5

Regularity Results for the Transport Equation

In this chapter we discuss regularity results for the transport equations used in this monograph. We begin with a discussion on smooth first integrals and how they are characterized in terms of the operator of even continuation by the scattering relation. Once this is established we discuss transport equations, including matrix attenuations, and we show a corresponding regularity result (Theorem 5.3.6); this will cover all necessary applications in subsequent chapters. We introduce here the attenuated X-ray transform and we compute its adjoint, although we leave for Chapter 12 a more thorough discussion of its significance.

5.1 Smooth First Integrals

Let (M, g) be a compact non-trapping manifold with strictly convex boundary. Recall that for $w \in C^\infty(\partial_+ SM)$ we set (see Definition 4.2.1)

$$w^\sharp(x,v) = w(\varphi_{-\tau(x,-v)}(x,v)).$$

The function $w^\sharp$ is a first integral of the geodesic flow, i.e. it is constant along its orbits. From the properties of τ we know that $w^\sharp$ is smooth on $SM \setminus \partial_0 SM$, but it may not be smooth at the glancing region $\partial_0 SM$. In this section we will characterize when smoothness holds. We can easily guess a necessary condition. Indeed, since $w^\sharp(x,v) = w \circ \alpha(x,v)$ for $(x,v) \in \partial_- SM$ where α is the scattering relation in Definition 3.3.4, we see that if $w^\sharp \in C^\infty(SM)$, then the function

$$w^\sharp|_{\partial SM} = \begin{cases} w(x,v), & (x,v) \in \partial_+ SM, \\ w \circ \alpha(x,v), & (x,v) \in \partial_- SM \end{cases}$$

must be smooth in ∂SM. We shall show that this condition is also sufficient.

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Following Pestov and Uhlmann (2005) we introduce the operator of even continuation with respect to α: for $w \in C^\infty(\partial_+ SM)$ define

$$A_+ w(x,v) := \begin{cases} w(x,v), & (x,v) \in \partial_+ SM, \\ w \circ \alpha(x,v), & (x,v) \in \partial_- SM. \end{cases}$$

Clearly $A_+ \colon C^\infty(\partial_+ SM) \to C(\partial SM)$. We also introduce the space

$$C^\infty_\alpha(\partial_+ SM) := \{w \in C^\infty(\partial_+ SM) \colon A_+ w \in C^\infty(\partial SM)\}.$$

The main result of this section is the following characterization.

Theorem 5.1.1 (Pestov and Uhlmann (2005)) *Let (M,g) be a compact non-trapping manifold with strictly convex boundary. Then*

$$C^\infty_\alpha(\partial_+ SM) = \{w \in C^\infty(\partial_+ SM) \colon w^\sharp \in C^\infty(SM)\}.$$

Proof We assume (M,g) isometrically embedded in a closed manifold (N,g) of the same dimension as M. Assuming that $A_+ w \in C^\infty(\partial SM)$, we need to show that $w^\sharp \in C^\infty(SM)$. Consider some smooth extension W of $A_+ w = w^\sharp|_{\partial SM}$ into SN. Writing $F(t,x,v) = \frac{1}{2} W(\varphi_t(x,v))$, it follows that

$$\begin{aligned} w^\sharp(x,v) &= \frac{1}{2}\left[W(\varphi_{\tau(x,v)}(x,v)) + W(\varphi_{-\tau(x,-v)}(x,v))\right] \\ &= F(\tau(x,v),x,v) + F(-\tau(x,-v),x,v). \end{aligned}$$

Recall that we already know that $w^\sharp$ is smooth in $SM \setminus \partial_0 SM$, so let us discuss what happens at the glancing region. Fix some $(x_0,v_0) \in \partial_0 SM$ and use Lemma 3.2.9 to write

$$w^\sharp(x,v) = F\big(Q(\sqrt{a(x,v)},x,v),x,v\big) + F\big(Q(-\sqrt{a(x,v)},x,v),x,v\big)$$

near (x_0,v_0) in SM. Setting $G(r,x,v) := F(Q(r,x,v),x,v)$, we have

$$w^\sharp(x,v) = G\big(\sqrt{a(x,v)},x,v\big) + G\big(-\sqrt{a(x,v)},x,v\big)$$

near (x_0,v_0) in SM, where G is smooth near $(0,x_0,v_0)$ in $\mathbb{R} \times SN$. Now

$$G(r,x,v) + G(-r,x,v) = H\big(r^2,x,v\big),$$

where H is smooth near $(0,x_0,v_0)$ (cf. Exercise 3.2.12). This finally shows that

$$w^\sharp(x,v) = H(a(x,v),x,v)$$

near (x_0,v_0) in SM, proving that $w^\sharp$ is smooth near (x_0,v_0) in SM. Since $(x_0,v_0) \in \partial_0 SM$ was arbitrary, we have $w^\sharp \in C^\infty(SM)$. □

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We make right away an application of this result to the function u^f in Definition 4.2.1 solving $Xu^f = -f$ and $u^f|_{\partial_- SM} = 0$. If u is a function on SM we denote the even and odd parts with respect to v by

$$u_+(x,v) = \frac{1}{2}(u(x,v) + u(x,-v)), \quad u_-(x,v) = \frac{1}{2}(u(x,v) - u(x,-v)).$$

Theorem 5.1.2 *Let (M,g) be a non-trapping manifold with strictly convex boundary and let $f \in C^\infty(SM)$. If f is even then u_-^f is smooth in SM. Similarly, if f is odd then u_+^f is smooth in SM.*

Proof Assume f is even (the proof for f is odd is almost identical). Since X maps odd/even functions to even/odd functions, we have $Xu_-^f = -f$.

By Proposition 3.3.1 there is $h \in C^\infty(SM)$ such that $Xh = -f$. Thus $w := h - u_-^f$ is a first integral, i.e. $Xw = 0$. We claim that w is smooth and hence so is u_-^f (if f is odd then $h - u_+^f$ would be smooth).

Let a denote the flip $\mathrm{a}(x,v) = (x,-v)$. Since $\mathrm{a} \circ \varphi_t = \varphi_{-t} \circ \mathrm{a}$ and f is even, we have

$$\begin{aligned} u^f(x,-v) &= \int_0^{\tau(x,-v)} f(\varphi_t(\mathrm{a}(x,v)))\,dt = \int_0^{\tau(x,-v)} f(\varphi_{-t}(x,v))\,dt \\ &= -\int_0^{-\tau(x,-v)} f(\varphi_t(x,v))\,dt. \end{aligned}$$

Hence

$$w = h - \frac{1}{2}\int_0^{\tau(x,v)} f(\varphi_t(x,v))\,dt - \frac{1}{2}\int_0^{-\tau(x,-v)} f(\varphi_t(x,v))\,dt,$$

and therefore for $(x,v) \in \partial SM$ we have

$$w(x,v) = h(x,v) - \frac{1}{2}\int_0^{\tilde{\tau}(x,v)} f(\varphi_t(x,v))\,dt.$$

By Lemma 3.2.6, $\tilde{\tau} \in C^\infty(\partial SM)$ and as a consequence $w|_{\partial SM}$ is smooth. By Theorem 5.1.1, $w \in C^\infty(SM)$ and the result follows. □

5.2 Folds and the Scattering Relation

The original proof of Theorem 5.1.1 was based on a result in Hörmander (1983–1985, Theorem C.4.4), which is in turn underpinned by a result similar to Lemma 3.2.10. In this section we explain the original approach in Pestov and Uhlmann (2005) as it is geometrically quite illuminating.

We start with a general definition from differential topology; for what follows we refer to Hörmander (1983–1985, Appendix C) for details.

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Definition 5.2.1 Let $f : M \to N$ be a smooth map between manifolds of the same dimension n. We say that f has a *Whitney fold* at $m \in M$ if $df_m : T_m M \to T_{f(m)} N$ has rank $n - 1$ and given smooth n-forms ω_M and ω_N that are non-vanishing at m and $f(m)$, respectively, we have

$$f^* \omega_N = \lambda\, \omega_M,$$

where $\lambda \in C^\infty(M)$ is such that $\lambda(m) = 0$ and $d\lambda|_{\ker df|_m} \neq 0$.

Remark 5.2.2 This definition is a little different from the one given in Hörmander (1983–1985, Appendix C), but it is easily seen to be equivalent (and a bit easier to use for computations). Note that the function λ is well defined up to a non-vanishing C^∞-multiple, so the conditions imposed on λ are indeed independent of the choices of n-forms. To gain more insight, note that if $df|_m$ has rank $n - 1$, we can choose local coordinates in N such that the map f can be represented as $f = (f_1, \ldots, f_n)$ with $df_n = 0$ at m. Then $df_1, \ldots, df_{n-1}$ are linearly independent at m, so we can choose local coordinates in M with $y_j = f_j$, $j < n$. It follows that we can represent f as

$$f(y) = (y_1, \ldots, y_{n-1}, f_n(y)).$$

Using this representation and the canonical volume form in Euclidean space we see that $\lambda(y) = \partial f_n(y)/\partial y_n$, so to have a fold at m we need $\partial^2 f_n(0)/\partial y_n^2 \neq 0$.

If f has a fold at $m \in M$, there exists an involution $\sigma : M \to M$ (locally defined) such that $\sigma^2 = \mathrm{Id}$, $\sigma \neq \mathrm{Id}$, $f \circ \sigma = f$ and the set of fixed points L of σ coincides with the set of points near m where df has rank $n - 1$. In fact, f has a very simple normal form near m, that is, in suitable coordinates f has a local expression at zero:

$$f(y_1, \ldots, y_n) = \left(y_1, \ldots, y_{n-1}, y_n^2\right).$$

Moreover, the involution is just given by $\sigma(y', y_n) = (y', -y_n)$ where $y' = (y_1, \ldots, y_{n-1})$, and L is determined by $y_n = 0$. Using this normal form it is not hard to show that the following result holds:

Theorem 5.2.3 *(Hörmander, 1983–1985, Theorem C.4.4) Suppose f has a fold at m and let u be C^∞ in a neighbourhood of $m \in M$. Then, there exists $v \in C^\infty$ in a neighbourhood of $f(m) \in N$ with $v \circ f = u$ if and only if $u \circ \sigma = u$.*

One implication in the theorem is straightforward: if v exists with $v \circ f = u$, then $u \circ \sigma = v \circ f \circ \sigma = v \circ f = u$, so the content of the theorem is the converse statement.

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Let us return now to the situation we are interested in, namely, let (M,g) be a compact non-trapping manifold with strictly convex boundary. Consider a slightly larger manifold M_0 engulfing M so that (M_0,g) is still non-trapping with strictly convex boundary and let τ_0 be the exit time of M_0. The existence of such (M_0,g) follows right away from Proposition 3.3.1 since $Xf > 0$ is an open condition (strict convexity of the boundary is also open under small perturbations).

We define a map $\phi\colon \partial SM \to \partial_- SM_0$ by

$$\phi(x,v) := \varphi_{\tau_0(x,v)}(x,v).$$

This map is C^∞ since $\tau_0|_{SM}$ is C^∞. Here is the main claim about ϕ:

Proposition 5.2.4 *The map ϕ has a Whitney fold at every point of the glancing region $\partial_0 SM$. Moreover, the relevant involution is the scattering relation α.*

Proof Let us first check that $\phi \circ \alpha = \phi$. Indeed

$$\phi(\alpha(x,v)) = \varphi_{\tau_0(\varphi_{\tilde\tau(x,v)}(x,v))}(\varphi_{\tilde\tau(x,v)}(x,v)) = \varphi_{\tau_0(\varphi_{\tilde\tau(x,v)}(x,v))+\tilde\tau(x,v)}(x,v)$$

and since $\tau_0(\varphi_{\tilde\tau(x,v)}(x,v)) = \tau_0(x,v) - \tilde\tau(x,v)$ the claim follows.

To prove that ϕ has a Whitney fold at $\partial_0 SM$, we first show that given $(x,v) \in \partial_0 SM$, we have

$$\ker d\phi_{(x,v)} \oplus T_{(x,v)}\partial_0 SM = T_{(x,v)}\partial SM. \tag{5.1}$$

To this end, we consider $\xi \in T_{(x,v)}\partial SM$ and we compute using the chain rule

$$d\phi_{(x,v)}(\xi) = d\tau_0(\xi)X(\phi(x,v)) + d\varphi_{\tau_0(x,v)}(\xi), \tag{5.2}$$

and from this it follows that $\mathbb{R}X(x,v) = \ker d\phi_{(x,v)}$ since $d\tau_0(X(x,v)) = -1$ and $d\varphi_{\tau_0(x,v)}(X(x,v)) = X(\phi(x,v))$. Note that if $d\phi_{(x,v)}(\xi) = 0$, then $\xi \in \mathbb{R}X(x,v)$ since $d\varphi_{\tau_0(x,v)}$ is a linear isomorphism. Since we are assuming that ∂M is strictly convex, (5.1) follows directly from Lemma 3.6.2.

To complete the proof we need to show the non-degeneracy condition in Definition 5.2.1. As a top dimensional form on $\partial_- SM_0$ we take $j_0^*(i_X d\Sigma^{2n-1})$, where j_0 denotes inclusion of ∂SM_0. Using Lemma 3.6.5 we see that this form does not vanish at $\phi(x,v)$. Using (5.2) we compute its pull-back under ϕ to be

$$\phi^*\big(j_0^*\big(i_X d\Sigma^{2n-1}\big)\big) = j^*\big(i_X d\Sigma^{2n-1}\big),$$

since the geodesic flow preserves $d\Sigma^{2n-1}$. This is checked exactly as in the proof of Proposition 3.6.8.

Using Lemma 3.6.5 again we deduce that we can use $\lambda = \mu$, so to complete the proof we need to show that $d\mu_{(x,v)}(X(x,v)) \neq 0$ for $(x,v) \in \partial_0 SM$. But if ρ is a boundary defining function as in Lemma 3.1.10, we have seen that $\mu(x,v) = \langle \nabla\rho(x), v\rangle$ for $(x,v) \in \partial SM$ and $d\mu_{(x,v)}(X(x,v)) = \mathrm{Hess}_x(\rho)(v,v) = -\Pi_x(v,v) < 0$ for $(x,v) \in \partial_0 SM$. □

We now explain how to use Theorem 5.2.3 to give a proof of Theorem 5.1.1. Consider a function $w \in C^\infty(\partial_+ SM)$ such that $A_+ w \in C^\infty(\partial SM)$. Clearly $A_+ w$ is invariant under α and thus by Theorem 5.2.3, there is a smooth function v defined in a neighbourhood of $\phi(\partial SM)$ such that $v \circ \phi = w$.

Consider the map $\Psi\colon SM \to \partial_- SM$ given by $\Psi(x,v) = \varphi_{\tau(x,v)}(x,v)$ and the analogous one $\Psi_0\colon M_0 \to \partial_- SM_0$ using τ_0. Note that $w^\sharp = w \circ \alpha \circ \Psi$ and that $\phi \circ \alpha \circ \Psi = \Psi_0|_{SM}$. Hence

$$w^\sharp = w \circ \alpha \circ \Psi = v \circ \phi \circ \alpha \circ \Psi = v \circ \Psi_0|_{SM},$$

and since v and $\Psi_0|_{SM}$ are C^∞ it follows that $w^\sharp$ is C^∞ as desired.

5.3 A General Regularity Result

Let (M,g) be a non-trapping manifold with strictly convex boundary and let $\mathcal{A}\colon SM \to \mathbb{C}^{m\times m}$ be a matrix-valued smooth function. We sometimes refer to $\mathcal{A}$ as a *matrix attenuation*.

We would like to study regularity results for solutions $u\colon SM \to \mathbb{C}^m$ to equations of the form

$$Xu + \mathcal{A}u = f,$$

where $f \in C^\infty(SM, \mathbb{C}^m)$ and $u|_{\partial SM} = 0$. We shall show that under these conditions u must be C^∞.

As we have done before, consider (M,g) isometrically embedded in a closed manifold (N,g) and extend $\mathcal{A}$ smoothly to N. Under these assumptions $\mathcal{A}$ on N defines a *smooth* cocycle over the geodesic flow φ_t of (N,g). The cocycle takes values in the group $GL(m,\mathbb{C})$ and is defined as follows: let $C\colon SN \times \mathbb{R} \to GL(m,\mathbb{C})$ be determined by the following matrix ODE along the orbits of the geodesic flow

$$\frac{d}{dt}C(x,v,t) + \mathcal{A}(\varphi_t(x,v))C(x,v,t) = 0, \qquad C(x,v,0) = \mathrm{Id}.$$

The function C is a *cocycle*:

$$C(x,v,t+s) = C(\varphi_t(x,v),s)\,C(x,v,t)$$

for all $(x,v) \in SN$ and $s,t \in \mathbb{R}$.

Exercise 5.3.1 Prove the cocycle property by using uniqueness for ODEs and the fact that φ_t is a flow.

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Having this cocycle is just as convenient as having φ_t defined for $t \in \mathbb{R}$ in SN. We shall see that using this we can reduce smoothness questions to τ; a recurrent theme.

Consider as before (M_0, g) non-trapping with strictly convex boundary and containing (M, g) in its interior. Let τ_0 be the exit time of M_0.

Lemma 5.3.2 *The function $R\colon SM \to GL(m,\mathbb{C})$, defined by*

$$R(x,v) := [C(x,v,\tau_0(x,v))]^{-1},$$

is smooth and satisfies

$$XR + \mathcal{A}R = 0,$$
$$X(R^{-1}) - R^{-1}\mathcal{A} = 0.$$

Proof Since $\tau_0|_{SM}$ is smooth and the cocycle C is smooth, the smoothness of R follows right away. To check that R satisfies the stated equation, we use that $\tau_0(\varphi_t(x,v)) = \tau_0(x,v) - t$ together with the cocycle property to obtain

$$\begin{aligned} R(\varphi_t(x,v)) &= [C(\varphi_t(x,v), \tau_0(\varphi_t(x,v))]^{-1} \\ &= C(x,v,t)[C(x,v,\tau_0(x,v))]^{-1}. \end{aligned}$$

Differentiating at $t = 0$ yields

$$XR = -\mathcal{A}R.$$

It also follows that $X(R^{-1}) = -R^{-1}(XR)R^{-1} = R^{-1}\mathcal{A}$. □

In subsequent chapters, we will discuss the attenuated X-ray transform in detail, but for now we give the most basic definitions as they are useful for phrasing the main regularity result for the transport equation with general matrix attenuation. In the scalar case, the *attenuated X-ray transform $I_a f$* of a function $f \in C^\infty(SM,\mathbb{C})$ with attenuation coefficient $a \in C^\infty(SM,\mathbb{C})$ can be defined as the integral

$$I_a f(x,v) := \int_0^{\tau(x,v)} f(\varphi_t(x,v)) \exp\left[\int_0^t a(\varphi_s(x,v))\, ds\right] dt$$

for $(x,v) \in \partial_+SM$. Alternatively, we may set $I_a f := u|_{\partial_+SM}$ where u is the unique solution of the transport equation

$$Xu + au = -f \text{ in } SM, \quad u|_{\partial_-SM} = 0.$$

The last definition generalizes without difficulty to the case of a general matrix attenuation $\mathcal{A}$. Let $f \in C^\infty(SM,\mathbb{C}^m)$ be a vector-valued function and consider the following transport equation for a function $u\colon SM \to \mathbb{C}^m$,

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$$Xu + \mathcal{A}u = -f \text{ in } SM, \quad u|_{\partial_- SM} = 0.$$

On a fixed geodesic the transport equation becomes a linear ODE with zero final condition, and therefore this equation has a unique solution that will be denoted by $u = u^f_{\mathcal{A}}$ in this chapter.

Definition 5.3.3 The attenuated X-ray transform of $f \in C^\infty(SM, \mathbb{C}^m)$ is given by

$$I_{\mathcal{A}} f := u^f_{\mathcal{A}}|_{\partial_+ SM}.$$

It is a simple task to write an integral formula for $u^f_{\mathcal{A}}$ using a matrix integrating factor as in Lemma 5.3.2.

Lemma 5.3.4 *With R as in Lemma 5.3.2 we have*

$$u^f_{\mathcal{A}}(x,v) = R(x,v) \int_0^{\tau(x,v)} (R^{-1} f)(\varphi_t(x,v))\, dt \;\; \textit{for } (x,v) \in SM.$$

Proof Let $u = u^f_{\mathcal{A}}$. A computation using $XR^{-1} = R^{-1}\mathcal{A}$ (which follows easily from $XR + \mathcal{A}R = 0$) and $Xu + \mathcal{A}u = -f$ yields

$$X(R^{-1}u) = (XR^{-1})u + R^{-1}Xu = -R^{-1}f.$$

Since $R^{-1}u|_{\partial_- SM} = 0$, the lemma follows. □

Remark 5.3.5 It is useful for future purposes to understand how the formula in the lemma changes if we consider a different integrating factor, i.e. another invertible matrix R_1 satisfying $XR_1 + \mathcal{A}R_1 = 0$. Since

$$X\big(R^{-1}R_1\big) = X\big(R^{-1}\big)R_1 + R^{-1}X(R_1) = R^{-1}\mathcal{A}R_1 - R^{-1}\mathcal{A}R_1 = 0,$$

we derive

$$R_1 = RW^\sharp,$$

where $W = R^{-1}R_1|_{\partial_+ SM}$.

Lemma 5.3.4 shows that $u^f_{\mathcal{A}}$ is, in general, as smooth as τ, i.e. smooth everywhere except perhaps at the glancing region $\partial_0 SM$. However, the next result will show that if $I_{\mathcal{A}} f = 0$, then $u^f_{\mathcal{A}}$ is C^∞.

Theorem 5.3.6 (Paternain et al. (2012)) *Let (M,g) be a non-trapping manifold with strictly convex boundary. Let $\mathcal{A} \in C^\infty(SM, \mathbb{C}^{m\times m})$ and $f \in C^\infty(SM, \mathbb{C}^m)$ be such that $I_{\mathcal{A}} f = 0$. Then $u^f_{\mathcal{A}} \in C^\infty(SM, \mathbb{C}^m)$.*

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Proof It is enough to show that the function $r := R^{-1}u^f_{\mathcal{A}}$ is smooth. According to Lemma 5.3.4, r satisfies

$$Xr = -R^{-1}f \ \text{ in } SM, \quad r|_{\partial SM} = 0.$$

Choose $h \in C^\infty(SM, \mathbb{C}^m)$ such that $Xh = -R^{-1}f$. We know such a function exists either by appealing to Proposition 3.3.1 or by using the enlargement M_0 of M, extending $R^{-1}f$ smoothly to N and setting

$$h(x,v) = \int_0^{\tau_0(x,v)} (R^{-1}f)(\varphi_t(x,v))\,dt \ \text{ for } (x,v) \in SM.$$

Recall that $\tau_0|_{SM}$ is smooth. Thus the function $h - r$ satisfies $X(h - r) = 0$ and since $(h - r)|_{\partial SM} = h|_{\partial SM} \in C^\infty(\partial SM, \mathbb{C}^m)$, Theorem 5.1.1 gives that $h - r$ is smooth in SM and thus r is smooth as desired. □

We conclude this section with a brief discussion as to what happens if we swap the choice of boundary conditions in the transport equation. Suppose that we consider the equation

$$Xu + \mathcal{A}u = f \ \text{ in } SM, \quad u|_{\partial_+ SM} = 0.$$

Note the change of sign in the right-hand side of the transport equation and the fact that we now demand u to vanish on the influx boundary. Let us call w^f the unique solution.

Lemma 5.3.7 *We have the following identity on $\partial_+ SM$:*

$$w^f \circ \alpha = R^{-1}\,u^f,$$

where R is the unique integrating factor for $\mathcal{A}$ with $R|_{\partial_- SM} = \mathrm{Id}$.

Exercise 5.3.8 Prove the lemma.

5.4 The Adjoint $I^*_{\mathcal{A}}$

Let (M, g) be a non-trapping manifold with strictly convex boundary and let $\mathcal{A}\colon SM \to \mathbb{C}^{m\times m}$ be a smooth matrix attenuation. In this section we shall compute the adjoint $I^*_{\mathcal{A}}$ of

$$I_{\mathcal{A}}\colon L^2(SM, \mathbb{C}^m) \to L^2_\mu(\partial_+ SM, \mathbb{C}^m).$$

We endow $\mathbb{C}^m$ with its standard Hermitian inner product, so the L^2 spaces are defined using this inner product and the usual volume forms $d\Sigma^{2n-1}$ and $d\mu = \mu\, d\Sigma^{2n-2}$.

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Using the same arguments as in Proposition 4.1.2 one shows:

Proposition 5.4.1 *The operator $I_{\mathcal{A}}$ extends to a bounded operator*

$$I_{\mathcal{A}}\colon L^2(SM,\mathbb{C}^m) \to L^2_\mu(\partial_+SM,\mathbb{C}^m).$$

Moreover, the following stronger result holds: $I_{\mathcal{A}}$ extends to a bounded operator

$$I_{\mathcal{A}}\colon L^2(SM,\mathbb{C}^m) \to L^2(\partial_+SM,\mathbb{C}^m).$$

Exercise 5.4.2 Prove the proposition.

Lemma 5.4.3 *If $R\colon SM \to GL(m,\mathbb{C})$ is such that $XR + \mathcal{A}R = 0$, then*

$$I_{\mathcal{A}}^* h = (R^*)^{-1}(R^*h)^\sharp.$$

Proof Recall that given R we can write

$$I_{\mathcal{A}} f = u_{\mathcal{A}}^f|_{\partial_+SM} = R(x,v)\int_0^{\tau(x,v)} (R^{-1}f)(\varphi_t(x,v))\,dt$$

for $(x,v) \in \partial_+SM$. Let us compute using Santaló's formula:

$$\begin{aligned}
(I_{\mathcal{A}} f, h) &= \int_{\partial_+SM} \langle I_{\mathcal{A}} f, h\rangle_{\mathbb{C}^m}\,d\mu\\
&= \int_{\partial_+SM} d\mu \left\langle \int_0^\tau (R^{-1}f)(\varphi_t(x,v))\,dt, R^*h\right\rangle_{\mathbb{C}^m}\\
&= \int_{\partial_+SM} d\mu \int_0^\tau \left\langle R^{-1}f, (R^*h)^\sharp\right\rangle_{\mathbb{C}^m} (\varphi_t(x,v))\,dt\\
&= \int_{SM} \left\langle R^{-1}f, (R^*h)^\sharp\right\rangle_{\mathbb{C}^m} d\Sigma^{2n-1}\\
&= (f, (R^*)^{-1}(R^*h)^\sharp),
\end{aligned}$$

and thus $I_{\mathcal{A}}^* h = (R^*)^{-1}(R^*h)^\sharp$ as desired. □

Remark 5.4.4 Observe that $U = (R^*)^{-1}$ solves the matrix transport equation $XU - \mathcal{A}^*U = 0$ and since $(R^*h)^\sharp$ is a first integral of the geodesic flow, $f = I_{\mathcal{A}}^* h$ solves

$$\begin{cases} Xf - \mathcal{A}^* f = 0,\\ f|_{\partial_+SM} = h. \end{cases}$$

We conclude this chapter by discussing the closely related X-ray transform with a matrix weight.

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Definition 5.4.5 Let (M,g) be a compact non-trapping manifold with strictly convex boundary. Given a smooth matrix weight $\mathbb{W}\colon SM \to GL(m,\mathbb{C})$, the matrix weighted X-ray transform is the map

$$I_{\mathbb{W}}\colon C^\infty(SM,\mathbb{C}^m) \to C^\infty(\partial_+ SM,\mathbb{C}^m),$$
$$I_{\mathbb{W}} f(x,v) = \int_0^{\tau(x,v)} (\mathbb{W} f)(\varphi_t(x,v))\, dt,$$

where $(x,v) \in \partial_+ SM$.

Note that one always has

$$I_{\mathbb{W}} f = u^{\mathbb{W} f}|_{\partial_+ SM},$$

where $u = u^{\mathbb{W} f}$ is the unique solution of

$$Xu = -\mathbb{W} f \text{ in } SM, \qquad u|_{\partial_- SM} = 0.$$

The following result shows that one can always reduce a matrix weighted transform $I_{\mathbb{W}}$ for $\mathbb{W} \in C^\infty(SM,GL(m,\mathbb{C}))$ into an attenuated X-ray transform $I_{\mathcal{A}}$ for a general attenuation $\mathcal{A} \in C^\infty(SM,\mathbb{C}^{m\times m})$, and vice versa. We note that there is a slight abuse of notation, but we hope that it will be clear from the context whether the transform involves a weight or an attenuation.

Lemma 5.4.6 *Let (M,g) be a compact non-trapping manifold with strictly convex boundary, and let $f \in C^\infty(SM,\mathbb{C}^m)$.*

(a) Given any $\mathbb{W} \in C^\infty(SM,GL(m,\mathbb{C}))$, one has

$$I_{\mathbb{W}} f = \mathbb{W} I_{\mathcal{A}} f|_{\partial_+ SM},$$

where $\mathcal{A} := \mathbb{W}^{-1}(X\mathbb{W}) \in C^\infty(SM,\mathbb{C}^{m\times m})$.

(b) Given any $\mathcal{A} \in C^\infty(SM,\mathbb{C}^{m\times m})$, one has

$$I_{\mathcal{A}} f = \mathbb{W}^{-1} I_{\mathbb{W}} f|_{\partial_+ SM},$$

where $\mathbb{W}$ is any solution in $C^\infty(SM,GL(m,\mathbb{C}))$ of $X\mathbb{W} - \mathbb{W}\mathcal{A} = 0$ in SM (e.g. $\mathbb{W}$ could be obtained from Lemma 5.3.2).

Proof (a) If $\mathcal{A}$ has the given form, then

$$(X+\mathcal{A})\big(\mathbb{W}^{-1}u^{\mathbb{W} f}\big) = \big(X\big(\mathbb{W}^{-1}\big) + \mathcal{A}\mathbb{W}^{-1}\big)u^{\mathbb{W} f} + \mathbb{W}^{-1}Xu^{\mathbb{W} f} = -f.$$

Since $u^{\mathbb{W} f}|_{\partial_- SM} = 0$, one has $u^f_{\mathcal{A}} = \mathbb{W}^{-1}u^{\mathbb{W} f}$ and thus $I_{\mathbb{W}} f = \mathbb{W} I^f_{\mathcal{A}}|_{\partial_+ SM}$.

(b) If $\mathbb{W}$ is as stated, then

$$X\big(\mathbb{W} u^f_{\mathcal{A}}\big) = (X\mathbb{W})u^f_{\mathcal{A}} + \mathbb{W}\big(-\mathcal{A}u^f_{\mathcal{A}} - f\big) = -\mathbb{W} f.$$

Thus $\mathbb{W} u^f_{\mathcal{A}} = u^{\mathbb{W} f}$ and $\mathbb{W} I_{\mathcal{A}} f|_{\partial_+ SM} = I_{\mathbb{W}} f$. □

Remark 5.4.7 Using the argument in Proposition 4.1.2, one can show that $I_{\mathbb{W}}$ is bounded $L^2(SM, \mathbb{C}^m) \to L^2(\partial_+ SM, \mathbb{C}^m)$ and thus it is also bounded $L^2(SM, \mathbb{C}^m) \to L^2_\mu(\partial_+ SM, \mathbb{C}^m)$. The adjoint

$$I_{\mathbb{W}}^* : L^2_\mu(\partial_+ SM, \mathbb{C}^m) \to L^2(SM, \mathbb{C}^m)$$

is easily computed as above and it is given by

$$I_{\mathbb{W}}^* h = \mathbb{W}^* h^\sharp.$$