Unfortunately the secondary school education on this continent is lagging far behind and we cannot hope to be successful in confronting our freshmen even with a much less sophisticated introduction to higher mathematics. In the third or the fourth year we offer courses on similar lines to the honours students of mathematics and physics referring for instance to the book by P. Halmos, Finite-dimensional Vector Spaces (or one of the many introductions to linear algebra) as a text, comparable in extent if not in actual execution, to Julia's book, but more specialized than Feigl's introduction to higher mathematics. Now, if we accept as an axiom that students of equal ages in different civilized countries are of equal intelligence, no reason can be seen why it should not be possible to raise the level of mathematical preparation of university freshmen to a degree such that also on this side of the Atlantic a reasonable course on linear algebra can be digested at the beginning, instead of the end of their mathematical courses.

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<u>The Theory of Matrices</u>, by F.R. Gantmacher. Chelsea, New York, 1959. Vol. I. x + 374 pages; vol. II. ix + 276 pages. $6.00 per volume.

Among all the books on Linear Algebra and Matrices that have been published during the last ten years in almost all languages, the present work, first published in Russian in 1954, is probably the most complete treatise on the subject. It has some of the features of a comprehensive handbook: its chapters are to a certain extent independent and the earlier ones can be read by anybody acquainted with determinants and the usual preliminaries from elementary algebra. The last three chapters, dealing with questions arising in probability, systems of differential equations, and the Routh-Hurwitz problem are of a more specialized nature.

Some eminent representatives of the abstract algebraic school have recently expressed their strong disapproval of the teaching of formal matrix theory, giving as reason that proofs of most theorems in linear algebra can be presented more briefly and more lucidly if the notion of matrix is avoided as long as there is no question of determinants. The present book does not attempt a development of linear algebra the abstract way. Its main object is finite matrices with number entries, real or complex, as they occur in ever so many problems of classical analysis, numerical analysis, mechanics, probability, etc. In the reviewer's opinion there can be no doubt that the methods used in Gantmacher's work are adequate and useful for all those who have to deal with problems of this kind. But also the casual reader, primarily interested in algebra, will find information not easily to be obtained elsewhere.

Chapter I (pp. 1-22) introduces rectangular matrices, rank (determinant definition), addition, multiplication, linear homogeneous transformations, the Binet-Cauchy formula, the algebra of square matrices. Chapter II (pp. 23-49) explains Gauss's elimination method. By means of the compound matrix symbols of chapter I one is led to a simple proof of Sylvester's determinant identity, and to the representation of a square matrix as a product with two triangular factors. Further there is a section on the technique of operating with block matrices. Chapter III (pp. 50-75) prepares the geometrical theory of linear operators in n-dimensional vector space. (In the last formula on p. 57 e_k should be replaced by g_i.) This turn to the abstract, however, is only apparent; the matrix representation gives the start and remains the end in view. Chapter IV deals with matrix polynomials (pp. 76-94) and contains a section on the by now well known method of D. K. Faddeev for simultaneous calculation of the coefficients of the characteristic polynomial, and of the adjoint matrix by a double recurrence formula*. Chapter V (pp. 95-129): Functions of Matrices, with the usual applications to systems of linear differential equations with constant coefficients. In the last section we find a discussion of stability and asymptotical stability in the sense of Lyapounov of the solution of a system of differential equations of the first order. (The reference "[14], p. 13" on p. 125 seems to be misleading, probably a misprint.) Chapter VI (pp. 130-174) deals with equivalence of polynomial matrices and the theory of elementary divisors; there are methods of constructing the matrix by which a given matrix can be transformed to its Jordan canonical form. Chapter VII (pp. 175-214) contains the geometrical structure theory of a linear operator ending with Krylov's method of transforming the characteristic equation of a matrix in such a way that the unknown λ occurs only in one column of the determinant. Chapter VIII (pp. 215-241): Matrix Equations, ending with a brief discussion of log A. Chapter IX (pp. 242-293): Linear Operators in the Unitary Space, contains, apart from the usual material, a generalization of Hadamard's inequality: Let $G(x_1, \ldots, x_m) = |(x_i \cdot x_j)|$ be the "Gramian" of m vectors in the unitary or euclidean space; then

$$G(x_1, \ldots, x_m) \leq G(x_1, \ldots, x_p)\, G(x_{p+1}, \ldots, x_m)$$

with equality if and only if each vector $x_1, \ldots, x_p$ is orthogonal to each of the vectors $x_{p+1}, \ldots, x_m$ or one of the factors on the right-hand side vanishes, which is the case if and only if its vectors are linearly dependent. In connection with the discussion of the orthogonalization process we find a brief digression into the infinite dimensional space. Chapter X (pp. 294-348), Quadratic and Hermitian Forms, contains the classical theory of one form and of a pencil of forms, extremal properties of the eigenvalues, a section on small oscillations, Hankel forms.

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* The method has also been discovered and published almost simultaneously and independently by J.-M. Souriau and by J. S. Frame.

Whereas the first volume represents more or less a compilation of the basic material, the second volume is definitely of a more advanced nature. It begins with chapter XI (pp. 1-23), Complex Symmetric Matrices, Skew-symmetric, and Orthogonal Matrices, subjects hardly ever dealt with systematically in a book. The elementary divisors and similarity normal forms of these classes of matrices are established. Chapter XII (pp. 24-49), making use of Kronecker's theory of minimal indices, derives a criterion for strict equivalence of two pencils of matrices, and for each pencil a strictly equivalent normal form, with an application to m linear differential equations of the first order in n unknown functions with constant coefficients. In chapter XIII (pp. 50-112), the author studies the spectral properties of matrices with non-negative elements which are the objects of interest in several different fields of mathematics, pure and applied. It begins with Frobenius' theorem stating that an irreducible non-negative matrix A always has a maximal positive characteristic root. (Reducibility refers to congruence transformation of A by a permutation matrix P, i.e. $A \to P'AP$.) There follow details on the normal form of reducible matrices, on stochastic matrices with applications to oscillation matrices (which are more thoroughly treated in another book by Gantmacher and M. G. Krein). A matrix is said to be oscillatory if it is totally non-negative and if one of its powers is totally positive. Chapter XIV (pp. 113-171) is devoted to applications of matrix theory in the theory of linear differential equations and reaches indeed much further in this direction than other books dealing with this matter. This chapter must therefore be considered as one of interest far beyond the bounds of a treatise on matrices. For some purely analytical theorems reference is made to Lyapounov's memoir "The general problem of stability of motion" which has been published as Annals of Mathematics Study No. 17 (Princeton, 1949). A system $dX/dt = P(t)X$ is said to be reducible if by a Lyapounov transformation $X = L(t)Y$ (where $L(t)$ and dL/dt are bounded and $0 < m < |L(t)|$) it can be reduced to a system with constant coefficients; it is shown that every system with periodic coefficients is reducible. For a system $dX/dt = AX$ with constant coefficients it is shown that by a Lyapounov transformation it can be reduced to a system whose matrix has Jordan normal form with real characteristic roots. The existence of the solution of a general system is demonstrated by the method of successive approximations, which leads to the infinite series of iterated integrals, called the matricant. Making use of the multiplicative properties of the matricant, the Volterra product integral is derived. There follows a theory in the case of complex analytic coefficients with isolated singularities. Attention is drawn to an error in the proof of Volterra's theorem according to which the multiplier matrix belonging to a closed path around a regular singularity is similar to $\exp(2\pi i P_{-1})$ where P_{-1} is the matrix of the residues of P. The author supports his statement by an explicit counterexample. The same argument leads to a refutation of G. D. Birkhoff's theorem of 1913. A canonical form of the solution in the neighbourhood of a regular singular point is then established. Finally mention is made of Lappo-Danilewski's work which is based on a definition of analytic functions of several matrices. The last chapter (pp. 172-250) deals with the problem of Routh and Hurwitz and related questions. It is absolutely comprehensive and original. It begins with a historical sketch. The subsequent exposition explains with most welcome attention to detail the criteria of Lyapounov, Routh, Hurwitz, Orlando, Liénard and Chipart, the method of quadratic forms of Hermite, making use of Frobenius' results on Hankel forms (cf. Chapter X); a digression on infinite Hankel matrices of finite rank; a proof of Routh-Hurwitz' theorem by means of the Cauchy indices of a rational function; Stieltjes' representation of Hurwitz polynomials by continued fractions and the connection with the moment problem. Finally we have a survey of several other topics connected with the main problem and its generalizations that have been dealt with mainly in the Russian literature (Chebyshev, Markov, Krein and Naimark, Chebotarëv and Meiman).

The system of notations used in this translation of Gantmacher's book is one rather generally accepted. There is, however, a slight inconsistency: whereas elsewhere in the book the transpose of a matrix is indicated by superscript T, we find that in vol. II, pp. 189-190 the (maybe more Continental) notation A' is used without previous warning.

At the end of each of the two volumes we find the same extensive (but by no means complete) bibliography, listing 67 books (18 in Russian) and 296 papers (116 in Russian), all, with only a few exceptions, published during the last 20 years; some are not referred to in the text, but are in more or less evident relations to one section or another. There is also an alphabetic index.

From this survey it may be seen that the book before us is an important work whose appearance in English will be welcomed by many mathematicians, pure and applied, in and outside the circle of those actually interested in the theory of matrices.

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Information Theory and Statistics, by Solomon Kullback. Wiley, New York, 1959. xvii + 395 pages. $ 12.50.

This is a book for statisticians. It discusses tests of statistical hypothesis from the point of view of information theory. The first 100 pages deal with the basic properties of information measures and their relevance to statistical inference; the remaining 250 pages are devoted to applying these concepts and analysing samples for hypothesis testing. The result is a unified and novel presentation of statistical procedures in hypothesis testing.