Bull. Aust. Math. Soc. **83** (2011), 353–368
doi:10.1017/S0004972711002103

DIFFERENTIAL SUBORDINATIONS FOR CLASSES OF MEROMORPHIC p-VALENT FUNCTIONS DEFINED BY MULTIPLIER TRANSFORMATIONS

R. M. EL-ASHWAH, M. K. AOUF and T. BULBOACĂ ✉

(Received 28 September 2009)

Abstract

We investigate several inclusion relationships and other interesting properties of certain subclasses of p-valent meromorphic functions, which are defined by using a certain linear operator, involving the generalized multiplier transformations.

2010 *Mathematics subject classification*: primary 30C45; secondary 30C80.

Keywords and phrases: multiplier transformations, meromorphic functions, differential subordination.

1. Introduction

For $n > -p$, let $\sum_{p,n}$ denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^k, \quad p \in \mathbb{N} = \{1, 2, 3, \ldots\},$$

which are analytic and p-valent in the punctured unit disc $\dot{U} = U \setminus \{0\}$, where $U = \{z \in \mathbb{C} : |z| < 1\}$. For convenience, we write $\sum_p \equiv \sum_{p,-p+1}$.

If f and g are two analytic functions in U, we say that f is subordinate to g, written symbolically as $f(z) \prec g(z)$, if there exists a *Schwarz function* w, which (by definition) is analytic in U with $w(0) = 0$, and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$.

It is well known that, if $f(z) \prec g(z)$, then $f(0) = g(0)$ and $f(U) \subset g(U)$. Further, if the function g is univalent in U, then we have the following equivalence (see [9]; see also [10, p. 4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \prec g(U).$$

For the functions $f_j \in \sum_{p,n}$, $j = 1, 2$, given by

$$f_j(z) = z^{-p} + \sum_{k=n}^{\infty} a_{k,j} z^k,$$

we define the *Hadamard (or convolution) product* of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{k=n}^{\infty} a_{k,1} a_{k,2} z^n.$$

Define the linear operator $I_p^m(n; \lambda, l) : \sum_{p,n} \to \sum_{p,n}$, where $\lambda \geq 0$, $l > 0$, and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, by

$$I_p^m(n; \lambda, l) f(z) = z^{-p} + \sum_{k=n}^{\infty} \left[\frac{\lambda(k+p)+l}{l}\right]^m a_k z^k. \tag{1.1}$$

Then, we can write (1.1) as

$$I_p^m(n; \lambda, l) f(z) = (\Phi_{n;\lambda,l}^{p,m} * f)(z),$$

where

$$\Phi_{n;\lambda,l}^{p,m}(z) = z^{-p} + \sum_{k=n}^{\infty} \left[\frac{\lambda(k+p)+l}{l}\right]^m z^k.$$

Using definition (1.1), it is easy to verify that the next formula holds for $\lambda > 0$:

$$\lambda z (I_p^m(n; \lambda, l) f(z))' = l I_p^{m+1}(n; \lambda, l) f(z) - (\lambda p + l) I_p^m(n; \lambda, l) f(z). \tag{1.2}$$

REMARK 1.1. (1) We note that $I_p^0(n; \lambda, l) f = f$ and

$$I_p^1(n; 1, 1) f(z) = \frac{(z^{p+1} f(z))'}{z^p} = (p+1) f(z) + z f'(z).$$

(2) For some special values of the parameters λ, l, m and p, we obtain the following operators studied by various authors:

(i) $I_p^m(n; 1, l) = I_p^m(n, l)$ (see Cho *et al.* [2]);
(ii) $I_p^m(n; 1, 1) = D_{n,p}^m$ (see Aouf and Hossen [1], and Liu and Srivastava [6]);
(iii) $I_1^m(0; 1, l) = D_l^m$ (see Cho *et al.* [3, 4]);
(iv) $I_1^m(0; 1, 1) = I^m$ (see Uralegaddi and Somanatha [18]).

Using differential subordinations as well as the linear operator $I_p^m(n; \lambda, l)$, we will introduce a subclass of $\sum_{p,n}$, as follows.

DEFINITION 1.2. (1) For the fixed parameters A and B, with $-1 \leq B < A \leq 1$, we say that a function $f \in \sum_{p,n}$ is in the class $\sum_{p,n}^m(\lambda, l; A, B)$, if it satisfies the subordination condition

$$-\frac{z^{p+1}(I_p^m(n; \lambda, l) f(z))'}{p} \prec \frac{1+Az}{1+Bz}, \quad l, \lambda > 0, m \in \mathbb{N}_0, n > -p. \tag{1.3}$$

(2) For convenience, we write

$$\sum_{p,n}^m(\lambda, l; \alpha) \equiv \sum_{p,n}^m\left(\lambda, l; 1 - \frac{2\alpha}{p}, -1\right), \quad 0 \leq \alpha < p,$$

https://doi.org/10.1017/S0004972711002103 Published online by Cambridge University Press

that is, $\sum_{p,n}^{m}(\lambda, l; \alpha)$ denotes the class of functions $f \in \sum_{p,n}$ satisfying

$$\operatorname{Re}\{-z^{p+1}(I_p^m(n; \lambda, l) f(z))'\} > \alpha, \quad z \in U.$$

REMARK 1.3. We have the next special cases of $\sum_{p,n}^{m}(\lambda, l; A, B)$, studied previously by different authors:

(i) $\sum_{p,0}^{m}(1, 1; A, B) = R_{m,p}(A, B)$ (see Liu and Srivastava [6]);
(ii) $\sum_{p,n}^{m}(1, 1; A, B) = \sum_{p,n}^{m}(A, B)$ (see Srivastava and Patel [16]);
(iii) $\sum_{p,0}^{0}(1, 1; A, B) = H(p; A, B)$ (see Mogra [11, 12]);
(iv) $\sum_{p,n}^{m}(1, l; A, B) = \sum_{p,n}^{m,l}(A, B)$, where $\sum_{p,n}^{m,l}(A, B)$ is the class of functions $f \in \sum_{p,n}$, satisfying

$$-\frac{z^{p+1}(I_p^m(n, l) f(z))'}{p} \prec \frac{1 + Az}{1 + Bz}, \quad l > 0, m \in \mathbb{N}_0, n > -p,$$

and $I_p^m(n, l) \equiv I_p^m(n; 1, l)$.

In the present paper we obtain several inclusion relationships for the function class $\sum_{p,n}^{m}(\lambda, l; A, B)$, and we investigate various other properties of functions belonging to the class $\sum_{p,n}^{m}(\lambda, l; A, B)$. Relevant connections of the results presented in this paper with those obtained in earlier works are also pointed out.

2. Preliminaries

To establish our main results, we will need the following lemmas and definition.

LEMMA 2.1 [5]. *Let the function h be convex (univalent) in U, with $h(0) = 1$. Suppose also that the function φ given by*

$$\varphi(z) = 1 + c_{p+n}z^{p+n} + c_{p+n+1}z^{p+n+1} + \cdots \tag{2.1}$$

is analytic in U. Then

$$\varphi(z) + \frac{z\varphi'(z)}{\delta} \prec h(z), \quad \operatorname{Re}\delta \geq 0, \delta \neq 0,$$

implies that

$$\varphi(z) \prec \psi(z) = \frac{\delta}{p+n} z^{-\delta/(p+n)} \int_0^z t^{\delta/(p+n)-1} h(t)\, dt \prec h(z), \tag{2.2}$$

and ψ is the best dominant of (2.2).

DEFINITION 2.2. We denote by $\mathcal{P}(\gamma)$ the class of functions φ given by

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + \cdots, \tag{2.3}$$

which are analytic in U and satisfy the inequality

$$\operatorname{Re}\varphi(z) > \gamma, \quad z \in U \ (0 \leq \gamma < 1).$$

LEMMA 2.3 [14]. *Let the function φ given by (2.3) be in the class $\mathcal{P}(\gamma)$. Then*

$$\operatorname{Re}\varphi(z) \geq 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|}, \quad z \in U \ (0 \leq \gamma < 1).$$

LEMMA 2.4 [17]. *For $0 \leq \gamma_1 < \gamma_2 < 1$, the inclusion*

$$\mathcal{P}(\gamma_1) * \mathcal{P}(\gamma_2) \subset \mathcal{P}(\gamma_3) \quad \text{where } \gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2),$$

holds and the result is the best possible. The symbol '$*$' stands for the previous mentioned Hadamard product *of the power series.*

LEMMA 2.5 [15]. *Let Φ be an analytic function in U, with $\Phi(0) = 1$ and* $\operatorname{Re}\Phi(z) > 1/2$, *$z \in U$. Then, for any function F analytic in U, the set $(\Phi * F)(U)$ is contained in the convex hull of $F(U)$, that is, $(\Phi * F)(U) \subset \operatorname{co} F(U)$.*

LEMMA 2.6 [19]. *For all real or complex numbers α_1, α_2, β_1, where $\beta_1 \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$,*

$$\int_0^1 t^{\alpha_2-1}(1-t)^{\beta_1-\alpha_2-1}(1-zt)^{-\alpha_1}\,dt = \frac{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_2)}{\Gamma(\beta_1)}\,{}_2F_1(\alpha_1,\alpha_2,\beta_1;z) \quad \text{for } \operatorname{Re}\beta_1 > \operatorname{Re}\alpha_2 > 0, \tag{2.4}$$

$${}_2F_1(\alpha_1,\alpha_2,\beta_1;z) = {}_2F_1(\alpha_2,\alpha_1,\beta_1;z), \tag{2.5}$$

$${}_2F_1(\alpha_1,\alpha_2,\beta_1;z) = (1-z)^{-\alpha_1}\,{}_2F_1\left(\alpha_1,\beta_1-\alpha_2,\beta_1;\frac{z}{z-1}\right), \tag{2.6}$$

and

$${}_2F_1\left(\alpha_1,\alpha_2,\frac{\alpha_1+\alpha_2+1}{2};\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(\frac{\alpha_1+\alpha_2+1}{2})}{\Gamma(\frac{\alpha_1+1}{2})\Gamma(\frac{\alpha_2+1}{2})}, \tag{2.7}$$

where ${}_2F_1$ represents the Gauss hypergeometric function.

3. Subordination theorems and the associated functional inequalities

Unless otherwise mentioned, we shall assume throughout the paper that n is an integer with $n > -p$, that $-1 \leq B < A \leq 1$, λ, $l > 0$, $m \in \mathbb{N}_0$, $\beta > 0$, and $p \in \mathbb{N}$.

THEOREM 3.1. *If the function $f \in \sum_{p,n}$ satisfies the subordination condition*

$$-\frac{(1-\beta)z^{p+1}(I_p^m(n;\lambda,l)f(z))' + \beta z^{p+1}(I_p^{m+1}(n;\lambda,l)f(z))'}{p} \prec \frac{1+Az}{1+Bz},$$

then

$$-\frac{z^{p+1}(I_p^m(n;\lambda,l)f(z))'}{p} \prec Q(z) \prec \frac{1+Az}{1+Bz}, \tag{3.1}$$

https://doi.org/10.1017/S0004972711002103 Published online by Cambridge University Press

where the function Q is given by

$$Q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1}{}_2F_1\left(1, 1, \frac{l}{\lambda\beta(p+n)} + 1; \frac{Bz}{1+Bz}\right), & B \neq 0, \\ 1 + \frac{l}{\lambda\beta(p+n)+l} Az, & B = 0, \end{cases}$$

and it is the best dominant of (3.1).

Furthermore, for all $k \in \mathbb{N}$, we have

$$\mathrm{Re}\left[-\frac{z^{p+1}(I_p^m(n; \lambda, l) f(z))'}{p}\right]^{1/k} > \rho^{1/k}, \quad z \in U, \tag{3.2}$$

where $\rho = Q(-1)$, and the inequality (3.2) is the best possible.

PROOF. If we consider the function φ defined by

$$\varphi(z) = -\frac{z^{p+1}(I_p^m(n; \lambda, l) f(z))'}{p}, \tag{3.3}$$

then φ has the form (2.1) and is analytic in U. Applying the identity (1.2) in (3.3), and differentiating the resulting equation with respect to z, we get

$$-\frac{(1-\beta)z^{p+1}(I_p^m(n; \lambda, l) f(z))' + \beta z^{p+1}(I_p^{m+1}(n; \lambda, l) f(z))'}{p} = \varphi(z) + \frac{\beta\lambda}{l} z\varphi'(z) \prec \frac{1+Az}{1+Bz}.$$

Now by using Lemma 2.1 for $\gamma = l/(\lambda\beta)$, we deduce that

$$\begin{aligned} -\frac{z^{p+1}(I_p^m(n; \lambda, l) f(z))'}{p} &\prec Q(z) \\ &= \frac{l}{\lambda\beta(p+n)} z^{-l/\lambda\beta(p+n)} \int_0^z t^{(l/\lambda\beta(p+n))-1} \frac{1+At}{1+Bt}\, dt \\ &= \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1}{}_2F_1\left(1,1,\frac{l}{\lambda\beta(p+n)} + 1; \frac{Bz}{1+Bz}\right), & B \neq 0, \\ 1 + \frac{l}{\lambda\beta(p+n)+l} Az, & B = 0, \end{cases} \end{aligned}$$

where we made a changes of variables, followed by the use of the identities (2.4), (2.5), and (2.6) (with $b = 1$ and $c = a + 1$). Hence, assertion (3.1) is proved.

In order to prove assertion (3.2), it is sufficient to show that

$$\inf\{\mathrm{Re}\, Q(z) : |z| < 1\} = Q(-1).$$

https://doi.org/10.1017/S0004972711002103 Published online by Cambridge University Press

Indeed, for $|z| \leq r < 1$,

$$\operatorname{Re} \frac{1+Az}{1+Bz} \geq \frac{1-Ar}{1-Br}, \quad |z| \leq r < 1.$$

Setting

$$G(s, z) = \frac{1+Asz}{1+Bsz}$$

and

$$d\nu(s) = \frac{l}{\lambda\beta(p+n)} s^{l/\lambda\beta(p+n)}\, ds, \quad 0 \leq s \leq 1,$$

which is a positive measure on [0, 1], we get

$$Q(z) = \int_0^1 G(s, z)\, d\nu(s),$$

so that

$$\operatorname{Re} Q(z) \geq \int_0^1 \frac{1-Asr}{1-Bsr}\, d\nu(s) = Q(-r), \quad |z| \leq r < 1.$$

Letting $r \to 1^-$ in the above inequality, and using the elementary inequality

$$\operatorname{Re} w^{1/k} \geq (\operatorname{Re} w)^{1/k}, \quad \operatorname{Re} w > 0,\ k \in \mathbb{N},$$

we obtain (3.2). Finally, inequality (3.2) is the best possible, as the function Q is the best dominant of (3.1). □

REMARK 3.2. Putting $\lambda = l = 1$ in Theorem 3.1, we obtain the result of Srivastava and Patel [16, Theorem 1].

For $\lambda = l = 1$, $n = 0$, and $\beta = 1$, Theorem 3.1 yields the following result, which improves the corresponding one of Liu and Srivastava [7, Theorem 1].

COROLLARY 3.3. *The inclusions*

$$R_{m+1,p}(A, B) \subset R_{m,p}(A, B) \subset R_{m,p}(1-2\rho, -1)$$

hold, where

$$\rho = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1, 1, \dfrac{1}{p}+1; \dfrac{B}{B-1}\right), & B \neq 0, \\ 1 - \dfrac{A}{p+1}, & B = 0, \end{cases}$$

and the result is the best possible.

Putting $A = 1 - 2\alpha/p$, $B = -1$, $\beta = \lambda = l = 1$, $m = 0$ and $n = -p+2$ in Theorem 3.1, and using (2.7), we get the following result.

https://doi.org/10.1017/S0004972711002103 Published online by Cambridge University Press

COROLLARY 3.4. *If the function $f \in \sum_{p,-p+2}$ satisfies the inequality*

$$\mathrm{Re}\{-z^{p+1}[(p+2)f'(z)+zf''(z)]\} > \alpha, \quad z \in U\ (0 \le \alpha < p),$$

then

$$\mathrm{Re}[-z^{p+1}f'(z)] > \alpha + (p-\alpha)\left(\frac{\pi}{2}-1\right), \quad z \in U,$$

and the result is the best possible.

REMARK 3.5. Taking $\alpha = -p(\pi-2)/(4-\pi)$ in the above corollary, we obtain that if the function $f \in \sum_{p,-p+2}$ satisfies

$$\mathrm{Re}\{-z^{p+1}[(p+2)f'(z)+zf''(z)]\} > -\frac{p(\pi-2)}{4-\pi}, \quad z \in U,$$

then $\mathrm{Re}[-z^{p+1}f'(z)] > 0$, $z \in U$ (see Pap [13]).

THEOREM 3.6. *If the function $f \in \sum_{p,n}^{m}(\lambda, l; \alpha)$, $0 \le \alpha < p$, then*

$$\mathrm{Re}\{-z^{p+1}[(1-\beta)(I_p^m(n;\lambda,l)f(z))' + \beta(I_p^{m+1}(n;\lambda,l)f(z))']\} > \alpha,$$

for $|z| < R$, where

$$R = \left[\sqrt{1+\left(\frac{\beta\lambda}{l}\right)^2(p+n)^2} - \frac{\beta\lambda}{l}(p+n)\right]^{1/(p+n)}. \tag{3.4}$$

The result is the best possible.

PROOF. If we let

$$-z^{p+1}(I_p^m(n;\lambda,l)f(z))' = \alpha + (p-\alpha)\varphi(z), \tag{3.5}$$

then φ has the form (2.1), and is analytic with positive real part in U. Using the identity (1.2) in (3.5), and differentiating the resulting equation with respect to z,

$$\begin{aligned} &-\frac{z^{p+1}[(1-\beta)(I_p^m(n;\lambda,l)f(z))' + \beta(I_p^{m+1}(n;\lambda,l)f(z))'] + \alpha}{p-\alpha} \\ &\quad = \varphi(z) + \frac{\beta\lambda}{l} z\varphi'(z). \end{aligned} \tag{3.6}$$

Applying in (3.6) the estimate (see [8])

$$\frac{|z\varphi'(z)|}{\mathrm{Re}\,\varphi(z)} \le \frac{2(p+n)r^{p+n}}{1-r^{2(p+n)}}, \quad |z| = r < 1,$$

we get

$$\begin{aligned} &\mathrm{Re}\left\{-\frac{z^{p+1}[(1-\beta)(I_p^m(n;\lambda,l)f(z))' + \beta(I_p^{m+1}(n;\lambda,l)f(z)'] + \alpha}{p-\alpha}\right\} \\ &\quad \ge \left[1 - \frac{2\beta\lambda}{l}\frac{(p+n)r^{p+n}}{1-r^{2(p+n)}}\right]\mathrm{Re}\,\varphi(z), \end{aligned} \tag{3.7}$$

https://doi.org/10.1017/S0004972711002103 Published online by Cambridge University Press

and it is easy to see that the right-hand side of (3.7) is positive, provided that $r < R$, where R is given by (3.4).

In order to show that the bound R is the best possible, we consider the function $f \in \sum_{p,n}$ defined by

$$-z^{p+1}(I_p^m(n;\lambda,l)f(z))' = \alpha + (p-\alpha)\frac{1+z^{p+n}}{1-z^{p+n}}.$$

Then

$$-\frac{z^{p+1}[(1-\beta)(I_p^m(n;\lambda,l)f(z))' + \beta(I_p^{m+1}(n;\lambda,l)f(z))'] + \alpha}{p-\alpha}$$
$$= \frac{1-z^{2(p+n)} + \frac{2\beta\lambda}{l}(p+n)z^{p+n}}{(1-z^{p+n})^2} = 0,$$

for $z = R\exp(i\pi/(p+n))$, which completes the proof of the theorem. □

REMARK 3.7. Putting $\lambda = l = 1$ in Theorem 3.6, we obtain the result of Srivastava and Patel [16, Theorem 2].

For $\beta = 1$, Theorem 3.6 reduces to the following result.

COROLLARY 3.8. *If the function $f \in \sum_{p,n}^m(\lambda, l;\alpha)$, $0 \le \alpha < p$, then $f \in \sum_{p,n}^{m+1}(\lambda, l;\alpha)$ for $|z| < \widetilde{R}$, where*

$$\widetilde{R} = \left[\sqrt{1+\left(\frac{\lambda}{l}\right)^2(p+n)^2} - \frac{\lambda}{l}(p+n)\right]^{1/(p+n)}.$$

The result is the best possible.

THEOREM 3.9. *Let $f \in \sum_{p,n}^m(\lambda, l; A, B)$, and let*

$$F_{p,c}(f)(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)\,dt, \quad c > 0. \tag{3.8}$$

Then

$$-\frac{z^{p+1}(I_p^m(n;\lambda,l)F_{p,c}(f)(z))'}{p} \prec \Theta(z) \prec \frac{1+Az}{1+Bz}, \tag{3.9}$$

where Θ is defined by

$$\Theta(z) = \begin{cases} \dfrac{A}{B} + \left(1-\dfrac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1,\dfrac{c}{p+n}+1;\dfrac{Bz}{1+Bz}\right), & B \ne 0, \\ 1 + \dfrac{Ac}{c+p+n}z, & B = 0, \end{cases}$$

and it is the best dominant of (3.9).

https://doi.org/10.1017/S0004972711002103 Published online by Cambridge University Press

Furthermore,

$$\operatorname{Re}\left[-\frac{z^{p+1}(I_p^m(n;\lambda,l)F_{p,c}(f)(z))'}{p}\right] > k, \quad z \in U,$$

where $k = \Theta(-1)$, *and this inequality is the best possible.*

PROOF. Setting

$$\varphi(z) = -\frac{z^{p+1}(I_p^m(n;\lambda,l)F_{p,c}(f)(z))'}{p}, \tag{3.10}$$

then φ has the form (2.1), and is analytic in U. Using in (3.10) the operator identity

$$z(I_p^m(n;\lambda,l)F_{p,c}(f)(z))' = cI_p^m(n;\lambda,l)f(z) - (c+p)(I_p^m(n;\lambda,l)F_{p,c}(f)(z)),$$

and differentiating the resulting equation with respect to z, we find that

$$-\frac{z^{p+1}(I_p^m(n;\lambda,l)f(z))'}{p} = \varphi(z) + \frac{z\varphi'(z)}{c} \prec \frac{1+Az}{1+Bz}.$$

Now, the remaining part of the proof follows by employing the same techniques that we used in the proof of Theorem 3.1. □

REMARK 3.10. (1) Setting $n = 0$ and $l = \lambda = 1$ in Theorem 3.9, we obtain the following result which improves the corresponding work of Liu and Srivastava [7, Theorem 2]. If $c > 0$ and $f \in R_{m,p}(A,B)$, then

$$F_{p,c}(R_{m,p}(A,B)) \subset R_{m,p}(1-2\zeta,-1) \subset R_{m,p}(A,B),$$

where

$$\zeta = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1, 1, \dfrac{c}{p}+1; \dfrac{B}{B-1}\right), & B \neq 0, \\ 1 - \dfrac{Ac}{c+p}, & B = 0. \end{cases} \tag{3.11}$$

The result is the best possible.

(2) Observing that

$$z^{p+1}(I_p^m(n;\lambda,l)F_{p,c}(f)(z))' = \frac{c}{z^c}\int_0^z t^{c+p}(I_p^m(n;\lambda,l)f(t))'\,dt, \tag{3.12}$$

whenever $f \in \sum_{p,n}$ and $c > 0$, the above remark can be restated as follows. If $c > 0$ and $f \in R_{m,p}(A,B)$, then

$$\operatorname{Re}\left[-\frac{c}{pz^c}\int_0^z t^{c+p}(I_p^m(n;\lambda,l)f(t))'\,dt\right] > \zeta, \quad z \in U,$$

where ζ is given by (3.11).

https://doi.org/10.1017/S0004972711002103 Published online by Cambridge University Press

According to (3.12), and taking in the above theorem $A = 1 - 2\alpha/p$, $B = -1$, and $m = 0$, we obtain the following special case.

COROLLARY 3.11. *If $c > 0$ and if $f \in \sum_{p,n}$ satisfies the inequality*

$$\mathrm{Re}[-z^{p+1} f'(z)] > \alpha, \quad z \in U \ (0 \le \alpha < p),$$

then

$$\mathrm{Re}\left[-\frac{c}{z^c}\int_0^z t^{c+p} f'(t)\,dt\right] > \alpha + (p-\alpha)\left[{}_2F_1\left(1, 1, \frac{c}{p+n}+1; \frac{1}{2}\right) - 1\right], \quad z \in U,$$

and the inequality is the best possible.

Using the technique of Srivastava and Patel [16, Theorem 4], we can prove the next theorem.

THEOREM 3.12. *Let the function $f \in \sum_{p,n}$, and suppose that $g \in \sum_{p,n}$ satisfies the inequality*

$$\mathrm{Re}[z^p I_p^m(n; \lambda, l) g(z)] > 0, \quad z \in U.$$

If

$$\left|\frac{I_p^m(n; \lambda, l) f(z)}{I_p^m(n; \lambda, l) g(z)} - 1\right| < 1, \quad z \in U \ (m \in \mathbb{N}_0, l, \lambda > 0),$$

then

$$\mathrm{Re}\left[-\frac{z(I_p^m(n; \lambda, l) f(z))'}{I_p^m(n; \lambda, l) f(z)}\right] > 0,$$

for $|z| < R_0$, where

$$R_0 = \frac{\sqrt{g(p+n)^2 + 4p(2p+n)} - 3(p+n)}{2(2p+n)}. \tag{3.13}$$

PROOF. Letting

$$w(z) = \frac{I_p^m(n; \lambda, l) f(z)}{I_p^m(n; \lambda, l) g(z)} - 1 = k_{p+n} z^{p+n} + k_{p+n+1} z^{p+n+1} + \cdots, \tag{3.14}$$

then w is analytic in U, with $w(0) = 0$, $|w(z)| < 1$ for all $z \in U$, and $w(z) = k_{p+m} z^{p+m} + k_{p+m+1} z^{p+m+1} + \cdots$. Defining the function ψ by

$$\psi(z) = \begin{cases} \dfrac{w(z)}{z^{p+m}}, & z \in \dot{U}, \\[2ex] \dfrac{w^{(p+m)}(0)}{(p+m)!}, & z = 0, \end{cases}$$

then ψ is analytic in $\dot{U}$ and continuous in U, hence it is analytic in the whole unit disc U. If $r \in (0, 1)$ is an arbitrary number, since $|w(z)| < 1$ for all $z \in U$, we deduce that

$$|\psi(z)| \le \max_{|z|=r} \left| \frac{w(z)}{z^{p+m}} \right| \le \max_{|z|=r} \frac{|w(z)|}{|z|^{p+m}} < \frac{1}{r^{p+m}}, \quad |z| \le r < 1.$$

By letting $r \to 1^-$ in the above inequality, we get $|\psi(z)| < 1$ for all $z \in U$, that is, $w(z) = z^{p+n}\psi(z)$, where the function ψ is analytic in U, and $|\psi(z)| < 1$, $z \in U$.

Therefore, (3.14) leads us to

$$I_p^m(n; \lambda, l) f(z) = I_p^m(n; \lambda, l) g(z)(1 + z^{p+n}\psi(z)), \quad z \in U,$$

and differentiating logarithmically the above relation, we obtain

$$\frac{z(I_p^m(n; \lambda, l) f(z))'}{I_p^m(n; \lambda, l) f(z)} = \frac{z(I_p^m(n; \lambda, l) g(z))'}{I_p^m(n; \lambda, l) g(z)} + \frac{z^{p+n}[(p+n)\psi(z) + z\psi'(z)]}{1 + z^{p+n}\psi(z)}. \tag{3.15}$$

Setting $\varphi(z) = z^p(I_p^m(n; \lambda, l) g(z))$, we see that the function φ has the form (2.1), is analytic in U with $\operatorname{Re}\varphi(z) > 0$, for all $z \in U$, and

$$\frac{z(I_p^m(n; \lambda, l) g(z))'}{I_p^m(n; \lambda, l) g(z)} = \frac{z\varphi'(z)}{\varphi(z)} - p.$$

Hence, from (3.15) we find that

$$\operatorname{Re}\left[-\frac{z(I_p^m(n; \lambda, l) f(z))'}{I_p^m(n; \lambda, l) f(z)}\right] \ge p - \left|\frac{z\varphi'(z)}{\varphi(z)}\right| - \left|\frac{z^{p+n}[(p+n)\psi(z) + z\psi'(z)]}{1 + z^{p+n}\psi(z)}\right|. \tag{3.16}$$

Now, by using in (3.16) the known estimates (see [8])

$$\left|\frac{\varphi'(z)}{\varphi(z)}\right| \le \frac{2(p+n)r^{p+n-1}}{1 - r^{2(p+n)}}, \quad |z| = r < 1,$$
$$\left|\frac{(p+n)\psi(z) + z\psi'(z)}{1 + z^{p+n}\psi(z)}\right| \le \frac{p+n}{1 - r^{(p+n)}}, \quad |z| = r < 1,$$

we conclude that

$$\operatorname{Re}\left[-\frac{z(I_p^m(n; \lambda, l) f(z))'}{I_p^m(n; \lambda, l) f(z)}\right] \ge \frac{p - 3(p+n)r^{p+n} - (2p+n)r^{2(p+n)}}{1 - r^{2(p+n)}},$$

for $|z| = r < 1$, which is positive provided that $r < R_0$, where R_0 is given by (3.13). □

THEOREM 3.13. *Let* $-1 \le B_i < A_i \le 1$, $i = 1, 2$, *and suppose that each of the functions* $f_i \in \sum_p$ *satisfies the subordination condition*

$$(1-\beta)z^p I_p^m(\lambda, l) f_i(z) + \beta z^p I_p^{m+1}(\lambda, l) f_i(z) \prec \frac{1 + A_i z}{1 + B_i z}, \quad i = 1, 2, \tag{3.17}$$

https://doi.org/10.1017/S0004972711002103 Published online by Cambridge University Press

where $I_p^m(\lambda, l) \equiv I_p^m(-p+1; \lambda, l)$. *Then*

$$(1-\beta)z^p I_p^m(\lambda, l)G(z) + \beta z^p I_p^{m+1}(\lambda, l)G(z) \prec \frac{1+(1-2\eta)z}{1-z},$$

where

$$G(z) = I_p^m(\lambda, p)(f_1 * f_2)(z)$$

and

$$\eta = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1-B_1)(1-B_2)}\left[1 - {}_2F_1\left(1, 1, \frac{l}{\beta\lambda}+1; \frac{1}{2}\right)\right].$$

The result is the best possible when $B_1 = B_2 = -1$.

PROOF. Since each of the functions $f_i \in \sum_p$, $i = 1, 2$, satisfies condition (3.17), then by letting

$$\varphi_i(z) = (1-\beta)z^p I_p^m(\lambda, l) f_i(z) + \beta z^p I_p^{m+1}(\lambda, l) f_i(z), \quad i = 1, 2, \tag{3.18}$$

we have

$$\varphi_i \in \mathcal{P}(\gamma_i) \quad \text{where } \gamma_i = \frac{1-A_i}{1-B_i} \ (i = 1, 2).$$

Using identity (1.2) in (3.18),

$$I_p^m(\lambda, l) f_i(z) = \frac{l}{\beta\lambda} z^{-p-l/\beta\lambda} \int_0^z t^{(l/\beta\lambda)-1} \varphi_i(t)\, dt, \quad i = 1, 2,$$

which, according to the definition of G, yields

$$I_p^m(\lambda, l)G(z) = \frac{l}{\beta\lambda} z^{-p-l/\beta\lambda} \int_0^z t^{(l/\beta\lambda)-1} \varphi_0(t)\, dt,$$

where

$$\begin{aligned} \varphi_0(z) &= (1-\beta)z^p I_p^m(\lambda, l)G(z) + \beta z^p I_p^{m+1}(\lambda, l)G(z) \\ &= \frac{l}{\beta\lambda} z^{-l/\beta\lambda} \int_0^z t^{(l/\beta\lambda)-1} (\varphi_1 * \varphi_2)(t)\, dt. \end{aligned} \tag{3.19}$$

Since $\varphi_i \in \mathcal{P}(\gamma_i)$, $i = 1, 2$, it follows from Lemma 2.4 that

$$\varphi_1 * \varphi_2 \in \mathcal{P}(\gamma_3) \quad \text{where } \gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2). \tag{3.20}$$

By using (3.20) and (3.19), from Lemmas 2.3 and 2.6, we get

$$\begin{aligned} \operatorname{Re} \varphi_0(z) &= \frac{l}{\beta\lambda} z^{-l/\beta\lambda} \int_0^1 u^{(l/\beta\lambda)-1} \operatorname{Re}(\varphi_1 * \varphi_2)(uz)\, du \\ &\geq \frac{l}{\beta^\lambda} \int_0^1 u^{(l/\beta^\lambda)-1} \left[2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u|z|}\right] du \\ &> \frac{l}{\beta\lambda} \int_0^1 u^{(l/\beta\lambda)-1} \left[2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u}\right] du \end{aligned}$$

$$= 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\left[1 - \frac{l}{\beta\lambda}\int_0^1 u^{(l/\beta\lambda)-1}(1+u)^{-1}\,du\right]$$
$$= 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\left[1 - \frac{1}{2}\,{}_2F_1\left(1, 1, \frac{l}{\beta\lambda} + 1; \frac{1}{2}\right)\right] = \eta, \quad z \in U.$$

When $B_1 = B_2 = -1$, consider the functions $f_i \in \sum_p$, $i = 1, 2$, which satisfy assumptions (3.17) and are defined by

$$I_p^m(\lambda, l) f_i(z) = \frac{l}{\beta\lambda} z^{-l/\beta\lambda} \int_0^z t^{(l/\beta\lambda)-1}\left(\frac{1 + A_i t}{1 - t}\right) dt, \quad i = 1, 2.$$

Thus, from (3.19) and Lemma 2.6, it follows that

$$\begin{aligned}\varphi_0(z) &= \frac{l}{\beta\lambda}\int_0^1 u^{(l/\beta\lambda)-1}\left[1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{1 - uz}\right] du \\ &= 1 - (1 + A_1)(1 + A_2) + (1 + A_1)(1 + A_2)(1 - z)^{-1} \\ &\qquad \times {}_2F_1\left(1, 1, \frac{l}{\beta\lambda} + 1; \frac{z}{z - 1}\right) \\ &\to 1 - (1 + A_1)(1 + A_2) + \frac{1}{2}(1 + A_1)(1 + A_2){}_2F_1\left(1, 1, \frac{l}{\beta\lambda} + 1; \frac{1}{2}\right),\end{aligned}$$

as $z \to -1$, which completes the proof. □

Taking $A_i = 1 - 2\alpha_i$, $B_i = -1$ $(i = 1, 2)$, $m = 0$ and $l = \lambda = 1$ in Theorem 3.13, we obtain the following result which refines the work of Yang [20, Theorem 4].

COROLLARY 3.14. *If the functions $f_i \in \sum_p$, $i = 1, 2$, satisfy the inequality*

$$\mathrm{Re}\{(1 + \beta p) z^p f_i(z) + \beta z^{p+1} f_i'(z)\} > \alpha_i, \quad z \in U\ (0 \le \alpha_i < 1, i = 1, 2), \tag{3.21}$$

then

$$\mathrm{Re}\{(1 + \beta p) z^p (f_1 * f_2)(z) + \beta z^{p+1}(f_1 * f_2)(z)\} > \eta_0, \quad z \in U,$$

where

$$\eta_0 = 1 - 4(1 - \alpha_1)(1 - \alpha_2)\left[1 - \frac{1}{2}\,{}_2F_1\left(1, 1, \frac{1}{\beta} + 1; \frac{1}{2}\right)\right].$$

The result is the best possible.

THEOREM 3.15. *If the function $f \in \sum_{p,n}$ satisfies the subordination condition*

$$(1 - \beta) z^p I_p^m(n; \lambda, l) f(z) + \beta z^p I_p^{m+1}(n; \lambda, l) f(z) \prec \frac{1 + Az}{1 + Bz},$$

then

$$\mathrm{Re}[z^p I_p^m(n; \lambda, l) f(z)]^{1/q} > \rho^{1/q}, \quad z \in U\ (q \in \mathbb{N}),$$

where $\rho = Q(-1)$ is given as in Theorem 3.1. The result is the best possible.

PROOF. Defining the function φ by

$$\varphi(z) = z^p I_p^m(n; \lambda, l) f(z), \tag{3.22}$$

we see that φ has the form (2.1) and is analytic in U. Using identity (1.2) in (3.22), and differentiating the resulting equation with respect to z, we obtain

$$(1-\beta) z^p I_p^m(n; \lambda, l) f(z) + \beta z^p I_p^{m+1}(n; \lambda, l) f(z) = \varphi(z) + \frac{\beta\lambda}{l} z\varphi'(z) \prec \frac{1+Az}{1+Bz}.$$

Now, by following similar steps to the proof of Theorem 3.1, and using the elementary inequality

$$\operatorname{Re} w^{1/q} \geq (\operatorname{Re} w)^{1/q}, \quad \operatorname{Re} w > 0, \, q \in \mathbb{N},$$

we obtain the result asserted by Theorem 3.15. □

From Corollary 3.14 and Theorem 3.15, for the special case $n = -p + 1$, $m = 0$, $A = 1 - 2\eta_0$, $B = -1$ and $q = 1$, we deduce the next result.

COROLLARY 3.16. *Let the functions $f_i \in \sum_p$ $(i = 1, 2)$, satisfy inequality (3.21). Then*

$$\operatorname{Re}[z^p (f_1 * f_2)(z)] > \eta_0 + (1-\eta_0)\left[{}_2F_1\left(1, 1, \frac{1}{\beta} + 1; \frac{1}{2}\right) - 1 \right], \quad z \in U,$$

where η_0 is given as in Corollary 3.14. The result is the best possible.

THEOREM 3.17. *If the function $g \in \sum_{p,n}$ satisfies the inequality*

$$\operatorname{Re}[z^p g(z)] > \tfrac{1}{2}, \quad z \in U, \tag{3.23}$$

then, for any function $f \in \sum_{p,n}^m(\lambda, l, A; B)$, we have

$$f * g \in \sum_{p,n}^{m}(\lambda, l; A, B).$$

PROOF. It is easy to check that

$$-\frac{z^{p+1}(I_p^m(n; \lambda, l)(f * g)(z))'}{p} = \left[-\frac{z^{p+1}(I_p^m(n; \lambda, l) f(z))'}{p} \right] * [z^p g(z)].$$

According to this relation, by applying Lemma 2.5 for the functions

$$F(z) = -\frac{z^{p+1}(I_p^m(n; \lambda, l) f(z))'}{p}$$

and $\Phi(z) = z^p g(z)$, and using the fact that the function $h(z) = (1 + Az)/(1 + Bz)$ is convex (univalent) in U, we deduce the conclusion of the theorem. □

https://doi.org/10.1017/S0004972711002103 Published online by Cambridge University Press

References

[1] M. K. Aouf and H. M. Hossen, 'New criteria for meromorphic p-valent starlike functions', *Tsukuba J. Math.* **17** (1993), 481–486.

[2] N. E. Cho, O. S. Kwon and H. M Srivastava, 'Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators', *J. Math. Anal. Appl.* **242** (2004), 470–480.

[3] N. E. Cho, O. S. Kwon and H. M Srivastava, 'Inclusion and argument properties for certain subclasses of meromorphic functions associated with a family of multiplier transformations', *J. Math. Anal. Appl.* **300** (2004), 505–520.

[4] N. E. Cho, O. S. Kwon and H. M. Srivastava, 'Inclusion relationships for certain subclasses of meromorphic functions associated with a family of multiplier transformations', *Integral Transforms Spec. Funct.* **16**(18) (2005), 647–659.

[5] D. J. Hallenbeck and St. Ruscheweyh, 'Subordinations by convex functions', *Proc. Amer. Math. Soc.* **52** (1975), 191–195.

[6] J.-L. Liu and H. M. Srivastava, 'A linear operator and associated families of multivalent functions', *J. Math. Anal. Appl.* **259** (2001), 566–581.

[7] J.-L. Liu and H. M. Srivastava, 'Subclasses of meromorphically multivalent functions associated with a certain linear operator', *Math. Comput. Modelling* **39** (2004), 35–44.

[8] T. H. MacGregor, 'Radius of univalence of certain analytic functions', *Proc. Amer. Math. Soc.* **14** (1963), 514–520.

[9] S. S. Miller and P. T. Mocanu, 'Second order differential inequalities in the complex plane', *J. Math. Anal. Appl.* **65** (1978), 289–305.

[10] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, 225 (Marcel Dekker Inc., New York, 2000).

[11] M. L. Mogra, 'Meromorphic multivalent functions with positive coefficients. I', *Math. Japon.* **35**(1) (1990), 1–11.

[12] M. L. Mogra, 'Meromorphic multivalent functions with positive coefficients. II', *Math. Japon.* **35**(6) (1990), 1089–1098.

[13] M. Pap, 'On certain subclasses of meromorphic m-valent close-to-convex functions', *Pure Math. Appl.* **9** (1998), 155–163.

[14] D. Ž. Pashkouleva, 'The starlikeness and spiral-convexity of certain subclasses of analytic functions', in: *Current Topics in Analytic Function Theory* (eds. H. M. Srivastava and S. Owa) (World Scientific Publishing Company, Singapore, 1992).

[15] R. Singh and S. Singh, 'Convolution properties of a class of starlike functions', *Proc. Amer. Math. Soc.* **106** (1989), 145–152.

[16] H. M. Srivastava and J. Patel, 'Applications of differential subordination to certain classes of meromorphically multivalent functions', *J. Inequal. Pure Appl. Math.* **6**(3) (2005), 15.

[17] J. Stankiewicz and Z. Stankiewicz, 'Some applications of the Hadamard convolution in the theory of functions', *Ann. Univ. Mariae Curie-Sklodowska Sect. A* **40** (1986), 251–265.

[18] B. A. Uralegaddi and C. Somanatha, 'New criteria for meromorphic starlike univalent functions', *Bull. Aust. Math. Soc.* **43** (1991), 137–140.

[19] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions: With an Account of the Principal Transcendental Functions*, 4th edn (Cambridge University Press, Cambridge, 1927).

[20] D.-G. Yang, 'Certain convolution operators for meromorphic functions', *South East Asian Bull. Math.* **25** (2001), 175–186.

R. M. EL-ASHWAH, Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt
e-mail: r_elashwah@yahoo.com

M. K. AOUF, Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
e-mail: mkaouf127@yahoo.com

T. BULBOACĂ, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania
e-mail: bulboaca@math.ubbcluj.ro