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A NOTE ON A THEOREM ON LATTICES IN LIE GROUPS

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ABSTRACT. The aim of this note is to clarify some statements in the book: *Discrete Subgroups of Lie groups*: Springer, 1972.

The purpose of this note is to respond to a review by James E. Humphreys in Mathematical Reviews on the article by A. N. Starkov: *A counterexample to a theorem on Lattices in Lie groups*. Vestnik Moskov, Ser. I Mat. Mekh. 1984, no. 5, 68-69, MR86F:22013). We quote that review in its entirety here:

> "Corollary 8.28 in the book of M. S. Raghunathan [*Discrete subgroups of Lie groups*, Springer, New York, 1972; MR 58 #22394a] makes a somewhat technical assertion about a lattice Γ in a non-semisimple Lie group $\mathscr{G}$. The present author regards this as false, without however pointing to any particular gap in the proof, and sketches a proposed counterexample modelled on an example in a closely related paper of L. Auslander [Ann. of Math. (2) 71 (1960), 579-590; MR 22 #12161]. However, some details are omitted, e.g., the proof that the exhibited group Γ is in fact discrete in $\mathscr{G}$. {Raghunathan has informed the reviewer that he and others were aware long ago of an error in the proof of Corollary 8.25 in his book, which in turn invalidates the proof of Corollary 8.28. It would be useful, in the reviewer's opinion, to have a definitive account of what is actually valid or invalid in this circle of ideas – lest it remain indeed a circle.}"

(For statement 8.25, see section 2; for statement 8.28, see proposition 1.3 in this note). In fact, the statement 8.25 in [5] is not true, and we shall give a counterexample in section 2. The statement 8.28 in [5] is correct[(1)], a proof independent of 8.25 was given by G. D. Mostow ([2], Lemma 3.9, p. 421). The methods used by Mostow in his proof are very useful. Here, we shall present a slight variation of his proof in section 1, so bring a closer relation to 8.25 in [5].

[(1)]Modulo an apparent misprint in the original statement, i.e. N should be also a normal subgroup of G.

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1.1 LEMMA. (Mostow [2], Lemma 3.8). *Let G be an analytic group which is a compact extension of an analytic normal solvable group R, i.e. G/R is compact. Assume that G does not contain any non-trivial normal semi-simple factor. Let* Ad *be the adjoint representation of G in the automorphism group* $\mathrm{Aut}(\mathscr{L}(R))$ *of the Lie algebra $\mathscr{L}(R)$ of R. Let U be the unipotent radical of the Zariski closure of* Ad G *in the algebraic group* $\mathrm{Aut}(\mathscr{L}(G))$. *Then the nilradical M of G is the identity component of the* $\{x \in G; \mathrm{Ad}\, x \in U\}$. *Furthermore M is a maximal connected nilpotent subgroup of G.*

PROOF. We may assume that the maximal nilpotent analytic normal subgroup N of R is simply connected. Otherwise, let C be the compact part (Torus part) of the center of N. Then C is also in the center of G. Once we show that the lemma is true in G/C, it is easy to see that it is also true in the general case. So, in the following we shall assume that N is simply connected. R/N is connected abelian, so it is isomorphic to the product of a vector group and a torus group. Let F be the inverse image of the vector group in R. Then F is simply connected and R/F is compact. Therefore, G and R both split over F; $F \subset F \rtimes D = R \subset F \rtimes DTK$, where D, T are torus groups and K is a compact semi-simple analytic subgroup. D and T are central in DTK. It is clear that the radical R' of G is the subgroup RDT.

We shall first study the kernel P of the adjoint representation Ad. The identity component P_0 of P does not contain any nontrivial semi-simple Levi-factor. Otherwise it will violate the assumption that G has no semi-simple factor. Therefore P_0 is contained in the radical $R' = RDT$. Because P/P_0 is discrete and normal in G/P_0, so it is central. In particular, the commutator $[P, P]$ of P is in P_0. Use the expression $G = FDTK$, by a calculation, we see that $[G, R'] \subset F \subset R$, $[R', R'] \subset F \subset R$. This implies $[[[P, P], P], P]$ is trivial, i.e. P is nilpotent. Because U is a normal nilpotent subgroup of $(\mathrm{Ad}\, G)^*$. So, $M' = \mathrm{Ad}^{-1}(U) = \{x{:}\mathrm{Ad}\, x \in U\}$ is nilpotent. We conclude that the identity component M of M' is a normal nilpotent analytic subgroup of G.

Now, we shall prove that M is a maximal connected nilpotent subgroup of G. Let L be any connected nilpotent subgroup of G which contains the nil radical N of R. Then $(RL)^-$ is solvable. Furthermore, $[R, L] \subset N$ (For a proof, see [1], Corollary 2, p. 51). Therefore L is a normal nilpotent subgroup of the solvable group $(RL)^-$. $(RL)^-$ is solvable, the unipotent elements in the adjoint representation of $(RL)^-$ in $\mathrm{Aut}(\mathscr{L}(R))$ forms a group. Because G is a compact extension of solvable group. And U in $(\mathrm{Ad}\, G)^*$ is the group generated by parts of Ad x where x comes from R. Since L acts on $\mathscr{L}(R)$ unipotently (by adjoint action). Therefore U Ad L is a group of unipotent elements. However, $(\mathrm{Ad}\, G)^* = U \rtimes \Phi$, where Φ consists of only semi-simple elements. Therefore, $U \,\mathrm{Ad}\, L \subset U$, i.e., $\mathrm{Ad}\, L \subset U$. Hence $L \subset M$. This completes the proof that M is a maximal analytic nilpotent subgroup of G.

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1.2 REMARK. In addition to the assumptions in the Lemma 1.1, assume that G/R is semi-simple, then $M = N$. Another sufficient condition for $M = N$ is that R is an exponential group. The following is the statement 8.28 in [5].

1.3 PROPOSITION. *Let G be a connected Lie group,* $\Gamma \subset G$ *a lattice. Let R be the radical of G and N the maximum connected closed nilpotent normal subgroup of G. Let S be a semi-simple subgroup of G* such that $G = S \cdot R$. *Let* σ *denote the action of S on R. Assume that the kernel of* σ *has no compact factor in its identity component. Let* $\pi: G \to G/R$ *and* $\pi': G \to G/N$ *be the natural maps. Then* $R/R \cap \Gamma$ *and* $N/N \cap \Gamma$ *are both compact. Moreover* $\pi(\Gamma)$ *and* $\pi'(\Gamma)$ *are lattices in* G/R *and* G/N *respectively.*

PROOF (Mostow). Let $R_1 = (\Gamma R)_0^-$, the identity component of the closure of ΓR. Then $R_1 \subset KR$ where K is a maximal compact analytic semi-simple subgroup of G (see [2] Lemma 3.41 d) or [6]).

By the Theorem 8.24 of [5], R_1 is a solvable analytic group. Since $R_1/\Gamma \cap R_1 \cong \Gamma R_1/\Gamma$ has finite invariant measure, $\Gamma \cap R_1$ is a uniform lattice in R_1 by a theorem of Mostow (Theorem 6.2 of [4] or Theorem 3.5 of [5]). By 1.1 and 1.2, the nilradical N of G is again a nilradical of R_1. Notice that $N \subset R_1$, $N \cap \Gamma = N \cap (\Gamma \cap R_1)$. Therefore $N/N \cap \Gamma$ is compact again by a theorem of Mostow ([3], or Theorem 3.3 of [5]). From the fact $N/N \cap \Gamma$ is compact, the map $N/N \cap \Gamma \to N\Gamma/\Gamma$ is a homeomorphism. Therefore $N\Gamma$ is locally compact, it is a closed subgroup of G (cf. Theorem 1.13 of [5]). Because $N\Gamma$ is closed; $\pi'(\Gamma)$ is a lattice in $\pi'(G)$. In $\pi'(G)$, $\pi'(R)$ is the nilradical. So $\pi'(R)\pi'(\Gamma)$ is closed. This implies RP is closed. We conclude $R/R \cap \Gamma$ is compact, i.e., $\pi(\Gamma)$ is closed.

2.1. The following is the statement 8.25 in [5]: Let G be a Lie group of the form $K \cdot R$ where K is a compact group and R is a connected normal solvable closed subgroup on which K acts with a finite kernel. Let Γ be a lattice in G. Then G/Γ and $N/N \cap \Gamma$ are compact where N is the maximum normal connected nilpotent subgroup of R. Following the original proof, let U be the closure of ΓR and U_0 be the identity component of U. Then U_0 is a compact extension of the solvable group R. Let N_1 be the maximal connected nilpotent subgroup of U_0. It is clear that $N_1\Gamma$ is closed and $N_1/N_1 \cap \Gamma$ is compact. Also $N \subset N_1$. However, in general N is properly contained in N_1. Also $N\Gamma$ is not always closed. The original proof was based upon the false assertion that $N = N_1$ to draw the conclusion that $N\Gamma$ is closed and $N/N \cap \Gamma$ is compact. Now, we shall give an example to show this is not true in general.

2.2 EXAMPLES. Let $A = \{(x, y, z); x, y, z$ real numbers$\}$. Define the multiplication in A by $(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy')$. This makes A a nilpotent group which is isomorphic with the group of upper

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triangular matrices. Let $B = \{a, b, e^{2\pi it}):a, b, t$ any real numbers}. Define the multiplication in B by

$$(a, b, e^{2\pi it})(a', b', e^{2\pi it'})$$
$$= (a + a' \cos 2\pi t - b' \sin 2\pi t, b + a' \sin 2\pi t + b' \cos 2\pi t, e^{2\pi i(t+t')}).$$

Then B is the group of rigid motion of the plane. Let $G = A \times B$. Let R be the subgroup of $A \times B$, $R = \{ (t, x, y, a, b, e^{2\pi it}):t, x, y, a, b$ real numbers}. Then G is a compact extension of R, $G = R \rtimes C$, $C = \{ (0, 0, 0, 0, 0, e^{2\pi it}):t$ reals}. The nilradical N of $R = \{ (0, y, z, a, b, 1):y, z, a, b$ are reals}. The nilradical N_1 of G is $A \times R^2 = A \times \{ (a, b, 1):a, b$ reals}. Hence N is properly contained in N_1.

We given another example: Let G be the semi-direct product of the two-dimension vector group V and the 2 dimension torus group C^2. A typical element in G is of the form $(x, y\ e^{2\pi is}, e^{2\pi it})$. The action of C^2 on V is defined by:

$$\eta(e^{2\pi is}, e^{2\pi it})(x, y) = (x \cos 2\pi s - y \sin 2\pi s, x \sin 2\pi s + y \cos 2\pi s)$$

So, in fact; $V \rtimes (C \times (1))$ is the rigid motion of the plane. Let R be the subgroup of G, $R = V \times D$, where $D \subset C^2$, $D = \{ (e^{2\pi is}, e^{2\pi is}):s$ real}. Then the nilradical N of R is the subgroup V. But the nilradical N_1 of G is $V \rtimes ((1) \times \mathbf{C})$. Hence N is a proper subgroup of N_1. Now, we shall define the lattice Γ of G. Let β be a real number rationally independent to π. Hence the group $\{e^{\beta mi}:m$ integer} is a dense subgroup of the circle group C. Let $\Gamma = \{ (m, n, 1, e^{m\beta i}):m, n$ integers}. Then Γ is a lattice in G. But $N\Gamma$ is dense but not closed in N_1. Also $N/N \cap P$ is not compact. Therefore 8.25 in [5] is not true in general.

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