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ORTHONORMAL EXPANSIONS AND THE PRINCIPLE OF UNIFORM BOUNDEDNESS

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Let $\{\phi_v : v \in \mathbb{N} \text{ (non-negative integers)}\} \subseteq C[0, 1]$ be a complete orthonormal sequence of complex-valued functions in $L^2[0, 1]$, $\{\lambda_v : v \in \mathbb{N}\}$ and $\{\lambda_{v\mu} : v, \mu \in \mathbb{N}\}$ be sequences of complex numbers. In this paper, the necessary and sufficient conditions are developed for the series $\sum_{v=0}^{\infty} \lambda_v \widehat{f}(v)\phi_v$ to converge and also $\lim_{\mu\to\infty} \sum_{v=0}^{\infty} \lambda_{v\mu}\widehat{f}(v)\phi_v$ to exist, in $C[0, 1]$ for each $f \in L^1[0, 1]$ where $\widehat{f}(v) = \int_0^1 f(t)\overline{\phi_v(t)}dt$, $v \in \mathbb{N}$.

Let $C[0, 1]$ denote the Banach space of continuous complex-valued functions with the usual sup norm and $\mathbb{C}$ be the complex number field. Further, let $\{\phi_v : v \in \mathbb{N}\} \subseteq C[0, 1]$ be a complete orthonormal sequence of complex valued functions in $L^2[0, 1]$. For any $f \in L^1[0, 1]$, let $\widehat{f}(v) = \int_0^1 f(t)\overline{\phi_v(t)}dt$ be the vth Fourier coefficient of f with respect to the sequence $\{\phi_v : v \in \mathbb{N}\}$. It is well known that the formal Fourier series $\sum_{v=0}^{\infty} \widehat{f}(v)\phi_v$ of $f \in L^1[0, 1]$ may not converge in $C[0, 1]$. In Section 1, we consider a sequence $\{\lambda_v : v \in \mathbb{N}\}$ of complex numbers and develop the necessary and sufficient conditions for the series $\sum_{v=0}^{\infty} \lambda_v\widehat{f}(v)\phi_v$ to converge in $C[0, 1]$ for each $f \in L^1[0, 1]$. In Section 2, we consider a doubly infinite sequence $\{\lambda_{v\mu} : v, \mu \in \mathbb{N}\}$ and investigate the necessary and sufficient conditions for the series $\sum_{v=0}^{\infty} \lambda_{v\mu}\widehat{f}(v)\phi_v$ to converge for each $\mu \in \mathbb{N}$ and also the $\lim_{\mu\to\infty} \sum_{v=0}^{\infty} \lambda_{v\mu}\widehat{f}(v)\phi_v$ to exist in $C[0, 1]$ for each $f \in L^1[0, 1]$. Aljančić [10] and Husain [2] have similar results for general orthonormal expansions of class C and L^∞ respectively. Husain [3] and others have also obtained similar results for Fourier series of summable functions.

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Throughout, let S denote the linear span of $\{\phi_v : v \in \mathbb{N}\}$. Then S is a dense subset of $L^2[0, 1]$ and hence also a dense subset of $L^1[0, 1]$. Let $\{\lambda_v : v \in \mathbb{N}\}$ be a sequence of complex numbers. For each $n \in \mathbb{N}$, we define the linear operator $S_n : L^1[0, 1] \to C[0,1]$ by

$$(1) \qquad S_n(f) = \sum_{v=0}^{n} \lambda_v \widehat{f}(v)\phi_v\ , \qquad f \in L^1[0, 1].$$

and for any $x \in [0, 1]$,

$$(2) \qquad (S_n f)\, x = \sum_{v=0}^{n} \lambda_v \widehat{f}(v)\phi_v(x) = \int_0^1 f(t)K_n(x,t)dt\ ,$$

where

$$(3) \qquad K_n(x,t) = \sum_{v=0}^{n} \lambda_v \phi_v(x)\overline{\phi_v(t)}\ .$$

Note that for each fixed $x \in [0, 1]$, $K_n(x,\cdot) \in C[0, 1]$ and hence $K_n(x,\cdot) \in L^\infty[0, 1]$. Let

$$\|K_n\|_\infty = \sup_x \left(\operatorname{ess\,sup}_t |K_n(x,t)|\right)\ .$$

The lemma below shows that $\|K_n\|_\infty < \infty$.

LEMMA 1. *For each $n \in \mathbb{N}$, let $S_n : L^1[0, 1] \to C[0, 1]$ be defined by (1). Then*

(a) *S_n is a bounded linear operator with $\|S_n\| = \|K_n\|_\infty$.*

(b) *For any $g \in S$, $\lim_{n\to\infty} S_n(g)$ exists in $C[0, 1]$.*

PROOF: (a) The boundedness of S_n follows from the definition (1). To obtain $\|S_n\|$, let $x \in [0, 1]$ and define the linear functional $S_x : L^1[0, 1] \to \mathbb{C}$ by

$$S_x(f) = (S_n f)\, x = \int_0^1 f(t)K_n(x,t)dt\ , \qquad f \in L^1[0, 1].$$

Then S_x is a continuous linear functional on $L^1[0, 1]$ with $\|S_x\| = \operatorname{ess\,sup}_t |K_n(x,t)|$. Further, by (2),

$$\|S_n\| = \sup_{\|f\|_1 \leqslant 1} \|S_n(f)\|_\infty = \sup_{\|f\|_1 \leqslant 1} \left(\sup_{x\in[0,1]} |(S_n f)\, x|\right) = \sup_{x\in[0,1]} \left(\sup_{\|f\|_1 \leqslant 1} |S_x(f)|\right)$$
$$= \sup_{x\in[0,1]} \|S_x\| = \|K_n\|_\infty\ .$$

To prove (b), let $g \in S$. Then there exists a $N \in \mathbb{N}$ and complex numbers $\{C_v : v = 0, 1, 2, \cdots, N\}$ such that $g = \sum_{v=1}^{N} C_v \phi_v$. By definition, for any $u \in \mathbb{N}$,

$$\widehat{\phi}_v(u) = \begin{cases} 1 & \text{if } v = u, \\ 0 & \text{if } v \neq u. \end{cases} \tag{4}$$

It follows by (1) that for any $n \geqslant N$,

$$S_n(g) = \sum_{v=0}^{N} C_v S_n(\phi_v) = \sum_{v=0}^{N} C_v \left(\sum_{u=0}^{n} \lambda_\mu \widehat{\phi}_v(u) \phi_u \right) = \sum_{v=0}^{N} C_v \lambda_v \phi_v.$$

Hence $\lim_{n \to \infty} S_n(g) = \sum_{v=0}^{N} C_v \lambda_v \phi_v$ exists in $C[0, 1]$. □

Theorem 1. *The series* $\sum_{v=0}^{\infty} \lambda_v \widehat{f}(v) \phi_v$ *converges in* $C[0, 1]$ *for each* $f \in L^1[0, 1]$ *if and only if* $\sup_n \|K_n\|_\infty < \infty$.

Proof: Consider the sequence $S_n : L^1[0, 1] \to C[0, 1]$ of bounded linear operators given by (1). If $\lim_{n \to \infty} S_n(f)$ exists in $C[0, 1]$ for each $f \in L^1[0, 1]$, then $\sup_n \|S_n(f)\|_\infty < \infty$ for each $f \in L^1[0, 1]$. Hence, by the uniform boundedness theorem ([4], p.267), $\sup_n \|K_n\|_\infty = \sup_n \|S_n\| < \infty$. Conversely, suppose $\sup_n \|K_n\|_\infty < \infty$. Let $f \in L^1[0, 1]$ and $\varepsilon > 0$ be given. Since S is dense in $L^1[0, 1]$, there exists a $g \in S$ with $\|f - g\|_1 \leqslant \varepsilon \left(2 \sup_n \|K_n\|_\infty \right)^{-1}$. If $g = \sum_{v=1}^{N} C_v \phi_v$, then as shown in Lemma 1, $S_n g = S_m g$ for all $n, m \geqslant N$ and

$$\begin{aligned} \|S_n f - S_m f\|_\infty &\leqslant \|S_n f - S_n g\|_\infty + \|S_n g - S_m g\|_\infty + \|S_m g - S_m f\|_\infty \\ &\leqslant 2 \sup_n \|K_n\|_\infty \|f - g\|_1 \leqslant \varepsilon. \end{aligned}$$

Thus $\{S_n f\}$ is a Cauchy sequence in $C[0, 1]$ and hence $\lim_{n \to \infty} S_n f$ exists in $C[0, 1]$. □

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In this section, we consider a fixed doubly indexed sequence $\{\lambda_{v\mu} : v, \mu \in \mathbb{N}\}$ of complex numbers and the corresponding associated operators $S_{n,\mu} : L^1[0, 1] \to C[0, 1]$ defined by

$$S_{n,\mu}(f) = \sum_{v=0}^{n} \lambda_{v\mu} \widehat{f}(v) \phi_v \,, \qquad f \in L^1[0, 1] \,, \quad n, \mu \in \mathbb{N}. \tag{5}$$

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Clearly, $S_{n,\mu}$ is a bounded linear operator and for any $x \in [0, 1]$

$$(6) \qquad (S_{n,\mu}f)x = \sum_{v=0}^{n} \lambda_{v\mu}\widehat{f}(v)\phi_v(x) = \int_0^1 f(t)K_{n,\mu}(x,t)dt,$$

where

$$(7) \qquad K_{n,\mu}(x,t) = \sum_{v=0}^{n} \lambda_{v\mu}\phi_v(x)\overline{\phi_v(t)}.$$

As before, note that for any fixed $x \in [0, 1]$, $K_{n,\mu}(x,\cdot) \in L^\infty[0, 1]$. Let for each $n, \mu \in \mathbb{N}$,

$$(8) \qquad \|K_{n,\mu}\|_\infty = \sup_{x\in[0,1]} \left(\operatorname{ess\,sup}_t |K_{n,\mu}(x,t)|\right).$$

It follows from Lemma 1 that for each n, μ in $\mathbb{N}$,

$$(9) \qquad \|S_{n,\mu}\| = \|K_{n,\mu}\|_\infty < \infty.$$

DEFINITION 1: The sequence $\{S_{n,\mu} : n, \mu \in \mathbb{N}\}$ of operators in (5) is μ-convergent in $C[0, 1]$ if and only if $\lim_{n\to\infty} S_{n,\mu}(f) \equiv \sum_{v=0}^{\infty} \lambda_{v\mu}\widehat{f}(v)\phi_v$ exists in $C[0, 1]$ for each $f \in L^1[0, 1]$ and each $\mu \in \mathbb{N}$.

As a restatement of Theorem 1, we have

THEOREM 2. *For each $n, \mu \in \mathbb{N}$, let $S_{n,\mu} : L^1[0, 1] \to C[0, 1]$ be defined by (5). Then the sequence $\{S_{n,\mu}\}$ is μ-convergent if and only if $\sup_n \|K_{n,\mu}\|_\infty < \infty$ for each $\mu \in \mathbb{N}$.*

DEFINITION 2: Let the sequence $\{S_{n,\mu} : n, \mu \in \mathbb{N}\}$ of operators in (5) be μ-convergent in $C[0, 1]$. For each $\mu \in \mathbb{N}$, define the operator $\widetilde{S}_\mu : L^1[0, 1] \to C[0, 1]$ by

$$(10) \qquad \widetilde{S}_\mu(f) = \lim_{n\to\infty} S_{n,\mu}(f), \qquad f \in L_1[0, 1].$$

It follows by (10) and the Banach-Steinhaus Theorem that for each $\mu \in \mathbb{N}$, $\widetilde{S}_\mu$ is a bounded linear operator and

$$(11) \qquad \left\|\widetilde{S}_\mu\right\| \leqslant \sup_n \|S_{n,\mu}\|.$$

LEMMA 2. *For each $n, \mu \in \mathbb{N}$, let $S_{n,\mu} : L^1[0, 1] \to C[0, 1]$ be defined by (5). If the sequence $\{S_{n,\mu}\}$ is μ-convergent in $C[0, 1]$ then the following statements are equivalent:*

(a) $\lim_{\mu\to\infty} \widetilde{S}_\mu(f)$ *exists in $C[0, 1]$ for each $f \in S$;*

(b) $\lim_{\mu\to\infty} \lambda_{v\mu}$ *exists for each $v \in \mathbb{N}$.*

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PROOF: Suppose (a) holds. Since the sequence $\{S_{n,\mu}\}$ is μ-convergent in $C[0,1]$, it follows that for any $\mu, v \in \mathbb{N}$, $\widetilde{S}_\mu \phi_v \in C[0,1]$. Hence by (4) for any $n \geqslant v$, $\mu \in \mathbb{N}$,

$$\widetilde{S}_\mu(\phi_v) = \lim_{n\to\infty} S_{n,\mu}(\phi_v) = \lim_{n\to\infty}\left(\sum_{w=0}^{n} \lambda_{w\mu}\widehat{\phi}_v(w)\phi_w\right) = \lambda_{v\mu}\phi_v;$$

consequently by (a), $\lim\limits_{\mu\to\infty} \lambda_{v\mu}\phi_v$ exists in $C[0,1]$ for each $v \in \mathbb{N}$. Since for $m, n \in \mathbb{N}$,

$$|\lambda_{vm} - \lambda_{vn}| = \|\lambda_{vm}\phi_v - \lambda_{vn}\phi_v\|_\infty \|\phi_v\|_\infty^{-1},$$

It follows that $\lim\limits_{\mu\to\infty} \lambda_{v\mu}$ exists in $\mathbb{C}$ for each $v \in \mathbb{N}$. Conversely, suppose (b) holds and $\lim\limits_{\mu\to\infty} \lambda_{v\mu} = \widetilde{\lambda}_v$ for each $v \in \mathbb{N}$. Let $f \in S$. Then there exists a nonnegative integer N and complex numbers $C_v (v = 0, 1, \cdots, N)$ such that $f = \sum\limits_{v=0}^{N} C_v\phi_v$. Thus for $n \geqslant N$ and $\mu \in \mathbb{N}$,

$$S_{n,\mu}(f) = \sum_{v=0}^{N} C_v S_{n,\mu}(\phi_v) = \sum_{v=0}^{N} C_v \lambda_{v\mu}\phi_v.$$

Hence, for any $\mu \in \mathbb{N}$, $\widetilde{S}_\mu(f) = \sum\limits_{v=0}^{N} C_v\lambda_{v\mu}\phi_v$. Consequently, by assumption (b) $\lim\limits_{\mu\to\infty} \widetilde{S}_\mu(f) = \sum\limits_{v=0}^{N} C_v\widetilde{\lambda}_\mu\phi_v$ exists in $C[0,1]$. Thus (b) implies (a). ∎

As an immediate consequence of Lemma 2 and the uniform boundedness theorem, we have:

THEOREM 3. *For each $n, \mu \in \mathbb{N}$, let $S_{n,\mu} : L^1[0,1] \to C[0,1]$ given in (5) be μ-convergent in $C[0,1]$. Then the sequence $\left\{\widetilde{S}_\mu(f) : \mu \in \mathbb{N}\right\}$ converges in $C[0,1]$ for each $f \in L^1[0,1]$ if and only if (a) $\sup\limits_\mu \left\|\widetilde{S}_\mu\right\| < \infty$ and (b) $\lim\limits_{\mu\to\infty} \lambda_{v\mu}$ exists for each $v \in \mathbb{N}$.*

Combining Theorem 2 and Theorem 3, we obtain:

COROLLARY 1. *The necessary and sufficient conditions for $\widetilde{S}_\mu(f) = \sum\limits_{v=0}^{\infty} \lambda_{v\mu}\widehat{f}(v)\phi_v \in C[0,1]$ for each $\mu \in \mathbb{N}$ and $\lim\limits_{\mu\to\infty} \widetilde{S}_\mu(f)$ exists in $C[0,1]$, for each $f \in L^1[0,1]$ are*

(i) $\sup\limits_n \|K_{n,\mu}\|_\infty < \infty$ *for each* $\mu \in \mathbb{N}$,

(ii) $\sup\limits_\mu \left\|\widetilde{S}_\mu\right\| < \infty$,

(iii) $\lim\limits_{\mu\to\infty} \lambda_{v\mu}$ *exists for each* $v \in \mathbb{N}$.

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As another view of the characterisation in Theorem 3, assume that for each $\mu \in \mathsf{N}$, $\widetilde{S}_\mu : L^1[0,1] \to C[0,1]$ given in Definition 2 is a bounded linear operator. It then follows that for each fixed $x \in [0,1]$, $\mu \in \mathsf{N}$, the mapping $T_{\mu,x} : L^1[0,1] \to \mathsf{C}$ defined by $T_{\mu,x}(f) = \left(\widetilde{S}_\mu f\right) x$ is a bounded linear functional on $L^1[0,1]$. Hence, by the well-known Riesz integral representation theorem (see [4], p.276) for each $x \in [0,1]$ and $\mu \in \mathsf{N}$, there exists a function $\widetilde{K}_\mu(x,\cdot) \in L^\infty[0,1]$ satisfying

$$\left(\widetilde{S}_\mu f\right) x = \int_0^1 f(t)\widetilde{K}_\mu(x,t)dt\,, \qquad x \in [0,1]\,,\ \ \mu \in \mathsf{N}\,.$$

Consequently, it follows as in Lemma 1 that for each $\mu \in \mathsf{N}$,

$$\left\|\widetilde{S}_\mu\right\| = \sup_x \left(\operatorname{ess\,sup}_t \left|\widetilde{K}_\mu(x,t)\right|\right) < \infty.$$

Setting $\left\|\widetilde{K}_\mu\right\|_\infty = \sup_x \left(\operatorname{ess\,sup}_t \left|\widetilde{K}_\mu(x,t)\right|\right)$, it follows from (9) and (11) that for each $\mu \in \mathsf{N}$,

$$\left\|\widetilde{S}_\mu\right\| = \left\|\widetilde{K}_\mu\right\|_\infty \leqslant \sup_n \|K_{n,\mu}\|_\infty\,. \tag{12}$$

Consequently, Theorem 3 may be rephrased as:

Theorem 4. *For $n, \mu \in \mathsf{N}$, let $S_{n,\mu} : L^1[0,1] \to C[0,1]$ given in (5) be μ-convergent in $C[0,1]$. Then the sequence $\left\{\widetilde{S}_\mu f : \mu \in \mathsf{N}\right\}$ converges in $C[0,1]$ for each $f \in L^1[0,1]$ if and only if (i) $\sup_\mu \left\|\widetilde{K}_\mu\right\|_\infty < \infty$ and (ii) $\lim_{\mu\to\infty} \lambda_{v\mu}$ exists for each $v \in \mathsf{N}$.*

As a consequence of (12), we also obtain:

Corollary 2. *If $\left\{\sup \|K_{n,\mu}\|_\infty : n,\mu \in \mathsf{N}\right\} < \infty$ and $\lim_{\mu\to\infty} \lambda_{v\mu}$ exists for each $v \in \mathsf{N}$, then for each $\mu \in \mathsf{N}$, the series $\sum_{v=0}^\infty \lambda_{v\mu}\widehat{f}(v)\phi_v$ converges and $\lim_{\mu\to\infty}\left(\sum_{v=0}^\infty \lambda_{v\mu}\widehat{f}(v)\phi_v\right)$ exists in $C[0,1]$ for each $f \in L^1[0,1]$.*

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