \sum_{g \in \mathbb{F}_2} e_g x e_g$$

for $x \in B(\ell^2(\mathbb{F}_2))$, the convergence of the sum on the right being in the strong operator topology. If $x \in C^c = C' \cap A$, then $\sigma(x) = \sum_{g \in \mathbb{F}_2} e_g x e_g = \sum_{g \in \mathbb{F}_2} x e_g = x$. Now $x = k + a$, where $k \in K(\ell^2(\mathbb{F}_2))$ and $a \in C_r^*(\mathbb{F}_2)$. Since $e_g \lambda_h e_g = 0$ for $g, h \in \mathbb{F}_2$ when h is not the identity of $\mathbb{F}_2$, $\sigma(a) \in \mathbb{C}.1$ and since $\sigma(k) \in C$, it follows that $\sigma(x) \in C$. Thus $C^c = C$.

2. By [11] (see also [13]) the representation of $C_r^*(\mathbb{F}_2) \odot C_r^*(\mathbb{F}_2)$ on $\ell^2(\mathbb{F}_2)$ given by

$$\sum a_i \otimes b_i \to \sum \lambda(a_i) \rho(b_i)$$

is not continuous relative to $\| \ \|_{min}$, which implies that there is a non-zero element in the kernel of the canonical homomorphism from $A \otimes_\alpha A$ to $A \otimes_{min} A$ and hence a non-zero element x in the kernel of the canonical homomorphism Ψ from $A \otimes_\beta A$ to $A \otimes_{min} A$. Since $K(\ell^2(\mathbb{F}_2))$ is nuclear, the restriction of $\| \ \|_\beta$ to $K(\ell^2(\mathbb{F}_2)) \odot A$ coincides with $\| \ \|_{min}$ and $\Psi|_{K(\ell^2(\mathbb{F}_2)) \otimes_\beta A}$ is isometric. Thus for $k \in K(\ell^2(\mathbb{F}_2))$,

$$\|x(k \otimes 1)\|_\beta = \|\Psi(x(k \otimes 1))\|_{min} = \|\Psi(x)\Psi(k \otimes 1)\|_{min} = 0,$$

which implies that $x(k \otimes 1) = 0$. Similarly $(k \otimes 1)x = 0$. For $c \in C$ with $c = k + \lambda 1$, where $k \in K(\ell^2(\mathbb{F}_2))$ and $\lambda \in \mathbb{C}$,

$$[x, c \otimes 1] = [x, k \otimes 1] = 0,$$

so that $x \in (C \otimes 1)^c$. Similarly $x \in (1 \otimes C)^c$, so that $x \in (C \otimes C)^c$. Since $\Psi(x) = 0$ and $\Psi|_{C \otimes C}$ is isometric, $x \notin C \otimes C$, which means that $C \otimes C$ is not maximal abelian in $A \otimes_\beta A$. $\square$

3.2. This result shows that without the requirement in the statement of Corollary 3 that C_1 have the extension property, $C_1 \otimes C_2$ may not be maximal abelian in $A_1 \otimes_\beta A_2$. By modifying the construction of the C*-algebra A above, it also follows that the conclusion of the Corollary can fail if C_2 does not contain an approximate identity for A_2.

To see this, let H be a separable infinite dimensional Hilbert space with orthonormal basis $\{\xi_i : i \in \mathbb{N}\}$, and let H_1 and H_2 be the closures of the linear subspaces of H spanned by $\{\xi_{2i} : i \in \mathbb{N}\}$ and $\{\xi_{2i-1} : i \in \mathbb{N}\}$, respectively. Then $H = H_1 \oplus H_2$. A self-adjoint unitary operator u on H is defined by

$$u\xi_{2i} = \frac{1}{\sqrt{2}}(\xi_{2i} + \xi_{2i-1}), \quad u\xi_{2i-1} = \frac{1}{\sqrt{2}}(\xi_{2i} - \xi_{2i-1}).$$

Let $\{e_{ij}\}$ be the set of rank one matrix units associated with the basis $\{\xi_i\}$. Then

$$u = \sum_i \frac{1}{\sqrt{2}}(e_{2i,2i} + e_{2i,2i-1} + e_{2i-1,2i} - e_{2i-1,2i-1}),$$

the sum on the right hand side converging in the strong operator topology. If $t \in B(H)$ is such that $tH_1 \subset H_1$ and $t|_{H_2} = 0$, then

$$t = \sum_{i,j} t_{ij} e_{2i,2j},$$

where $t_{ij} = (t\xi_{2j}|\xi_{2i})$, the convergence again being in the strong operator topology. Then

$$utu = \frac{1}{2} \sum_{i,j} t_{ij}(e_{2i,2j} + e_{2i,2j-1} + e_{2i-1,2j} + e_{2i-1,2j-1}).$$

Let $B = C_r^*(\mathbb{F}_2) \oplus \{0\}$ on H, where $C_r^*(\mathbb{F}_2)$ is acting in its identity representation on H_1 with H_1 (respectively H_2) and $\ell^2(\mathbb{F}_2)$ identified so that $\{\xi_{2i} : i \in \mathbb{N}\}$ (respectively $\{\xi_{2i-1} : i \in \mathbb{N}\}$) is the standard basis of $\ell^2(\mathbb{F}_2)$ in some enumeration, and let A_0 be the non-unital C*-algebra $uBu + K(H)$. Then $A_0/K(\ell^2(\mathbb{F}_2)) \cong C_r^*(\mathbb{F}_2)$.

For $t = \lambda_g \in C_r^*(\mathbb{F}_2)$ and $i \in \mathbb{N}$,

$$t_{ii} = (t\xi_{2i}|\xi_{2i}) = \begin{cases} 1 & (g = e) \\ 0 & (g \neq e) \end{cases},$$

where e denotes the identity element of $\mathbb{F}_2$. Moreover

$$e_{2i}u(t \oplus 0)ue_{2i} = \frac{1}{2}t_{ii}e_{2i,2i}$$

and

$$e_{2i-1}u(t \oplus 0)ue_{2i-1} = \frac{1}{2}t_{ii}e_{2i-1,2i-1}$$

for $t \in C_r^*(\mathbb{F}_2)$ and $i \in \mathbb{N}$. Let C_0 be the abelian C*-algebra generated by the set of projections $\{e_{ii} : i \in \mathbb{N}\}$ in A_0. The weak-operator closure L of C_0 is maximal abelian in $B(H)$ and if σ is the canonical projection onto L, then for $t \in C_r^*(\mathbb{F}_2)$,

$$\sigma(u(t \oplus 0)u) = \sum_i e_i u(t \oplus 0)ue_i = \frac{1}{2}\sum_i t_{ii}(e_{2i,2i} + e_{2i-1,2i-1}) = \lambda 1,$$

for some $\lambda \in \mathbb{C}$, where 1 is the identity operator on H. If $k + b \in C_0^c$ with $k \in K(H)$ and $b \in uBu$, then $k + b \in L^c = L$ and so $k + b = \sigma(k) + \sigma(b)$. Now $\sigma(k) \in C_0$ and $\sigma(b) = \lambda 1$ for some $\lambda \in \mathbb{C}$. Hence $\lambda 1 \in A_0$. Since A_0 is non-unital, $\lambda = 0$, which implies that $k + b = \sigma(k) \in C_0$. Thus $C_0^c = C_0$ and C_0 is maximal abelian in A_0.

Let $q : A_0 \to C_r^*(\mathbb{F}_2)$ be the quotient map. With λ and ρ as in 3.1, $\{\lambda, \rho \circ q\}$ is a commuting pair of representations of the pair $\{C_r^*(\mathbb{F}_2), A_0\}$ on $\ell^2(\mathbb{F}_2)$ with corresponding representation π_r of $C_r^*(\mathbb{F}_2) \odot A_0$ given by

$$\pi_r\left(\sum a_i \otimes b_i\right) = \sum \lambda(a_i)\rho(q(b_i)).$$

A C*-norm $\|\ \|_\alpha$ on $C_r^*(\mathbb{F}_2) \odot A_0$ is defined by

$$\|x\|_\alpha = \max\{\|x\|_{min}, \|\pi_r(x)\|\} \quad (x \in C_r^*(\mathbb{F}_2) \odot A_0).$$

As in the proof of Proposition 4, for any norm $\|\ \|_\beta$ on $C_r^*(\mathbb{F}_2) \odot A_0$ with $\|x\|_\beta \geq \|x\|_\alpha$, there is a non-zero $x \in C_r^*(\mathbb{F}_2) \otimes_\beta A_0$ such that $\Psi(x) = 0$, where Ψ is the canonical homomorphism from $C_r^*(\mathbb{F}_2) \otimes_\beta A_0$ to $C_r^*(\mathbb{F}_2) \otimes_{min} A_0$. Since $K(H)$ is nuclear, $\Psi|_{C_r^*(\mathbb{F}_2) \otimes_\beta K(H)}$ is isometric and, as in the proof of Proposition 4, $[x, a \otimes k] = 0$ for any $a \in C_r^*(\mathbb{F}_2)$ and $k \in K(H)$. Thus $x \in (C_1 \otimes C_0)^c \setminus (C_1 \otimes C_0)$ for any masa C_1 in $C_r^*(\mathbb{F}_2)$, in particular if C_1 is the abelian C*-subalgebra of $C_r^*(\mathbb{F}_2)$ with the extension property generated by one of the canonical unitary generators. It is easy to see directly that C_0 does not contain an approximate identity for A_0.

4. Some open problems. When H is the Hilbert space $\ell^2(\mathbb{N})$, $B(H)$ is non-nuclear and in particular the C*-norms $\|\ \|_{max}$ and $\|\ \|_{min}$ on $B(H) \odot C_r^*(\mathbb{F}_2)$ are distinct [13]. If C is a non-atomic masa of $B(H)$ isomorphic to $L^\infty(0, 1)$ (which does not have the extension property by [9]), is it true that $(C \otimes 1)^c = C \otimes C_r^*(\mathbb{F}_2)$ in $B(H) \otimes_{max} C_r^*(\mathbb{F}_2)$? Junge and Pisier [7] have shown that the C*-norms $\|\ \|_{max}$ and $\|\ \|_{min}$ on $B(H) \odot B(H)$ are distinct. Is it true that $(C \otimes 1)^c = C \otimes B(H)$ in $B(H) \otimes_{max} B(H)$ for any masa C of $B(H)$? Is $C_1 \otimes C_2$ maximal abelian in $B(H) \otimes_{max} B(H)$ for any masas C_1 and C_2? The case $C_1 \cong C_2 \cong \ell^\infty(\mathbb{N})$ is particularly intriguing. The question of whether masas isomorphic to $\ell^\infty(\mathbb{N})$ have the extension property relative to $B(H)$ was first investigated by Kadison and Singer [9], but remains at the time of writing one of the more significant unsolved problems in the subject, despite the attention of many distinguished workers. If the Kadison-Singer problem had a positive solution, it would follow by Corollary 3 that $\ell^\infty(\mathbb{N}) \otimes \ell^\infty(\mathbb{N})$ is maximal abelian in $B(H) \otimes_{max} B(H)$. If however it could be shown that $\ell^\infty(\mathbb{N}) \otimes \ell^\infty(\mathbb{N})$ is not maximal abelian in $B(H) \otimes_{max} B(H)$, then it would follow that $\ell^\infty(\mathbb{N})$ does not have the extension property.

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