Compositio Mathematica **115:** 21–35, 1999.

Verma modules over generalized Witt algebras

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Received 26 February 1996; accepted in final form 18 August 1997

Abstract. We construct and investigate a structure of Verma-like modules over generalized Witt algebras. We also prove Futorny-like theorem for irreducible weight modules whose dimensions of the weight spaces are uniformly bounded.

Mathematics Subject Classifications (1991): Primary 17B68; Secondary 17B10.

Key words: generalized Witt algebra, Verma module.

1. Introduction

The notion of generalized Witt algebras appears as a natural generalization of the well-known Witt algebra in [27]. Since that article only a few main properties of generalized Witt algebras have been studied. In positive characteristics complete classification of such algebras over an algebraically closed field is known, some simplicity criteria is obtained and the automorphism group is determined ([18, 25]). In zero characteristic, generalized Witt algebras are known as centerless higher rank Virasoro algebras ([15, 24]). In this case complete classification of pointed Harish–Chandra modules over such algebras is given ([24]), automorphism group is determined, all finite-dimensional subalgebras are described and some realization by differential operators is constructed ([15]). In fact, almost nothing is known about non-pointed weight modules over such algebras. At the same time representation theory of the Witt algebra and Virasoro algebra is very well developed, although there are a lot of interesting unsolved problems (see [3–4, 11–13, 19, 23]).

During the last decade a number of papers devoted to different generalizations of generalized Witt algebras have appeared (the history of this question is due to Kaplansky [9]) which can be found in [8, 10, 16–17, 22, 26]. Most of those algebras are very difficult and nothing but existence and simplicity is known about them.

The aim of this paper is to define and study some analogue of Verma modules ([14]), Verma type modules and generalized Verma modules ([6–7]) over generalized Witt algebras over complex field. All the results obtained in the paper remain valid over an arbitrary algebraically closed field of zero characteristics. Moreover, some of them take place even for higher rank Virasoro algebras (universal central extension of generalized Witt algebras). We also describe a support of special kind irreducible weight modules over generalized Witt algebras generalizing Futorny theorem on a support of irreducible module over classical simple complex finite-dimensional Lie algebras ([14]).

The paper is organized as follows: in Section 2 we collect all necessary preliminaries. In Section 3 we give some inner characterization of Cartan subalgebra. In Section 4 we prove a Futorny-like theorem which describes a support of simple weight module over generalized Witt algebras of strictly finite type (see Section 2 for precise definition). In Section 5 we study Verma modules over generalized Witt algebras and obtain complete information about their structure in Theorem 3. In Section 6 we classify all non-standard Borel subsets associated with generalized Witt algebra and obtain information on a structure of corresponding Verma-type modules (Theorem 4 and Theorem 5) and generalized Verma modules (Theorem 6 in Section 7).

2. Notations and preliminary results

Let $\mathbb{C}$ denote the complex numbers, $\mathbb{Q}$ denote all rationals, $\mathbb{Z}$ denote all integers and $\mathbb{N}$ denote all positive integers.

Consider an additive subgroup $P \subset \mathbb{C}$ of finite rank $n \in \mathbb{N}$. Let $\mathfrak{G}$ denote a $\mathbb{C}$-space with the base e_x, $x \in P$. For arbitrary $x, y \in P$ define

$$[e_x, e_y] = (y - x)e_{x+y}, \tag{1}$$

Clearly this operation can be extended on $\mathfrak{G}$ by linearity. One can see that (1) defines on $\mathfrak{G}$ a structure of Lie algebra, moreover, $\mathfrak{G}$ is simple infinite-dimensional Lie algebra as soon as P is non-trivial. Following [27] we will call $\mathfrak{G}$ generalized Witt algebra of rank n.

Denote by $\mathfrak{H} = \langle e_0 \rangle$ the Cartan subalgebra of $\mathfrak{G}$. We will call elements $x \in P$ roots of $\mathfrak{G}$. A $\mathbb{C}$-space $\langle e_x \rangle$ corresponding to a root $x \in P$ will be called root subspace.

Let V be a $\mathfrak{G}$-module. For $\lambda \in \mathbb{C}$ define

$$V_\lambda = \{v \in V : e_0 v = \lambda v\}.$$

Every non-zero vector $v \in V_\lambda$ will be called a weight vector of weight λ.

Every non-zero subspace V_λ will be called a weight subspace corresponding to a weight λ. A module V will be called a weight module provided

$$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda.$$

Let V be a weight module. We will write supp V for the set of all weights of V. We will say that a weight module V is of strictly finite type if

$$\sup_{\lambda \in \operatorname{supp} V} \dim V_\lambda < \infty.$$

For an element

$$g = \sum_{x \in P} g_x e_x \in \mathfrak{G}$$

we will denote by $\operatorname{supp} g$ the set $\{x \in P : g_x \neq 0\}$.

It is easy to verify that any irreducible module having a weight vector is a weight module and that all submodules and quotients of a weight module are also weight modules.

Consider the universal enveloping algebra $U(\mathfrak{G})$. Trivially, every $\mathfrak{G}$-module is also an $U(\mathfrak{G})$-module and vice-versa. So $U(\mathfrak{G})$ is a $\mathfrak{G}$-module with natural action. For $g \in U(\mathfrak{G})$ we will write $H(g) = s$ if $[e_0, g] = sg$.

For the rest of the paper $\mathfrak{W}$ will denote the classical Witt algebra with the standard base $\{e_i\}_{i \in \mathbb{Z}}$ and Lie brackets given by

$$[e_i, e_j] = (j - i)e_{i+j}.$$

EXAMPLE 1. The easiest example of generalized Witt algebra $\mathfrak{G}$ which is not isomorphic to $\mathfrak{W}$ can be obtained for $P = \{a + bi : a, b \in \mathbb{Z}\}$.

3. Cartan subalgebra

Our first goal is to give an invariant definition of the Cartan subalgebra $\mathfrak{H}$.

Consider the adjoint representation ad $\mathfrak{G}$ defined by ad $g \cdot a = [g, a]$ for every $g, a \in \mathfrak{G}$.

LEMMA 1. *Let* $\mathfrak{G}$ *be a generalized Witt algebra. An element* ad h *is semisimple* (*i.e. possess an eigenbase*) *if and only if* $h \in \mathfrak{H}$.

Proof. Every $h \in \mathfrak{H}$ can be written in the form $h = ce_0$ for some $c \in \mathbb{C}$. By definition of $\mathfrak{G}$ we have

$$\text{ad}\, h \cdot e_x = [h, e_x] = c[e_0, e_x] = cxe_x$$

for every $x \in P$ and thus h is semisimple.

To prove the necessity we consider some linear order $<$ on abelian group P. Then for every $x, y, z \in P$ holds

$$x < y \Rightarrow x + z < y + z.$$

For every $a \in \mathfrak{G}$, $a \notin \mathfrak{H}$ at least one of the following conditions holds:

(1) $\min_{x \in \operatorname{supp} a} x \notin 0$,

(2) $\max_{x \in \operatorname{supp} a} x \notin 0$.

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Consider the first case (one can rewrite the same arguments for the second one). Let $b \in \mathfrak{G}$ be an element such that

$$\min_{y \,\in\, \mathrm{supp}\, b} y < \min_{x \,\in\, \mathrm{supp}\, a} x, \qquad \min_{y \,\in\, \mathrm{supp}\, b} y < 0.$$

It follows immediately that

$$\min_{z \,\in\, \mathrm{supp}\, [a,b]} z < \min_{y \,\in\, \mathrm{supp}\, b} y$$

and we obtain $[a, b] \neq \lambda b$ for any $\lambda \in \mathbb{C}$. Thus $\mathrm{ad}\, a$ is not semisimple. □

It follows from [15] that there is continuum many of pairwise non-isomorphic generalized Witt algebras for each finite rank $n > 1$.

4. Irreducible $\mathfrak{G}$-modules with finite-dimensional weight spaces

In this section we obtain a weak analogue of Futorny theorem on support of irreducible weight module over generalized Witt algebra. The original theorem ([7]) states that:

THEOREM 1. *Let $\mathfrak{G}$ be a classical simple finite-dimensional Lie algebra and V be an irreducible weight $\mathfrak{G}$-module. Then either V has a semiprimitive element or* $\mathrm{supp}\, V$ *has the following form*:

$$\mathrm{supp}\, V = \lambda + \sum_{n_i \,\in\, \mathbb{Z}, \alpha_i \,\in\, \pi} n_i \alpha_i,$$

where $\lambda \in \mathrm{supp}\, V$ and π denotes a base of standard root system of $\mathfrak{G}$.

We present the following analog of this result for irreducible weight modules over generalized Witt algebras:

THEOREM 2. *Let V be an irreducible module of strictly finite type over generalized Witt algebra $\mathfrak{G}(\mathfrak{G} \not\simeq \mathfrak{W})$. Then either* $\mathrm{supp}\, V = \lambda + P$ *for some $\lambda \in \mathbb{C}$ or* $\mathrm{supp}\, V = P \backslash \{0\}$.

To prove this theorem we need the following key lemma:

LEMMA 2. *For $\lambda \in \mathbb{C}$, $\lambda \neq 0$ let $M(\lambda)$ be Verma module over Witt algebra $\mathfrak{W}$ with highest weight λ. Denote by $L(\lambda)$ its unique irreducible quotient. Then*

$$\sup_{\mu \,\in\, \mathrm{supp}\, L(\lambda)} \dim L(\lambda)_\mu = \infty.$$

Proof. Denote by v_λ some primitive generator of $M(\lambda)$ and by N the maximal submodule of $M(\lambda)$.

If $M(\lambda)$ is irreducible $(N = 0)$ we have $\dim M(\lambda)_{\lambda-k} = P(k)$, where $P(k)$ denotes the classical partition function ([1]). Thus

$$\sup_{\mu \in \operatorname{supp} M(\lambda)} \dim M(\lambda)_\mu = \infty.$$

Suppose now that $N \neq 0$. According to [19, Theorem A] there can be two possibilities:

(1) $N \simeq M(\lambda')$ for some $\lambda' < \lambda$. Since for any fixed $s \in \mathbb{N}$ holds

$$P(k) - P(k-s) \to \infty, \quad k \to \infty \tag{2}$$

it follows that

$$\begin{aligned} \sup_{\mu \in \operatorname{supp} L(\lambda)} \dim L(\lambda)_\mu &= \lim_{k\to\infty} (M(\lambda)_{\lambda-k} - M(\lambda')_{\lambda-k}) \\ &= \lim_{k\to\infty} (P(k) - P(k-s)) = \infty \end{aligned}$$

and the statement of lemma follows.

(2) $\lambda \in \{s_m, t_m\}$ for some $m \in \mathbb{N}$, where

$$s_m = -\tfrac{1}{2}(3m^2 + m), \qquad t_m = -\tfrac{1}{2}(3m^2 - m).$$

In this case there is an exact sequence:

$$\begin{aligned} &\cdots \xrightarrow{d_{m+i+1}} M(s_{m+i}) \oplus M(t_{m+i}) \xrightarrow{d_{m+i}} \cdots \xrightarrow{d_{m+2}} \\ &\xrightarrow{d_{m+2}} M(s_{m+1}) \oplus M(t_{m+1}) \xrightarrow{d_{m+1}} M(\lambda) \xrightarrow{\varepsilon} L(\lambda) \to 0 \end{aligned}$$

(ε is a canonical projection and homomorphisms d_i defined in [19]).

Let $\nu_k \in \{s_k, t_k\}$ for some $k \in \mathbb{N}$. Following [19], we can write the character formula for $L(\nu_k)$:

$$\operatorname{ch} L(\nu_k) = \operatorname{ch} M(\nu_k) + (-1)^k \sum_{i>k} (-1)^i (\operatorname{ch} M(s_i) + \operatorname{ch} M(t_i)).$$

It is convenient to rewrite it in the form

$$\operatorname{ch} L(\nu_k) = \operatorname{ch} M(\nu_k) - \operatorname{ch} L(\nu_{k+1}) - \operatorname{ch} M(\xi_{k+1}),$$

where $\{\nu_{k+1}, \xi_{k+1}\} = \{s_{k+1}, t_{k+1}\}$. Thus we obtain

$$\operatorname{ch} L(\nu_{k+1}) + \operatorname{ch} L(\nu_k) = \operatorname{ch} M(\nu_k) - \operatorname{ch} M(\xi_{k+1})$$

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and it follows immediately from (2) that our statement is true at least for one of the modules $L(\nu_{k+1})$ or $L(\nu_k)$. In fact, the statement of lemma is true for $L(s_1)$ and $L(t_1)$.

On the other hand we have the following equality:

$$\begin{aligned}\operatorname{ch} L(\nu_k) &= \operatorname{ch} M(\nu_k) - \operatorname{ch} M(s_{k+1}) - \operatorname{ch} M(t_{k+1}) \\ &\quad + \operatorname{ch} M(\nu_{k+2}) + \operatorname{ch} L(\xi_{k+2})\end{aligned}$$

for $\{\nu_{k+2}, \xi_{k+2}\} = \{s_{k+2}, t_{k+2}\}$. Thus

$$\begin{aligned}\operatorname{ch} L(\xi_{k+2}) &= \operatorname{ch} M(s_{k+1}) - \operatorname{ch} M(\nu_k) - \operatorname{ch} M(\nu_{k+2}) \\ &\quad + \operatorname{ch} M(t_{k+1}) + \operatorname{ch} L(\nu_k).\end{aligned}$$

We can assume that $\lambda = \xi_{k+2}$. From the above formulae it follows immediately that in order to prove that $\dim L(\xi_{k+2})_{\xi_{k+2}-s}$ is not bounded it is sufficient to show that for any fixed $a, b, c \in \mathbb{N}$ such that $a < b$ holds

$$(P(s) - P(s-b)) - (P(s+c) - P(s+c-a)) \to \infty, \quad s \to \infty.$$

Let $F(s) = P(s) - P(s-1)$. One can see that $F(s)$ is a partition function with generating function

$$\varphi(x) = \prod_{i=2}^{\infty} \frac{1}{1-x^i}.$$

Using asymptotic theorem ([1, Theorem 6.2]) we obtain $F(s) \sim C(\sqrt{s})^{-3} \exp(M\sqrt{s})$ for some constants C and M. Thus

$$\frac{F(s)}{F(s-1)} \to 1, \quad s \to \infty.$$

It follows immediately that for any fixed $a, b, c \in \mathbb{N}$ such that $a < b$ holds

$$\frac{P(s) - P(s-b)}{P(s+c) - P(s+c-a)} \to \frac{b}{a} > 1, \quad s \to \infty.$$

The last observation, together with $P(s) - P(s-b) \to \infty$, $s \to \infty$, gives us

$$(P(s) - P(s-b)) - (P(s+c) - P(s+c-a)) \to \infty, \quad s \to \infty$$

which completes the proof. □

Proof of Theorem 2. Let $\lambda \in \operatorname{supp} V$. Clearly $\operatorname{supp} V \subset \lambda + P$.

It is sufficient to show that any $\mathfrak{G}$-module V such that there is $b \in \operatorname{supp} V + P$, $b \notin \operatorname{supp} V$, $b \neq 0$ is not of strictly finite type. Take arbitrary $\lambda \in \operatorname{supp} V$. Let $z =$

$b-\lambda$. Consider a subalgebra $\mathfrak{G}_1 \simeq \mathfrak{W}$ of $\mathfrak{G}$ generated by all e_y, $y\in P\cap \mathbb{Q}z$. Then $V_1 = U(\mathfrak{G}_1)V_\lambda$ is a weight $\mathfrak{G}_1$-module of strictly finite type. It follows from the Mathieu classification of irreducible modules over $\mathfrak{W}$ ([13]) that for any simple weight $\mathfrak{W}$-module $M\neq 0$ with finite-dimensional weight spaces one of the following holds

(1) $\operatorname{supp} M = \mu + \mathbb{Z}$ for some weight μ;
(2) $\operatorname{supp} M = \mathbb{Z}\backslash\{0\}$;
(3) $\operatorname{supp} M = \mu + \mathbb{N}$ for some weight μ;
(4) $\operatorname{supp} M = \mu - \mathbb{N}$ for some weight μ.

This implies that V_1 has as a subquotient some non-trivial simple highest (lowest) weight module and thus is not of strictly finite type by Lemma 2. This contradiction proves our theorem. $\square$

5. Verma modules

In this section we construct and investigate the submodule structure of Verma modules over generalized Witt algebras. From now on we assume that generalized Witt algebra $\mathfrak{G}$ considered is not isomorphic to $\mathfrak{W}$.

We will call a subset $T\subset P$ a Borel subset provided T is a subsemigroup, $-T\cap T=\emptyset$ and $-T\cup T\cup\{0\}=P$. In a natural way one can associate with an arbitrary Borel subset T certain linear order $<_T$ on P defined as follows: $a<_T b$ if and only if $b-a\in T$. A Borel subset T will be called standard if for any $a,b\in T$, $0<_T a<_T b$ there is $n\in\mathbb{N}$ such that $b<_T na$. By [28] this natural correspondence between standard Borel subsets and orders on P satisfying the above condition (Archimed law) is bijective.

With each Borel subset T we associate the following partition of $\mathfrak{G}$:

$$\mathfrak{G} = \mathfrak{G}_T \oplus \mathfrak{H} \oplus \mathfrak{G}_{-T},$$

where $\mathfrak{G}_{\pm T} = \{g\in\mathfrak{G}\colon \operatorname{supp} g\subset \pm T\}$.

Let T be an arbitrary Borel subset. For $a\in\mathfrak{G}_T$, $h\in\mathfrak{H}$, $\lambda\in\mathfrak{H}^*$ and $z\in\mathbb{C}$ set

$$(a+h)(z) = \lambda(h)z.$$

In such a way we define a structure of $U(\mathfrak{H}\oplus\mathfrak{G}_T)$-module on $\mathbb{C}$. The module

$$M(\lambda) = U(\mathfrak{G}) \bigotimes_{U(\mathfrak{H}\oplus\mathfrak{G}_t)} \mathbb{C}$$

will be called a Verma-type module corresponding to T and λ. We will call $M(\lambda)$ a Verma module provided T is standard.

Obviously there exists continuum many of non-conjugated standard Borel subsets in P (the reader can find a criterion of isomorphism for such subsets in [28]).

The standard properties of Verma-type modules (in fact of Verma modules) are described in the following evident lemma.

LEMMA 3. (1) $M(\lambda)$ *is a weight module with* supp $M(\lambda) = \lambda \cup \lambda - T$.

(2) $M(\lambda)$ *has the unique maximal submodule* $N(\lambda)$ *and the unique irreducible quotient* $L(\lambda) \simeq M(\lambda)/N(\lambda)$.

(3) *If* V *is a weight* $\mathfrak{G}$*-module generated by an element* v *such that* $hv = \lambda(h)v$ *for all* $h \in \mathfrak{H}$ *and* $\mathfrak{G}_T v = 0$, *then there exists a canonical epimorphism* $\varphi: M(\lambda) \to V$ such that $\varphi(1 \otimes 1) = v$.

(4) *If* $M(\lambda)$ *is Verma module then*

$$\dim M(\lambda)_\mu = \begin{cases} 1, & \mu = \lambda; \\ \infty, & \mu \in \lambda - T. \end{cases}$$

Submodule structure of Verma modules over $\mathfrak{W}$ (and over Virasoro Lie algebra) is very non-trivial and completely obtained in [3, 19]. It happens that Verma modules over generalized Witt algebra have a rather simple structure described in the following theorem.

THEOREM 3. (1) *If* $\lambda \neq 0$ *then* $M(\lambda)$ *is irreducible*.

(2) *If* $\lambda = 0$ *then* $N(0) = \{v \in M(0), \operatorname{supp} v \cap 0 = \emptyset\}$, *and* $M(0)/N(0) \simeq \mathbb{C}$.

(3) $N(0)$ *is irreducible.*

Proof. First of all we will prove the second part of the theorem. In the case $\lambda = 0$ there exists, according to Lemma 3, an epimorphism $\varphi: M(0) \to \mathbb{C}$, where $\mathbb{C}$ is a trivial $\mathfrak{G}$-module. Hence

$$N(0) = \{v \in M(\lambda), \operatorname{supp} v \cap 0 = \emptyset\}$$

and $M(0)/N(0) \simeq \mathbb{C}$.

To prove the first part consider an element $v \in M(\lambda)_\mu$, $v \neq 0$. We can write it in the form

$$v = \sum_{i=1}^{k} a_i u_i v_\lambda,$$

where v_λ denotes a canonical generator of $M(\lambda)$, $a_i \in \mathbb{C}$, $1 \leqslant i \leqslant k$ and each u_i is of the form $u_i = e_{x_1} e_{x_2} \dots e_{x_{k_i}}$.

For every $i = 1, 2, \dots, k$ denote $P(u_i) = \{x_1, x_2, \dots, x_{k_i}\}$.

Let $N = U(\mathfrak{G})v$. Our goal is to prove $N = M(\lambda)$. Set $K(v) = \max_i k_i$. Denote

$$I = \{i \in \{1, 2, \dots, k\} : k_i = K(v)\} \quad \text{and} \quad P(I) = \bigcup_{i \in I} P(u_i).$$

Choose $x \in P(I)$ such that $x <_T y$ for every $x \neq y \in P(I)$. Then one can choose $y \in P$, $x <_T y$, $y <_T z$ for any $z \in P(I)$, $z \neq x$ such that $e_{-y} v \neq 0$. Continuing this procedure, if necessary, we can assume that $|I| = 1$ and $0 \notin \operatorname{supp} U(\mathfrak{G}_T) N_\mu$.

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Then in the same way one can choose an element $y \in P$ such that $e_{-y}v \neq 0$ and $K(e_{-y}v) < K(v)$. We obtain that $e_z v_\lambda \in N$ for some $z \in P$ and thus $v_\lambda \in N$ since $\lambda \neq 0$.

The last part follows from the proof of the first part. $\square$

Remark 1. In such a way we construct a family of irreducible $\mathfrak{G}$-modules that depends on $n+1$ real parameter. To be more formal, 'almost all' (in a sense of Lebesque measure) such parameters define non-isomorphic irreducible representations (Verma modules) over $\mathfrak{G}$. Moreover, all such modules possess both finite and infinite-dimensional weight spaces and are rather simple for calculations.

Remark 2. It seems that the modules constructed in this section are a direct analogue of Verma modules associated with Kac-Moody Lie algebras in spite of Verma type modules which will be considered in the next section (see [6] to compare the situation with Kac-Moody algebras). We also note that a decomposition of $\mathfrak{G}$ obtained here is not a triangular decomposition in sense of [14] since T is not finitely generated.

Remark 3. One can easily see that Theorem 3 holds also for Verma modules over higher rank Virasoro algebras (those are not isomorphic to classical Virasoro algebra) defined in [15].

6. Modules of Verma type

In this section we investigate a family of modules analogues to Verma type modules over affine Lie algebra induced from non-standard Borel subalgebra (see [5, 6] for more details). We also remark that in the affine case the crucial difference between standard and non-standard Borel subalgebras is a propriety to define a triangular decomposition of the algebra in sense of [14, 20–21]. As was noted in the previous section, even standard Borel subalgebras of generalized Witt algebras do not lead to triangular decomposition and thus the technique of [14] cannot be used in any case.

The modules of Verma type that are not the Verma modules defined in the previous section, correspond to non-standard Borel subsets. Their basic properties are very close to those of Verma modules, but as will be shown, they may have a more complicated submodule structure.

A non-standard Borel subset T corresponds to the linear order $<_T$ on P which is not Archimed. By [28], we can decompose $P = P_1 \oplus P_2$ – the ordered sum of two subgroups $P_1, P_2 \subset P$ in such a way that the restriction of $<_T$ on P_1 is Archimed and P_2 is convex (i.e. $0 < h < h'$ and $h' \in P_2$ implies $h \in P_2$). This leads to a description of classes of Verma type modules (=non-standard Borel subsets) in the lemma below.

We will call a subalgebra $\mathfrak{G}' \subset \mathfrak{G}$ normal provided $\mathfrak{G}'$ is a generalized Witt algebra and the set $P' = \{x \in P : e_x \in \mathfrak{G}'\}$ is convex (i.e. if $\alpha x \in P$ for some $\alpha \in \mathbb{Q}$ and $x \in P'$ then $\alpha x \in P'$).

Combining the arguments above we obtain the following classification of classes of Verma type modules over generalized Witt algebra.

LEMMA 4. (1) *There exists a natural one-to-one correspondence between non-standard Borel subsets and non-Archimed linear orders on the abelian group* P.

(2) *There exists a one-to-one correspondence between Borel subsets* T *and couple of corteges* $(P_1, P_2, \ldots, P_k)$, $(L_1, L_2, \ldots, L_k, L)$ *consisting of convex non-trivial subgroups* $P_i \subset P$ *such that* $P_i \subset P_{i-1}$, $P_i \neq P_{i-1}$, $2 \leqslant i \leqslant k$, *Archimed subsemigroups* L_i *of* $P_{i-1}(P_0 = P)$ *such that* $L_i \cap P_i = \emptyset$, $L_i \cup -L_i \cup P_i = P_{i-1}$ *and some standard Borel subset* L of P_k.

(3) *A couple of corteges above defines standard Borel subset if and only if* $k = 0$.

We fix cortege $(P_1, P_2, \ldots, P_k, L)$ corresponding to the set T and put $P_0 = P$.

It follows immediately from Lemma 4 that there exists a continuum family of non-isomorphic non-standard Borel subsets for any $n > 2$. There is also a countable family of such subsets for $n = 2$ and there are no such subsets for $n = 1$.

We will describe the submodule structure of Verma type modules over generalized Witt algebra separately in the following two cases

(1) $P_k \simeq \mathbb{Z}$.
(2) $P_k \not\simeq \mathbb{Z}$.

To proceed we need the following result.

LEMMA 5. *Let* $M(\lambda)$ *be the Verma type module with a generator* v_λ. *Set*

$$\mathfrak{G}_m = \langle e_x, x \in P_m \rangle, \quad m = 0, 1, \ldots, k.$$

For every $v \in M(\lambda)$ *there exists* $u \in U(\mathfrak{G})$ *such that* $0 \neq uv \in U(\mathfrak{G}_k)v_\lambda$.

Proof. To prove this statement it is sufficient to show that for every $i = 0, 1, \ldots, m-1$ the following assertion is true.

For every $v \in U(\mathfrak{G}_i)v_\lambda$ *there exists* $u \in U(g)$ *such that* $0 \neq uv \in U(\mathfrak{G}_{i+1})v_\lambda$.

If rank $P_{i+1} <$ rank $P_i - 1$ one can prove this using the same arguments as in proof of Theorem 3.

Let rank $P_{i+1} =$ rank $P_i - 1$ and $v \in U(\mathfrak{G}_i)v_\lambda$ be an element of weight μ. We can write v in the form

$$v = \sum_{j=1}^{s} a_j u_j w_j v_\lambda, \tag{3}$$

where $a_j \in \mathbb{C}$, each w_j is a monomial element from $U(\mathfrak{G}_{i+1})$ and $u_j = e_{x_1} e_{x_2} \ldots e_{x_{s_j}}$ for all possible j such that $e_{x_t} \notin U(\mathfrak{G}_{i+1})$ for any $t = 1, 2, \ldots, s_j$.

Since there is only finite number of e_y that appears in (3) and $\mathfrak{G}_{i+1}$ is infinite-dimensional it follows using the same arguments as in the proof of Theorem 3 that for some $z \in P$ such that $\mu + z \in \lambda + P_{i+1}$ holds $e_z v \neq 0$. This completes the proof. □

Using Lemma 5, one can easily obtain the following theorem.

THEOREM 4. *Let $P_k \not\simeq \mathbb{Z}$, then*

(1) *$M(\lambda)$ is irreducible if and only if $\lambda \neq 0$.*
(2) *If $\lambda = 0$, then $N(0) = \mathfrak{G}_{-T}M(0)$ and $M(0)/N(0) \simeq \mathbb{C}$.*
(3) *$N(0)$ is irreducible.*

Proof. By Lemma 5 for every $0 \neq v \in M(\lambda)$ there exists $u \in U(\mathfrak{G})$ such that $0 \neq uv \in U(\mathfrak{G}_k)v_\lambda$. But the $\mathfrak{G}_k$-module $U(\mathfrak{G}_k)v_\lambda$ is a Verma module, hence it is irreducible by Theorem 3 as soon as $\lambda \neq 0$. Thus so is $M(\lambda)$. This proves the first part of the theorem. One can easily reduce the third statement to Theorem 3 in the same way.

The second statement follows from the universal property of $M(\lambda)$. □

THEOREM 5. *Let $P_k \simeq \mathbb{Z}$ and $0 <_T \alpha$ is a generator of P_k. Then*

(1) *$M(\lambda)$ is irreducible if and only if*

$$\lambda \neq -\frac{(m^2-1)\alpha}{24},$$

where $m = 0, 1, 2, \ldots$

(2) *In the case $\lambda = (-(\alpha/24)((6k)^2 - 1))$ for $k \geqslant 1$,*

$$N(\lambda) \simeq M\left(\left(-\frac{\alpha}{24}((6k+2)^2-1)\right)\right).$$

(3) *In the case $\lambda = (-(\alpha/24)((6k+3)^2 - 1))$ for $k \geqslant 0$,*

$$N(\lambda) \simeq M\left(\left(-\frac{\alpha}{24}((6k+9)^2-1)\right)\right).$$

(4) *In the case $\lambda = (-(\alpha/24)((6k+2)^2 - 1))$ for $k \geqslant 0$,*

$$N(\lambda) \simeq M\left(\left(-\frac{\alpha}{24}((6k+10)^2-1)\right)\right).$$

(5) *In the case $\lambda = (-(\alpha/24)((6k+4)^2 - 1))$ for $k \geqslant 0$,*

$$N(\lambda) \simeq M\left(\left(-\frac{\alpha}{24}((6k+8)^2-1)\right)\right).$$

(6) *In the case $\lambda = (-(\alpha/24)((6k \pm 1)^2 - 1))$ for any $\mu = (-(\alpha/24)((6j \pm 1)^2 - 1))$, $j > k$ holds*

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$$M(\mu) \subset M(\lambda).$$

Proof. From Lemma 5 it follows easily that there exists a one-to-one correspondence between submodules of $M(\lambda)$ and $U(\mathfrak{G}_k)$-submodules of the module $M_k(\lambda) = U(\mathfrak{G}_k)v_\lambda$. Since $\mathfrak{G}_k \simeq \mathfrak{W}$ we obtain that $M_k(\lambda)$ is a Verma module over the Witt algebra and our result follows from [3, Theorem 1.9, 1.10].

COROLLARY 1. *Under the conditions of Theorem* 5 *we have the following resolutions* (ε *is a canonical projection*)

$$\text{(1)} \quad \xrightarrow{d_{j-i+1}} M(s_{j-i}) \oplus M(t_{j-i}) \xrightarrow{d_{j-i}} \cdots \xrightarrow{d_{j-2}}$$
$$\xrightarrow{d_{j-2}} M(s_{j-1}) \oplus M(t_{j-1}) \xrightarrow{d_{j-1}} M(\nu_j) \xrightarrow{\varepsilon} L(\nu_j) \to 0$$

for $\nu_j \in \{s_j, t_j\}$ *where*

$$s_j = \left(-\frac{\alpha}{2}(3j^2 + j)\right), \qquad t_j = \left(-\frac{\alpha}{2}(3j^2 - j)\right).$$

$$\text{(2)} \quad 0 \longrightarrow M(\nu_{s+2}) \xrightarrow{d_{s+2}} M(\nu_s) \xrightarrow{\varepsilon} L(\nu_s) \longrightarrow 0$$

for $\nu_s = (-(\alpha/24)((6s)^2 - 1))$, $s \geqslant 1$.

$$\text{(3)} \quad 0 \longrightarrow M(\nu_{s+1}) \xrightarrow{d_{s+1}} M(\nu_s) \xrightarrow{\varepsilon} L(\nu_s) \longrightarrow 0$$

for $\nu_s = (-(\alpha/24)((6s+3)^2 - 1))$, $s \geqslant 0$.

$$\text{(4)} \quad 0 \longrightarrow M(\delta_{s+1}) \xrightarrow{j_{k+1}} M(\gamma_s) \xrightarrow{\varepsilon} L(\gamma_s) \longrightarrow 0$$

for $\gamma_j = (-(\alpha/24)((6j+2)^2 - 1))$, $\delta_j = (-(\alpha/24)((6j+4)^2 - 1))$, $j \geqslant 0$.

$$\text{(5)} \quad 0 \longrightarrow M(\gamma_{s+1}) \xrightarrow{j_{k-1}} M(\delta_s) \xrightarrow{\varepsilon} L(\delta_s) \longrightarrow 0$$

for γ_s, δ_s *defined above*.

Proof. Follows immediately from Lemma 5 and [19, Theorem A]. □

PROPOSITION 1. *Let* $\lambda \neq 0$ *and* $x \in -T \backslash P_k$. *Then*

$$\dim L(\lambda)_{\lambda+x} = \infty.$$

Proof. Consider $U(\mathfrak{G}_k)$-module $M_k(\lambda)$ defined in proof of Theorem 5. By Lemma 2

$$\sup_{\mu \in \operatorname{supp} L_k(\lambda)} \dim L_k(\lambda)_\mu = \infty.$$

Thus for every $s \in \mathbb{N}$ we can choose some μ and the subspace $V \subset L_k(\lambda)_\mu$ such that V does not intersect with the maximal submodule of $M_k(\lambda)$ and $\dim V > s$. It follows immediately that there exists $z \in P$ such that

$$e_z V \subset M(\lambda)_{\lambda+x}.$$

Moreover, $e_z V \cap N(\lambda) = 0$. Thus

$$\dim L(\lambda)_{\lambda+x} \geqslant s. \qquad \square$$

Following Proposition 1 and Corollary 1 one can write the character formula for $L(\lambda)$ using the character for the irreducible quotients of the Verma modules over Witt algebra ([19]). This completes the submodule structure description of Verma-type modules over generalized Witt algebra.

For a $\mathfrak{G}$-module V we will denote

$$\mathrm{F}\operatorname{supp} V = \{\lambda \in \operatorname{supp} V : \dim V_\lambda < \infty\}.$$

Note, that in the last case ($P_k \simeq \mathbb{Z}$) we obtain a continuum family of irreducible $\mathfrak{G}$-modules V for which both $\mathrm{F}\operatorname{supp} V$ and $\operatorname{supp} V \setminus \mathrm{F}\operatorname{supp} V$ and infinite. Obviously such an effect is impossible for Verma modules. For the analogs in the Kac–Moody case, see [6].

7. Generalized Verma modules

In this section we construct another class of $\mathfrak{G}$-modules analogues to generalized Verma modules over simple finite-dimensional Lie algebras ([7]) and investigate their irreducibility.

We will call a subset $T \subset P$ parabolic if T is subsemigroup and $-T \cup T = P$.

Analogously to Sections 5 and 6 we have the following interpretation of parabolic subsets in terms of partial orders on T.

LEMMA 6. (1) *There exists a natural one-to-one correspondence between parabolic subsets and linear pre-orders on the abelian group P.*

(2) *There exists a one-to-one correspondence between Borel subsets T and couple of corteges $(P_1, P_2, \dots P_k)$, $(L_1, L_2, \dots, L_k)$ consisting of convex non-trivial subgroups $P_i \subset P$ such that $P_i \subset P_{i-1}$, $P_i \neq P_{i-1}$, $2 \leqslant i \leqslant k$ and Archimed subsemigroups L_i of P_{i-1} $(P_0 = P)$ such that $L_i \cap P_i = \emptyset$, $L_i \cup -L_i \cup P_i = P_{i-1}$.*

For a parabolic subset T consider the following subalgebras

$$\mathfrak{G}_T = \langle e_x : x \in -T \cap T\rangle \quad \text{and} \quad \mathfrak{G}^T = \langle e_x : x \in T\rangle.$$

Let V be an irreducible $\mathfrak{G}_T$-module. Setting $e_x v = 0$ for all $v \in V$ and $x \in T \setminus -T$, we make V into $\mathfrak{G}^T$-module. The module

$$M(V) = U(\mathfrak{G}) \bigotimes_{U(\mathfrak{G}^T)} V$$

https://doi.org/10.1023/A:1000531924778 Published online by Cambridge University Press

is called the generalized Verma module associated with V and T.

Remark 4. In the case $-T \cap T = \{0\}$ we get the definition of the Verma-type module.

Suppose that $\mathfrak{G}_T$ is non-trivial.

We will call a generalized Verma module $M(V)$ finitely-dense provided V is a non-trivial module of strictly finite type. Such modules seem to be an analog of the stratified modules over simple finite-dimensional Lie algebras ([2]).

EXAMPLE 2. One can choose the module V to be a non-trivial simple module from the intermediate series for $\mathfrak{G}$ (see [24]).

It happens that finitely-dense modules have a rather simple structure.

THEOREM 6. *Every finitely-dense generalized Verma module is irreducible.*

Proof. Let $M(V)$ be a finitely-dense generalized Verma module over $\mathfrak{G}$, $0 \neq v \in M(V)_\lambda$. Using the same arguments as in Lemma 5, one can show that there exists an element $u \in U(\mathfrak{G})$ such that $0 \neq uv \in V$. Thus there exists a one-to-one correspondence between the submodules of $M(V)$ and the $\mathfrak{G}_T$-submodules of V. Since V is irreducible so is $M(V)$. □

Finitely-dense generalized Verma modules present another class of irreducible $\mathfrak{G}$-modules having many finite and infinite-dimensional weight spaces.

Acknowledgements

The work is partially supported by ISSEP, grant PSU051083. The author would like to thank Prof. A.M. Vershik, Prof. Yu. A. Drozd and Prof. V.M. Futorny for helpful discussions that led to the improvements of this paper.

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