

## COUNTEREXAMPLES TO SMOOTHING CONVEX FUNCTIONS

BY

PATRICK ADRIAN NEALE SMITH

**ABSTRACT.** Greene and Wu have shown that any continuous strongly convex function on a Riemannian manifold can be uniformly approximated by infinitely differentiable strongly convex functions. This result is not true if the word 'strongly' is omitted; in this paper, we give examples of manifolds on which  $\mathcal{C}^k$  convex functions cannot be approximated by  $\mathcal{C}^{k+1}$  convex functions ( $k = 0, 1, 2, \dots$ ).

**Introduction.** Greene and Wu have proven ([2], Theorem 2) that any continuous strictly convex function on a  $\mathcal{C}^\infty$  Riemannian manifold can be uniformly approximated by  $\mathcal{C}^\infty$  strictly convex functions. They comment that they do not know if this theorem remains true when the assumption of strict convexity is replaced by simple convexity. The purpose of this paper is to show that the theorem fails in this case.

The examples used here are similar to those used by Fornaess in [1] to disprove a similar result for plurisubharmonic functions. We will construct, for  $k = 0, 1, \dots, \infty$ , a manifold  $M$  with points  $p, q \in M$  and a convex function  $\rho \in \mathcal{C}^k(M)$  such that  $\rho(p) \neq \rho(q)$ , but  $\sigma(p) = \sigma(q)$  whenever  $\sigma \in \mathcal{C}^{k+1}(M)$  is convex (here  $\mathcal{C}^{k+1}(M)$  means the real analytic functions on  $M$ ).

Greene and Wu have pointed out that it is still not known whether semilocal approximation is possible. That is, if  $\rho$  is a continuous convex function in a neighbourhood of a compact subset  $K$  of a Riemannian manifold  $M$  and  $\epsilon > 0$ , is there a  $\mathcal{C}^\infty$  convex function  $\sigma$  in a neighbourhood of  $K$  such that  $|\sigma - \rho| < \epsilon$  on  $K$ ? Such an approximation is always possible on locally Euclidean spaces. (This can be seen by a simple convolution argument.) On the other hand, the manifold used here for the case  $k = \infty$  is locally Euclidean, but has the property that semilocal approximation of  $\mathcal{C}^\infty$  convex functions by real analytic convex functions is not possible.

It should also be noted that convex functions can always be semilocally approximated by  $\mathcal{C}^\infty$  subharmonic functions (see Proposition 1 of [6]).

I would like to thank John Erik Fornaess for suggesting the connection between this problem and [1].

**The Examples.** We first recall that a function on a Riemannian manifold is said to be convex if it is convex along every geodesic, when considered as a function of arc

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length. The manifolds that we will consider will all be constructed by defining an open subset  $\Omega$  of  $\mathbf{R}^2$  and using translations to identify various points within  $\Omega$ . The metric used will be the Euclidean metric on  $\mathbf{R}^2$ , so the convex functions will be those functions on the manifold which are convex as functions on  $\Omega$ .

The basic construction is the following: for any  $(a, b) \in \mathbf{R}^2$ , we put

$$A(a, b) = \{|y - b| < 1\} \cup \{a - 3 < x < a - 2, |y - b| < 2\} \\ \cup \{a - 1 < x < a, |y - b| < 2\}$$

and obtain a manifold  $M(a, b)$  from  $A(a, b)$  by identifying, for  $\alpha \in (a - 3, a - 2)$  and  $\beta \in (b + 1, b + 2)$ ,

$$(\alpha, \beta) \text{ with } (\alpha + 2, \beta - 3) \\ (\alpha + 2, \beta) \text{ with } (\alpha, \beta - 3)$$

(see Figure 1). Also, put  $N(a, b) = \{(x, y) \in M(a, b) : a - 3 < x < a\}$ . We then have:

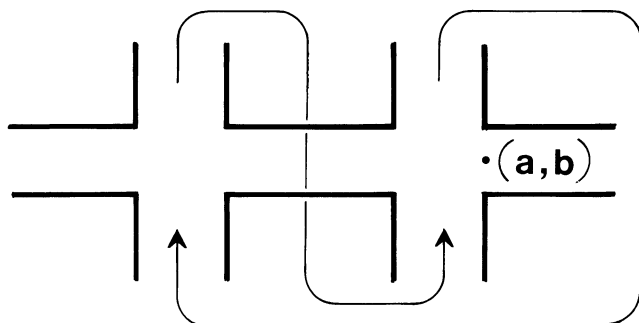


FIG. 1.  $M(a, b)$

LEMMA 1. Any convex function  $f$  on  $N(a, b)$  must be constant.

PROOF: For each  $x_0$  with  $a - 3 < x_0 < a - 2$ , the set  $\{x = x_0\} \cup \{x = 2 + x_0\}$  is a compact geodesic in  $N(a, b)$ , so  $f$  must be constant on this set. In particular,  $f(x_0, y) = f(2 + x_0, y)$  whenever  $|y - b| < 1$ .

Fix  $y_0$  with  $|y_0 - b| < 1$ ; then  $f(x, y_0)$  is a convex function of  $x$ . If  $a - 3 < x_1 < x_2 < a - 2$ , then  $x_1 < x_2 < 2 + x_1 < 2 + x_2$ ,  $f(x_1, y_0) = f(2 + x_1, y_0)$ , and  $f(x_2, y_0) = f(2 + x_2, y_0)$ . It follows from the convexity of  $f(x, y_0)$  that  $f(x_1, y_0) = f(x_2, y_0)$ . Thus,  $f$  is constant on  $\{(x, y_0) : a - 3 < x < a - 2\}$ , and hence also on  $\{(x, y_0) : a - 1 < x < a\}$ . Since  $f$  is convex, this forces  $f$  to be constant on each line segment  $\{(x, y_0) : a - 3 < x < a\}$ .

We have now shown that  $f$  is constant on the vertical lines  $\{x = x_0\}$ ,  $x_0 \in (a - 3, a - 2) \cup (a - 1, a)$ , and on the horizontal lines  $\{y = y_0\}$ ,  $y_0 \in (b - 1, b + 1)$ . Since these lines cover  $N(a, b)$ ,  $f$  must be constant.  $\square$

All of the manifolds we construct will contain one or more of these  $M(a, b)$ ; the ‘usual’ identifications on  $A(a, b)$  will refer to those used above to construct  $M(a, b)$ .

We will consider three separate cases. In order of increasing complexity, these are:  $k = \infty$ ,  $k = 0$ , and  $1 \leq k < \infty$ .

CASE 1 ( $k = \infty$ ). Here we take  $M = M(0, 0)$ . By Lemma 1, every real analytic convex function on  $M$  is constant on  $N(0, 0)$ , and hence is constant on all of  $M$ . On the other hand, it is easy to construct non-constant  $\mathcal{C}^\infty$  convex functions on  $M$ , so we see that  $\mathcal{C}^\infty$  convex functions on  $M$  can not be approximated by real analytic convex functions.

This example also shows that semilocal approximation of  $\mathcal{C}^\infty$  convex functions by real analytic convex functions fails even on locally Euclidean spaces. For if we take

$$K = \left\{ -\frac{8}{3} \leq x \leq -\frac{7}{3}, |y| < 2 \right\} \cup \left\{ -\frac{2}{3} \leq x \leq -\frac{1}{3}, |y| < 2 \right\} \\ \cup \left\{ -4 \leq x \leq 1, |y| \leq \frac{1}{2} \right\}$$

then  $K$  is a compact subset of  $M$  and any real analytic function convex in a neighbourhood of  $K$  must be constant on  $K$ , but this is not true for  $\mathcal{C}^\infty$  convex functions.

CASE 2 ( $k = 0$ ). Put

$$\Omega = A(0, 0) \cup A(-1, 5) \cup \{0 < x < 1, -2 < y < 7\}$$

and let  $M$  be the manifold obtained from  $\Omega$  by identifying  $(\alpha, 6 + \beta)$  with  $(\alpha, -2 + \beta)$  whenever  $0 < \alpha, \beta < 1$ , in addition to making the usual identifications on  $A(0, 0)$  and  $A(-1, 5)$  (see Figure 2).

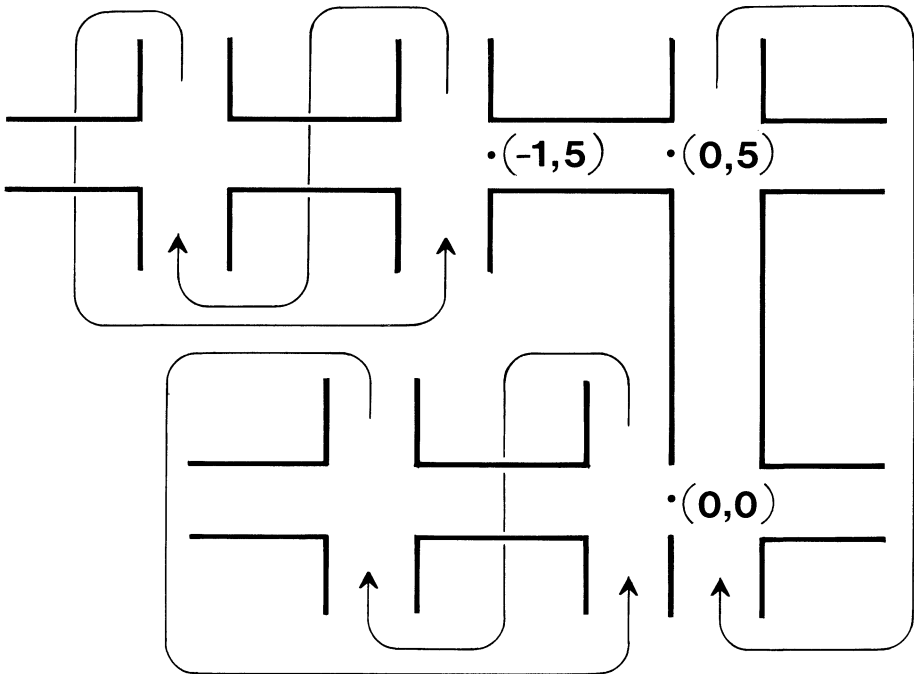


FIG. 2. The manifold used for  $k = 0$ .

LEMMA 2. *If  $\sigma$  is a  $\mathcal{C}^1$  convex function on  $M$ , then  $\sigma(0, 0) = \sigma(-1, 5)$ .*

PROOF: By Lemma 1,  $\sigma$  must be constant on  $N(0, 0)$  and  $N(-1, 5)$ . Hence,  $(\partial\sigma/\partial x)(0, 0) = (\partial\sigma/\partial x)(-1, 5) = 0$ .

For  $0 < x_0 < 1$ , the line  $\{x = x_0\}$  is a compact geodesic in  $M$ , so  $\sigma(x_0, 0) = \sigma(x_0, 5)$ . Thus,  $(\partial\sigma/\partial x)(0, 5) = (\partial\sigma/\partial x)(0, 0) = 0$  and  $\sigma(0, 5) = \sigma(0, 0)$ .

Since  $(\partial\sigma/\partial x)(-1, 5) = (\partial\sigma/\partial x)(0, 5) = 0$ , the convexity of  $\sigma$  implies that  $\sigma$  is constant on  $\{(x, 5) : -1 \leq x \leq 0\}$ , so  $\sigma(0, 0) = \sigma(0, 5) = \sigma(-1, 5)$ . □

Now define  $\rho$  on  $M$  by

$$\rho(x, y) = \begin{cases} 0 & \text{if } x \leq 0, y < 2 \\ x & \text{if } x \geq 0 \\ -1 & \text{if } x \leq -1, y > 3 \\ x & \text{if } x \geq -1, y > 3 \end{cases}$$

Then  $\rho$  is a  $\mathcal{C}^0$  convex function on  $M$ ,  $\rho(0, 0) = 0$ , and  $\rho(-1, 5) = -1$ .

CASE 3 ( $1 \leq k < \infty$ ). For each  $n \geq 2$ , choose an  $\epsilon_n$  with  $0 < \epsilon_n < 1/(n - 1) - 1/n$ , and put

$$\begin{aligned} \Omega = & A(0, 0) \cup \bigcup_{n=2}^{\infty} A(0, 5n) \cup \left\{ -\frac{5}{3} < x < -\frac{4}{3}, y > 9 \right\} \\ & \cup \bigcup_{n=2}^{\infty} \left[ \left\{ \frac{1}{n} < x < \left(\frac{1}{n}\right) + \epsilon_n, |y| < 2 \right\} \right. \\ & \left. \cup \{n^k < x < n^k + \epsilon_n, |y - 5n| < 2\} \right] \end{aligned}$$

$M$  will be the manifold obtained from  $\Omega$  by making the usual identifications on  $A(0, 0)$  and each  $A(0, 5n)$ , and also, for  $0 < \alpha < \epsilon_n, 0 < \beta < 1$ , identifying

$$\begin{aligned} (1/n + \alpha, 1 + \beta) & \text{ with } (n^k + \alpha, 5n - 2 + \beta) \\ (n^k + \alpha, 5n + 1 + \beta) & \text{ with } (1/n + \alpha, -2 + \beta) \end{aligned}$$

(see Figure 3).

LEMMA 3. *If  $\sigma$  is a  $\mathcal{C}^{k+1}$  convex function on  $M$ , then  $\sigma(0, 0) = \sigma(0, 10)$ .*

PROOF: Put  $g(x) = \sigma(x, 0)$  and  $g_n(x) = \sigma(x, 5n), n \geq 2$ . Then by Lemma 1, each  $g_n$  is constant on  $[-3, 0]$ , as is  $g$ .

Put  $f(y) = \sigma(-3/2, y)$ . Then  $f$  is a convex function on  $(9, \infty)$ , and is constant on each interval  $(5n - 1, 5n + 1), n \geq 2$ ; this implies that  $f$  is constant. Letting  $r$  be this constant, we then have  $g_n(x) = r$  for each  $x \in [-3, 0]$  and  $n \geq 2$ .

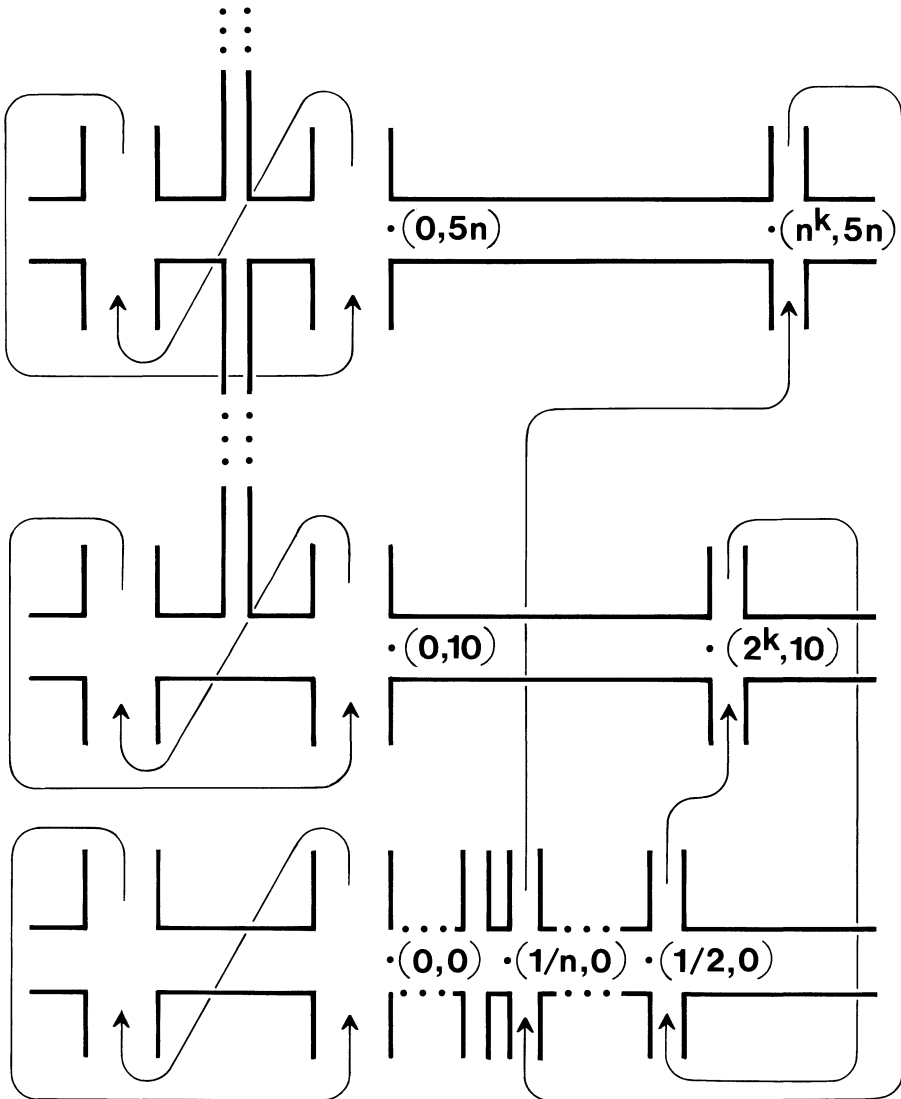


FIG. 3. The manifold used for  $1 \leq k < \infty$ .

From the construction of  $M$ , we must have  $g((1/n) + x) = g_n(n^k + x)$  whenever  $0 < x < \epsilon_n$ . Hence,  $g_n(n^k) = g(1/n)$  and  $g'_n(n^k) = g'(1/n)$ . Since  $g'_n(0) = 0$  and  $g'_n$  is increasing, this gives us

$$0 \leq g_n(n^k) - g_n(0) \leq n^k g'_n(n^k)$$

or

$$0 \leq g\left(\frac{1}{n}\right) - r \leq n^k g'\left(\frac{1}{n}\right)$$

But  $g$  is  $\mathcal{C}^{k+1}$  and constant on  $[-3, 0]$ , so  $g'(x)$  is  $o(x^k)$  near 0. Hence,  $\lim_{n \rightarrow \infty} n^k g'(1/n) = 0$ , so  $g(0) = r = g_2(0)$ , proving the lemma.  $\square$

We must now find a  $\mathcal{C}^k$  convex function  $\rho$  on  $M$  with  $\rho(0, 0) \neq \rho(0, 10)$ . To do this, we first choose, for each  $n \geq 2$ , a  $\mathcal{C}^\infty$  convex function  $g_n$  on  $R$  with

$$g_n(x) = \begin{cases} -k & \text{for } x \leq 0 \\ \left(x - n^k + \frac{1}{n}\right)^{k+1} & \text{for } x \geq n^k. \end{cases}$$

(This is possible, since we will then have  $0 < g_n(n^k) - g_n(0) < n^k g'_n(n^k)$  and  $g''_n(n^k) > 0$ ).

Once the  $g_n$  have been chosen, we put

$$\rho(x, y) = \begin{cases} 0 & \text{if } x \leq 0, |y| < 2 \\ x^{k+1} & \text{if } x \geq 0, |y| < 2 \\ -k & \text{if } x \leq 0, y > 8 \\ g_n(x) & \text{if } x \geq 0, |y - 5n| < 2, n \geq 2. \end{cases}$$

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UNIVERSITY OF TORONTO  
TORONTO, ONT. M5S 1A1