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CUSPS, TRIANGLE GROUPS AND HYPERBOLIC 3-FOLDS

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Abstract

We provide a number of explicit examples of small volume hyperbolic 3-manifolds and 3-orbifolds with various geometric properties. These include a sequence of orbifolds with torsion of order q interpolating between the smallest volume cusped orbifold ($q = 6$) and the smallest volume limit orbifold ($q \to \infty$), hyperbolic 3-manifolds with automorphism groups with large orders in relation to volume and in arithmetic progression, and the smallest volume hyperbolic manifolds with totally geodesic surfaces. In each case we provide a presentation for the associated Kleinian group and exhibit a fundamental domain and an integral formula for the co-volume. We discuss other interesting properties of these groups.

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0. Introduction

An *orbifold* is a space locally modelled on $\mathbb{R}^n$ modulo a finite group action. The canonical example is the orbit space of a group acting by homeomorphisms discontinuously on a manifold. In this paper we will basically be concerned with orientable hyperbolic 3-orbifolds of finite volume. These are the orbit spaces $\mathbf{Q}$ of discrete groups of orientation-preserving isometries of hyperbolic 3-space $\mathbf{H}^3$. Thus $\mathbf{Q} = \mathbf{H}^3/\Gamma$ where Γ is a co-finite volume discrete subgroup of $\mathrm{Isom}^+(\mathbf{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$. Discrete subgroups of $\mathrm{Isom}^+(\mathbf{H}^3)$, which are not

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virtually abelian, are called *Kleinian groups* ; see [3, 17, 21] for the basic definitions and properties of these groups. We say an orbifold **Q** is *cusped* if it is noncompact and of finite volume. The *singular set* of **Q** is the set of non-manifold points, that is, the projection to the orbit space of the set of fixed points of elements of $\Gamma\backslash\{\text{identity}\}$. The *degree* of a point in the singular set is the order of branching about that point.

A fundamental invariant of a hyperbolic 3-orbifold is its volume. The celebrated Mostow rigidity theorem [20] says that volume is a topological invariant. Isomorphic co-finite volume Kleinian groups have isometric orbit spaces. In two dimensions the signature formula (the Riemann-Hurwitz formula) gives complete information on the possible volumes of hyperbolic 2-orbifolds (the orbit spaces of Fuchsian groups). The situation is very different for hyperbolic 3-orbifolds: here there is no such formula, nor even a version of the Chern-Gauss-Bonnet formula in the manifold case (apart from Borel's formula in the general arithmetic case). From the work of Thurston and Jørgensen [21], we know that the set of volumes is well-ordered. In particular there is an orbifold of minimal volume, an orbifold of smallest volume which is the limit of other orbifold volumes, a smallest volume cusped orbifold, a smallest volume orbifold with singular set of degree p, and so forth. Some of these have been identified: for instance, the smallest volume cusped orbifold [18], smallest arithmetic orbifold [5], smallest limit orbifold [1] and smallest orbifold with singular set of degree 6 [8]. See also [14] for the smallest volume hyperbolic manifold with totally geodesic boundary (we shall use this knowledge later). Note however that the smallest volume orbifold has not yet been identified, although lower bounds for this volume are known, see [19, 9] for example. The latter estimates are off by at most one order of magnitude.

In this paper we provide a number of interesting examples of small volume hyperbolic 3-orbifolds. For instance we provide an infinite family of co-finite volume discrete groups $\{\Gamma^{o}_{1,6}(q) : q \geq 6\}$, whose orbit spaces are orbifolds which interpolate naturally between the smallest volume cusped orbifold and the smallest volume limit orbifold. Each group $\Gamma_{1,6}(q)$ has an elliptic element of order q. In [9] we conjecture these groups to be of minimal co-volume with respect to this property, and provide lower bounds for the volume of all such examples. It follows from the results here that this conjecture is asymptotically correct; see Section 3. We give presentations and co-volumes of these groups and show them to be 3-generator groups. It is surely the case that $\Gamma^{o}_{1,6}(7)$ is the smallest co-volume Kleinian group with a Fuchsian subgroup. Also we show that the torsion free subgroup of smallest index in the group $\Gamma^{o}_{1,6}(12)$ is actually the smallest co-volume Kleinian group whose orbit space contains a totally geodesically embedded surface (this follows from [14]).

We also present a naturally related sequence of examples (based on a different sequence of discrete groups) of hyperbolic 3-manifolds $N^3(q)$ whose volume growth is approximately $12q$ and whose automorphism groups have order $24q$ for q sufficiently large. (In fact the orders form an arithmetic progression). These manifolds contain totally geodesic surfaces (Accola-Maclachlan curves) and provide small volume examples of hyperbolic manifolds with various geometric properties.

Other of our examples are also interesting. We give two infinite families of discrete co-finite volume Kleinian groups, $\{\Gamma^o_{4,6}(q) : q \geq 6\}$ and $\{\Gamma^o_{2,4}(q) : q \geq 4\}$, each containing a single conjugacy class of elliptic element of order p and with the same (finite) limit volume; no two of the groups are isomorphic, yet the families have the same geometric limits. One understands of course that these orbifolds are obtained by Dehn surgeries on the limit orbifold; however our approach is completely explicit. Basically we show geometrically how one can open up the cusps of various cusped tetrahedral orbifolds using a truncating procedure, by slicing off an infinite volume piece with a triangle group. This produces orbifolds of successively larger volume with a finite volume limit. We carry out this procedure for all the nine singly-cusped tetrahedral orbifolds. We hope to return to the others (the fourteen with two or more cusps) in a sequel.

We would like to thank S. Kojima for helpful conversations which led to some of the ideas here, and T. Marshall for reading and correcting an earlier version.

1. The basic construction

There are a number of basic models for hyperbolic geometry. In what follows we shall use two models: the upper half space

$$\mathbf{H}^3 = \{(x_1, x_2, x_3) : x_3 > 0\}$$

with the metric $ds = |dx|/x_3$ of constant curvature -1, and the unit ball

$$\mathbf{B}^3 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < 1\}$$

with the metric $ds = 2|dx|/(1 - |x|^2)$, also of constant curvature -1. In either model we denote hyperbolic distance by $\rho(x, y)$. The usual stereographic

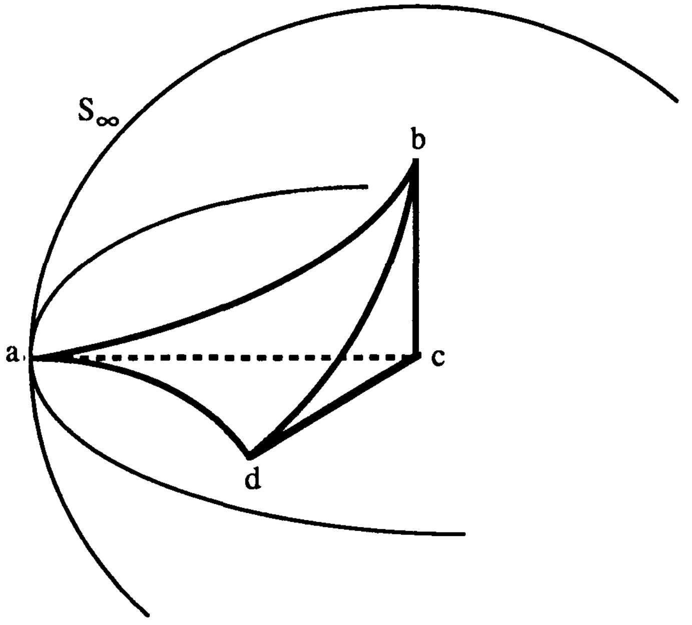

FIGURE 1. A hyperbolic tetrahedron.

projection provides an isometry between these spaces and we shall use them more or less interchangably; see [3] for details. In the first case $\overline{\mathbb{R}}^2 = \partial\mathbf{H}^3$ and in the second case $S^2 = \partial\mathbf{B}^3$, is called the *sphere at infinity* and denoted S_∞.

Let **T** be a hyperbolic tetrahedron, that is a convex hyperbolic polyhedron with four faces (sides) A, B, C and D and vertices **a**, **b**, **c** and **d** each opposite to a face, as illustrated in Figure 1.

Let $\sigma_a, \dots, \sigma_d$ denote the reflections in the faces A, ..., D respectively and let $\Gamma(\mathbf{T})$ denote the group of hyperbolic isometries generated by reflections in the faces of **T**, that is

$$\Gamma(\mathbf{T}) = \langle\sigma_a, \sigma_b, \sigma_c, \sigma_d\rangle.$$

Define $\Gamma^o(\mathbf{T})$ to be the index two normal subgroup preserving orientation. It is quite easy to see that $\Gamma^o(\mathbf{T})$ is generated by the products $\{\sigma_a\sigma_b, \sigma_a\sigma_c, \sigma_a\sigma_d\}$. As yet, we have not supposed that the group $\Gamma(\mathbf{T})$ is discrete. Notice however that the groups $\Gamma(\mathbf{T})$ and $\Gamma^o(\mathbf{T})$ are simultaneously discrete or nondiscrete.

Associated to each tetrahedron whose dihedral angles are integral divisors of π there is a Coxeter diagram, a graph constructed as follows: there are four vertices, each corresponding to a face of **T**, and an edge connecting two vertices indicates that the dihedral angle between the two faces is $\pi/(\text{number of edges} + 2)$. See Figure 2 for example.

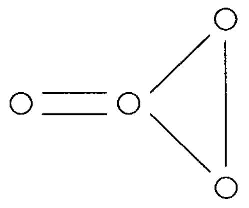

FIGURE 2. Coxeter Diagram.

Note that if there is no edge joining two vertices, then the dihedral angle is $\pi/2$. Up to the obvious symmetry induced by relabelling, each hyperbolic tetrahedron determines a unique Coxeter diagram, and conversely each Coxeter diagram determines at most one hyperbolic tetrahedron. (Naturally some diagrams cannot be realised, although it is not too difficult to see what the restrictions may be; however this will not concern us here).

Suppose now, as illustrated in Figure 1, that **a** is an ideal vertex of **T** (that is, $\mathbf{a} \in S_\infty$). Then necessarily the sum of the dihedral angles $\angle$BC+$\angle$BD+$\angle$CD is π. Let $\Gamma_a = \langle \sigma_b, \sigma_c, \sigma_d \rangle$. Using the upper half-space model $\mathbf{H}^3$ of hyperbolic geometry and normalising so that $\mathbf{a} = \infty$, one easily sees that Γ_a acts as a group of Euclidean isometries on any plane parallel to $\partial \mathbf{H}^3$, and thus $\Gamma_a | S_\infty$ is conjugate to a group of Euclidean isometries of the plane generated by reflections in the sides of a triangle. There is a complete classification of such discrete groups (as elementary Kleinian groups); see [3, 17]. They are simply $\mathbb{Z}_2$ extensions of the classical (2,3,6), (2,4,4) and (3,3,3) Euclidean triangle groups. In particular, the dihedral angles are of the form $(\pi/2, \pi/3, \pi/6)$, $(\pi/2, \pi/4, \pi/4)$ or $(\pi/3, \pi/3, \pi/3)$.

As is well known, there is a complete classification of those hyperbolic tetrahedra **T** with the property that the group $\Gamma(\mathbf{T})$ is discrete: see [15, 22, 7]. A good place to find this list is [19; pp 201-203]. There are nine closed examples (no ideal vertices) and 23 with at least one ideal vertex. Here we will be concerned with those groups with a single ideal vertex. There are exactly nine such groups and we list them below in order of increasing volume.

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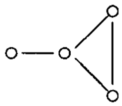
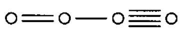
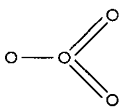
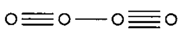
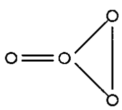
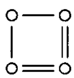
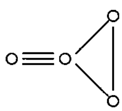
	Coxeter Diagram	Volume	Cusp Type
1.		$V_1 \approx 0.042289$	$\Delta(2,3,6)$
2.		$V_2 \approx 0.076330$	$\Delta(2,4,4)$
3.		$V_3 \approx 0.084578$	$\Delta(3,3,3)$
4.		$V_4 \approx 0.105723$	$\Delta(2,3,6)$
5.		$V_5 \approx 0.152661$	$\Delta(2,4,4)$
6.		$V_6 \approx 0.171502$	$\Delta(2,3,6)$
7.		$V_7 \approx 0.211446$	$\Delta(3,3,3)$
8.		$V_8 \approx 0.305332$	$\Delta(2,4,4)$
9.		$V_9 \approx 0.507471$	$\Delta(3,3,3)$

We now show how each cusped tetrahedron is the limit of a natural one-parameter family of hyperbolic prisms (five-sided hyperbolically convex polyhedra). Let **T** be one of the hyperbolic tetrahedra above. Again use the upper half-space model of hyperbolic geometry and normalise so the cusp is at ∞. Choose a pair of faces with dihedral angle π/r and let $\mathbf{T}_\theta$, $0 \leq \theta \leq \pi/r$, be that convex hyperbolic polygon obtained by continuously decreasing the angle π/r to θ, while fixing all the other dihedral angles. This is how to construct $\mathbf{T}_\theta$. Choose an infinite edge for which the dihedral angle is to remain constant. The faces meeting this edge will remain hyperplanes of $\mathbb{R}^3$ perpendicular to $\mathbb{R}^2$ and

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the bounded face of **T** lies on a sphere again perpendicular to $\mathbb{R}^2$. A face of $\mathbf{T}_\theta$ will lie on this same sphere Σ and so two of the dihedral angles on this face will remain constant as illustrated in Figure 3. We have only to determine the third face. Since we are decreasing the dihedral angle and since initially the angle sum at the ideal vertex was π, the third face must lie on a sphere perpendicular to $\mathbb{R}^2$. Thus there are three free parameters corresponding to the centre of this sphere (in $\mathbb{R}^2$) and its radius.

In the illustration below we are varying the dihedral angle along **ac**. The dihedral angles along **bc** and **bd** remain fixed (as the face **bcd** is constrained to lie on the same sphere).

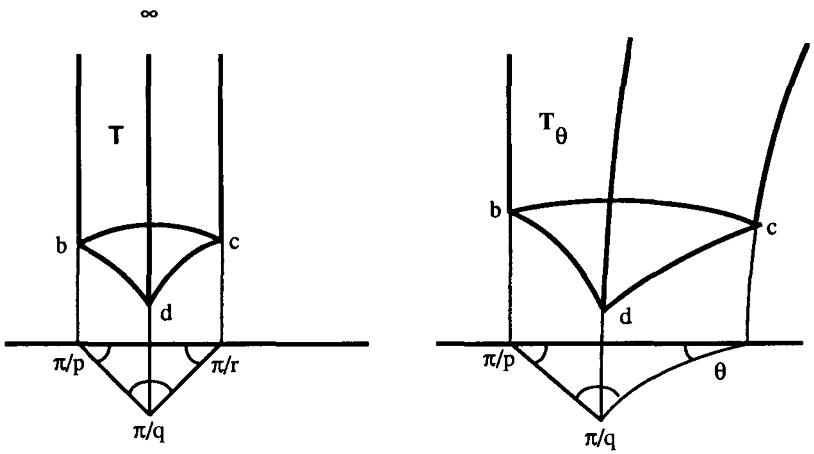

FIGURE 3. Opening the cusp of a hyperbolic tetrahedron

We have three things to arrange; namely the two dihedral angles **ad** and **bc** should remain constant and the dihedral angle **ac** must become θ. Let L denote the set of points $z \in \mathbb{R}^2$ which are the center of some sphere whose intersection with Σ and the hyperplane Π containing **bd** have the same dihedral angles as **T**. Then L is a simple unbounded curve in $\mathbb{R}^2$ passing through $\Sigma \cap \Pi$. To see this we may normalise so Σ and Π are hyperplanes, then, since there is one such point z, it is clear by scaling that L is a line. Here it is important that the sum of the dihedral angles at the vertex **d** exceeds π. Undoing the normalisation, the line becomes a curve which we know in our initial situation passes through ∞! (Here, as we undo the normalisation the spheres are mapped to spheres with the same angle intersection properties, however their Euclidean centers are not necessarily preserved.) One could also do this using the inversive product [3] and solving some complicated equations. Now as we vary the centre of the sphere along this line we see that as $z \to \infty$ (in the correct direction) the dihedral angle of intersection with the remaining face (our angle θ) tends to π/r (the initial angle). As z moves along this curve towards a point of intersection of Σ and Π the dihedral angle decreases to a point of tangency, with dihedral angle 0, and thereafter the sphere does not meet the hyperplane. Actually, in all but the case with Coxeter diagram 8, two of the dihedral angles on the face opposite the ideal vertex are $\pi/2$, making the process somewhat easier to visualise.

Using the ball model of hyperbolic geometry and extending the polyhedron in the obvious fashion, one can see that opening the cusp in this manner amounts to moving the ideal vertex beyond the sphere at infinity and changing one of the dihedral angles. The sum of the angles at one of the finite vertices is decreasing and this vertex can be thought of as moving toward the sphere at infinity.

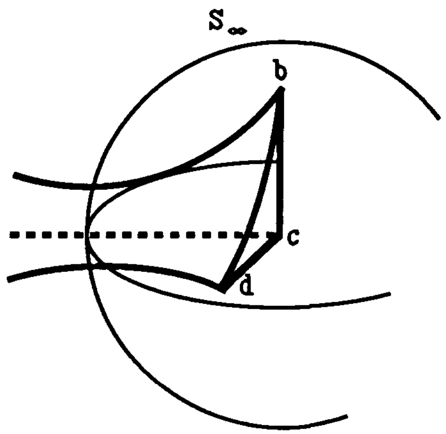

FIGURE 4

In this way we can see that there is a '$(\pi/p, \pi/q, \theta)$-triangle' subtended at infinity. We want to show that we can truncate this polygon by a hyperbolic hyperplane which is perpendicular to each of the three unbounded faces. To see this it is easiest to use the half-space model.

Let ß be the face of $\mathbf{T}_\theta$ which corresponds to the finite face of $\mathbf{T}$, that is, the face opposite the ideal vertex of $\mathbf{T}$. In all but Case 8 listed above, two of the dihedral angles at which ß meets the remaining faces of $\mathbf{T}_\theta$ are $\pi/2$ (since, by construction, they are the same as those of $\mathbf{T}$). Let v be the vertex where the two dihedral angles of $\pi/2$ meet. Move $\mathbf{T}_\theta$ around by an isometry so that the two

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unbounded faces (other than ß) which meet at v are hyperplanes of $\mathbb{R}^3$, and let v' be the point of $\mathbb{R}^2 = \partial\mathbf{H}^3$ which is the intersection of these two hyperplanes. Then ß is a portion of the surface of a sphere S centered at v'. Let Σ be the remaining unbounded face. Then the dihedral angle between ß and Σ is at most $\pi/3$ (because in no case are all the dihedral angles $\pi/2$). Increasing the radius of S we see that eventually the dihedral angle tends to π, while initially it is less than or equal to $\pi/3$. Thus, at some intermediate value it is $\pi/2$.

For Case 8, we need a slightly different argument. Two of the dihedral angles at which ß meets the unbounded faces are $\pi/3$ and the other is $\pi/2$. It is relatively clear how to construct the truncation, and we leave this to the reader.

After truncating $\mathbf{T}_\theta$ in this manner, we are left with a hyperbolic prism. Possibly some of the vertices b, c, d are ideal or are even 'beyond the sphere at infinity'. In the finite volume case, they will all be finite or ideal.

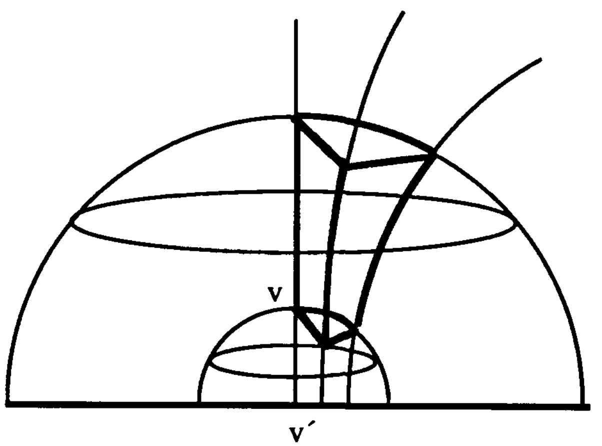


FIGURE 5. Truncating $\mathbf{T}_\theta$.

We label the additional three vertices of this polyhedron by x, y and z, as illustrated. We will now consider the group generated by reflections in this prism. As the process of opening up a cusp of a hyperbolic tetrahedron can be done in (possibly) three different ways for each distinct tetrahedron, we introduce some notation to clarify the group we are considering. For $i = 1, 2, \ldots, 8$, and $p \in \{2, 3, 4, 6\}$, and $0 \leq \theta \leq \pi/p$, let $\Gamma_{i,p}(\theta)$ be the group of hyperbolic isometries generated in the prism obtained by opening up the cusp of the tetrahedron with the ith Coxeter diagram (as listed above) along the edge going out to the cusp with dihedral angle π/p by an angle θ.

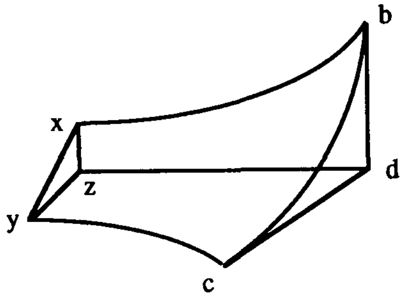


FIGURE 6. A hyperbolic prism

Notice that there is possible ambiguity, arising from the fact that there may be two edges with the same dihedral angle running out to the ideal vertex. This does not occur in Cases 5 and 8, by symmetry. In Cases 2, 3, 7 and 9 there is a distinction to be made, and there are two possibilities (again by symmetry). In each case however, one of the stabilisers of a finite vertex is the dihedral group (2,2,3) or (2,2,4), corresponding to the edge that meets the vertex at which two of the dihedral angles are $\pi/2$ (the other vertices having A_4, S_4 or A_5 as stabilisers). In these cases we will indicate using a prime (′) when we mean decrease the angle along the edge whose end stabilizer is not dihedral, as for instance $\Gamma_{2,4}(\theta)$ and $\Gamma'_{2,4}(\theta)$ are distinct. The former opened along the edge **ab** and the latter along **ad**, when the Coxeter diagram is labelled lexicographically. Also note that $\Gamma_{i,p}(\pi/p) = \Gamma(\mathbf{T}_i)$. The edges that end in dihedral stabilisers will provide infinite families of finite co-volume discrete groups.

We continue to denote the reflections in the faces A, B, C and D from **T** by $\sigma_a, \dots, \sigma_d$. The reflection in the additional face which was obtained by truncation is σ_e. We now have the following.

THEOREM 1.1. *The group $\Gamma_{i,p}(\theta)$ is discrete if and only if $\theta = \pi/q$ for some $q \geq p$. In this case the subgroup $\Delta'(i, p, q) = \langle \sigma_b, \sigma_c, \sigma_d \rangle$ is a $\mathbb{Z}_2$ extension of a hyperbolic triangle group $\Delta(i, p, q)$. If $\mathbf{T}_i$ has a cusp of signature (p, s, t), then $\Delta(i, p, q)$ is the (q, s, t)-triangle group. Moreover the group $\Gamma_{i,p}(\pi/q)$ has cofinite volume if and only if (i, p, q) is one of the following triples :*

(i) $(i, 6, q)$ *for* $i = 1, 4, 6$ *and* $q \geq 6$

(ii) $(i, 3, q)$ *for* $i = 3, 7, 9$ *and* $q \geq 3$

(iii) $(2, 4, q)$ *for* $q \geq 4$

(iv) $(5, 2, q)$ *for* $q \geq 3$

(v) $(1, 2, q)$ *or* $(2, 2, q)$ *for* $2 \leq q \leq 6$

(vi) $(1, 3, q)$ *or* $(3, 3, q)'$ *for* $3 \leq q \leq 6$

(vii) $(5, 4, q)$ *or* $(2, 4, q)'$ *for* $4 \leq q \leq 6$

(viii) $(4, 2, q)$ *for* $2 \leq q \leq 4$

(ix) $(4, 3, q)$ *or* $(7, 3, q)'$ *for* $q = 3, 4$

(x) $(6, 2, q)$ *or* $(8, 2, q)$ *for* $q = 2, 3$

(xi) $(6, 3, 3)$, $(9, 3, 3)'$ *or* $(8, 4, 4)$.

(Here the prime denotes $\Gamma'_{i,p}(\pi/q)$.*)*

PROOF. It easily follows from the Poincaré polyhedral theorem, see [17; Theorem H11], that if $\theta = \pi/q$ then the group is discrete. As the face E bounded by x, y and z is perpendicular to the sides B,C and D, the group generated by $\sigma_b, \sigma_c, \sigma_d$ is (conjugate to) a group of reflections in a hyperbolic triangle. This group is discrete if and only if $\theta = \pi/q$. (Notice that initially the sum of angles of this triangle is π, but as soon as q exceeds p the sum of angles is less than π.) One finds that the sum of the dihedral angles at each of the vertices of the prism is at least π only in the case of the indicated triples.

Since all the cusps are of the form (2, 3, 6), (2, 4, 4) or (3, 3, 3), we see that the triangle groups occuring above are the infinite families

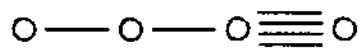
$$(2, 3, p),\ p \geq 7,\quad (2, 4, p),\ p \geq 5,\quad (3, 3, p),\ p \geq 4,\quad (4, 4, p),\ p \geq 3,$$

together with the groups (2, 5, 6), (2, 6, 6), (3, 4, 6), (3, 5, 6), (3, 6, 6).

Henceforth we shall write $\Gamma_{i,p}(q)$ for $\Gamma_{i,p}(\pi/q)$.

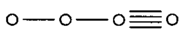
2. Presentations

The Poincaré theorem also provides us with a presentation for each of these groups. We give a complete list of the presentations in an appendix, and give here only a presentation for the groups $\Gamma_{1,p}(q)$, $p = 2, 3, 6$ and $q \geq 7$. As our primary interest is in the orientation-preserving subgroup, we also give a presentation for all the orientation-preserving subgroups of index 2. We indicate the families of groups obtained by opening up a particular tetrahedron by giving the Coxeter diagram of the tetrahedron.

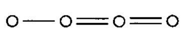


$$\Gamma_{1,6}(q) \cong \langle a, b, c, d, e : a^2 = b^2 = c^2 = d^2 = e^2 = (eb)^2 = (ec)^2 = (ed)^2 = 1$$
$$(ab)^3 = (ac)^2 = (ad)^2 = (bc)^3 = (bd)^2 = (cd)^q = 1\rangle$$
$$\Gamma_{1,3}(q) \cong \langle a, b, c, d, e : a^2 = b^2 = c^2 = d^2 = e^2 = (eb)^2 = (ec)^2 = (ed)^2 = 1$$
$$(ab)^3 = (ac)^2 = (ad)^2 = (bc)^q = (bd)^2 = (cd)^6 = 1\rangle$$
$$\Gamma_{1,2}(q) \cong \langle a, b, c, d, e : a^2 = b^2 = c^2 = d^2 = e^2 = (eb)^2 = (ec)^2 = (ed)^2 = 1$$
$$(ab)^3 = (ac)^2 = (ad)^2 = (bc)^3 = (bd)^q = (cd)^6 = 1\rangle$$

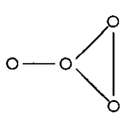
For the orientation-preserving subgroups, we have simplified the presentation a little. We subsequently will show that for all but a few exceptions, the groups $\Gamma^{o}_{i,p}(q)$ are actually 3-generator groups. We start out by letting $x = ab$, $y = bc$, $z = cd$, $w = de$. Clearly these four elements generate the index 2 subgroup consisting of those words of even length and thus we obtain the presentations listed below:

1. 

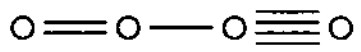
$$\Gamma^{o}_{1,6}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^3 = (xy)^2 = (xyz)^2 = y^3 = (yz)^2 = z^q = 1\rangle$$
$$\Gamma^{o}_{1,3}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^3 = (xy)^2 = (xyz)^2 = y^q = (yz)^2 = z^6 = 1\rangle$$
$$\Gamma^{o}_{1,2}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^3 = (xy)^2 = (xyz)^2 = y^3 = (yz)^q = z^6 = 1\rangle$$

2. 

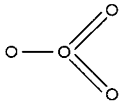
$$\Gamma^{o}_{2,2}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^3 = (xy)^2 = (xyz)^2 = y^4 = (yz)^q = z^4 = 1\rangle$$
$$\Gamma^{o}_{2,4}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^3 = (xy)^2 = (xyz)^2 = y^4 = (yz)^2 = z^q = 1\rangle$$
$$\Gamma^{o\prime}_{2,4}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^3 = (xy)^2 = (xyz)^2 = y^q = (yz)^2 = z^4 = 1\rangle$$

3. 

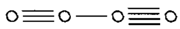
$$\Gamma^{o}_{3,3}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^3 = (xy)^2 = (xyz)^2 = y^3 = (yz)^3 = z^q = 1\rangle$$
$$\Gamma^{o\prime}_{3,3}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^3 = (xy)^2 = (xyz)^2 = y^q = (yz)^3 = z^3 = 1\rangle$$

4. 

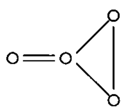
$\Gamma^{o}_{4,6}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^4 = (xy)^2 = (xyz)^2 = y^3 = (yz)^2 = z^q = 1 \rangle$
$\Gamma^{o}_{4,3}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^4 = (xy)^2 = (xyz)^2 = y^q = (yz)^2 = z^6 = 1 \rangle$
$\Gamma^{o}_{4,2}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^4 = (xy)^2 = (xyz)^2 = y^3 = (yz)^q = z^6 = 1 \rangle$

5. 

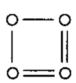
$\Gamma^{o}_{5,4}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^3 = (xy)^2 = (xyz)^2 = y^q = (yz)^4 = z^2 = 1 \rangle$
$\Gamma^{o}_{5,2}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^3 = (xy)^2 = (xyz)^2 = y^4 = (yz)^4 = z^q = 1 \rangle$

6. 

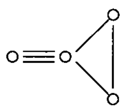
$\Gamma^{o}_{6,6}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^5 = (xy)^2 = (xyz)^2 = y^3 = (yz)^2 = z^q = 1 \rangle$
$\Gamma^{o}_{6,3}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^5 = (xy)^2 = (xyz)^2 = y^q = (yz)^2 = z^6 = 1 \rangle$
$\Gamma^{o}_{6,2}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^5 = (xy)^2 = (xyz)^2 = y^3 = (yz)^q = z^6 = 1 \rangle$

7. 

$\Gamma^{o}_{7,3}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^4 = (xy)^2 = (xyz)^2 = y^3 = (yz)^3 = z^q = 1 \rangle$
$\Gamma^{o\prime}_{7,3}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^4 = (xy)^2 = (xyz)^2 = y^q = (yz)^3 = z^3 = 1 \rangle$

8. 

$\Gamma^{o}_{8,4}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^3 = (xy)^2 = (xyz)^3 = y^4 = (yz)^2 = z^q = 1 \rangle$
$\Gamma^{o}_{8,2}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^3 = (xy)^2 = (xyz)^3 = y^4 = (yz)^q = z^4 = 1 \rangle$

9. 

$\Gamma^{o}_{9,3}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^5 = (xy)^2 = (xyz)^2 = y^3 = (yz)^3 = z^q = 1 \rangle$
$\Gamma^{o\prime}_{9,3}(q) \cong \langle x, y, z, w : (yzw)^2 = (zw)^2 = w^2 = x^5 = (xy)^2 = (xyz)^2 = y^q = (yz)^3 = z^3 = 1 \rangle$

LEMMA 2.1. *If $\alpha, \beta, \gamma, \delta$ are group elements satisfying the relations*

$$\alpha^2 = \beta^2 = \gamma^2 = \delta^2 = (\alpha\beta)^2 = (\beta\gamma)^2 = (\gamma\delta)^2 = (\delta\alpha)^k = 1,$$

where k is odd, then $\langle \alpha\beta, \beta\gamma, \gamma\delta, \delta\alpha \rangle$ is a 2-generator group, generated for example by $\gamma\alpha$ and $\beta\delta$.

PROOF. Let $x = \beta\alpha$, $y = \beta\delta$ and $u = \beta\gamma$. Then $\alpha\beta = x^{-1} = x$, $\gamma\delta = uy$ and $\delta\alpha = y^{-1}x = (xy)^{-1}$ and the given relations imply $x^2 = u^2 = (uy)^2 = (xy)^k = 1$. Note also that $\gamma\beta = ux$. Next $[ux, y] = (ux)^{-1}y^{-1}uxy = x^{-1}u^{-1}y^{-1}uxy = xuy^{-1}uxy = xyxy$ (since $uy^{-1}u = y$), and therefore $y^{-1}x = (xy)^{k-1} = [ux, y]^{(k-1)/2}$. As also $u = (ux)x^{-1}$, it follows that

$$\langle x, y, u\rangle = \langle ux, y\rangle$$

and therefore $\langle \alpha\beta, \beta\gamma, \gamma\delta, \delta\alpha\rangle = \langle \gamma\alpha, \beta\delta\rangle$.

COROLLARY 2.2. *The groups $\Gamma^{o}_{i,p}(q)$ are 3-generator groups for $i = 1, 2, 3, 5, 6, 8$ and 9 and all relevant p and q.*

PROOF. It suffices to show the reflection groups $\Gamma_{i,p}(q)$ are 4-generator groups. For this consider the presentations given in the appendix and take $(\alpha, \beta, \gamma, \delta) = (b, e, c, a)$. The relations of Lemma 2.1 are satisfied with $k = 3$ for $\Gamma_{i,p}(q)$, $i = 1, 2, 3, 5$ and 8, and with $k = 5$ for $\Gamma_{i,p}(q)$, $i = 6$ and 9, and the result follows from the lemma.

Here is a reduced presentation for the group $\Gamma^{o}_{1,6}(q)$.

PROPOSITION 2.3. *The group $\Gamma^{o}_{1,6}(q)$ has a complete presentation given by*

$$\begin{aligned}\Gamma^{o}_{1,6}(q) \cong \langle a, b, c : \ & a^2 = b^2 = c^p = (bc)^2 = (c^{-1}acac^{-1})^2 \\ & = (ac^{-1}acac^{-1})^2 = (c^{-1}acac^{-1}b)^3 = 1\rangle.\end{aligned}$$

In particular the subgroup $\langle b, c\rangle$ is the dihedral group D_p of order $2p$, and thus $\Gamma^{o}_{1,6}(q)$ is a factor group of the group $D_p * \mathbb{Z}_2$.

PROOF. The presentation can be deduced form the lemma and elimination of redundant relations. The subgroup $\langle b, c\rangle$ is a discrete Kleinian group generated by two elements of order two and is therefore a dihedral group of order $2p$. The remainder of the theorem follows immediately.

Lemma 2.1 also shows $\Gamma^{o}_{4,3}(q)$ is also 3-generator when $k = q$ is odd by taking $(\alpha, \beta, \gamma, \delta) = (b, d, a, c)$ in the proof above.

On the other hand if q is even, then the Abelianisation of $\langle ab, bc, cd, de\rangle$ in the group $\Gamma^{o}_{4,3}(q)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ which is obviously not a 3-generator group and hence neither is $\Gamma^{o}_{4,3}(q)$ for any even q. In the remaining three cases

$\Gamma_{4,2}(q)$, $\Gamma_{7,3}(q)$ and $\Gamma'_{7,3}(q)$, whenever q is divisible by 3 each of these groups has $\Gamma_{7,3}(3)$ as a factor group, with generators a, b, c, d, e satisfying the relations $a^2 = b^2 = c^2 = d^2 = e^2 = (eb)^2 = (ec)^2 = (ed)^2 = (ab)^4 = (ac)^2 = (ad)^2 = (bc)^3 = (bd)^3 = (cd)^3 = 1$. Letting $x = eb$, $y = ea$, $u = ec$ and $v = ed$, the orientation-preserving subgroup of $\Gamma_{7,3}(3)$ is

$$\begin{aligned} G \;=\; \langle x, y, u, v :\; & x^2 = u^2 = v^2 = (xy)^4 = (uy)^2 = (vy)^2 \\ & = (xu)^3 = (xv)^3 = (uv)^3 = 1\rangle. \end{aligned}$$

The commutator subgroup of this group is G' generated by xu, xv, y^2 and $xyxy^{-1}$, and the factor group is $G/G' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Also $G'/G'' \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}$ (generated by the images of xu, xv and y^2). In particular, if the relations

$$\begin{aligned} y^6 = [xu, xv] = [xu, y^2] = [xu, xyxy^{-1}] = [xv, y^2] \\ = [xv, xyxy^{-1}] = [y^2, xyxy^{-1}] = 1 \end{aligned}$$

are adjoined to those of G, the quotient is a group of order 108, being a split extension of the group $\langle xu, xv, y^2\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ by $\langle u, y^3\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, with every element of $\langle u, y^3\rangle$ being either centralised or inverted under conjugation by each element of $\langle u, y^3\rangle$. This factor group cannot be generated by 3 elements however as all elements have order 1, 2, 3 or 6, but the cube of every element of order 6 is the same as y^3, so any two elements generate a subgroup of order at most 18.

When q is not divisible by 3, let $x = eb$, $y = ea$, $u = ec$ and $v = ed$. The relations for $\Gamma_{7,3}(q)$ imply

$$x^2 = u^2 = v^2 = (xy)^4 = (uy)^2 = (vy)^2 = (xu)^3 = (xv)^3 = (uv)^q = 1.$$

This time $uvu = uxvxvxu = (uxv)x(uxv)^{-1}$ and $vuv = vxuxuxv = (uxv)^{-1}x(uxv)$, so that $\langle x, uxv\rangle$ contains uvu and vuv and therefore also $(uv)^3 = (uvu)(vuv)$. But now $uv \in \langle (uv)^3\rangle \le \langle x, uxv\rangle$ and thus $\langle x, uxv\rangle$ contains $v = (vuv)(uv)^{-1}$ and also u, thus showing $\langle x, y, u, v\rangle = \langle x, y, uxv\rangle$. Similarly in the case of $\Gamma'_{7,3}(q)$, we can show $xu \in \langle (xu)^3\rangle \le \langle x, uxv\rangle$ and thus $\langle x, y, u, v\rangle = \langle x, y, uxv\rangle$; while in the case of $\Gamma_{4,2}(q)$, we find $xv \in \langle (xv)^3\rangle \le \langle u, v, xy\rangle$ and thus $\langle x, y, u, v\rangle = \langle u, v, xy\rangle$. It follows that the groups $\Gamma^{\mathrm{o}}_{7,3}(q)$, $\Gamma'^{\mathrm{o}}_{7,3}(q)$ and $\Gamma^{\mathrm{o}}_{4,2}(q)$ are all 3-generator groups (when q is not divisible by 3). We record this as

THEOREM 2.4. *The groups* $\Gamma^{o}_{4,2}(q)$, $\Gamma^{o}_{7,3}(q)$ *and* $\Gamma'^{o}_{7,3}(q)$ *are* 3*-generator groups if and only if* $q \not\equiv 0 \bmod 3$. *The group* $\Gamma^{o}_{4,3}(q)$ *is a* 3*-generator group if and only if* q *is odd.*

3. Volumes

An important feature of the construction we have given is that the families of groups $\Gamma_{i,p}(q)$ have fundamental domains which are part of a one-parameter family of polyhedra corresponding to $\Gamma_{i,p}(\theta)$, and parametrised by the angle $0 \le \theta \le \pi/p$. According to a result of C. Hodgson [12], the element of volume along this family is

$$d\text{Vol} \;=\; -\tfrac{1}{2}\; l\, d\theta \tag{3.1}$$

Here l is the length of the edge at which the dihedral angle θ occurs. (Actually here we do not need the full strength of the result of [12]. All we require can be derived from Coxeter's results [6]; see for instance [16]. The result however is still nontrivial.)

We now need to derive l as a function of θ for each of the families of groups above.

Let us consider the specific and illustrative case $\Gamma_{1,p}(\theta)$, $p \in \{2, 3, 6\}$. We subsequently present the results in all cases.

First take the infinite family $\Gamma_{1,6}(\theta)$. Here we vary the angle along the edge **ab**. Using the same labelling as in Figure 6, we consider the hyperbolic quadrilateral $\langle x, b, d, z\rangle$. We see that all the angles are right angles except for $\angle(z, b, d)$ which is equal to $\pi/3$. The length l is determined by the length u, and u in turn is determined as the length of a side of a hyperbolic triangle for which we have all the angles. Using hyperbolic trigonometric formulae, see [3], we have

$$\sinh(u)\sinh(l) = \cos(\pi/3) = 1/2. \tag{3.2}$$

Considering the hyperbolic triangle $\langle x, y, z\rangle$ we have also

$$\cosh(u)\sin(\theta) = \cos(\pi/3) = 1/2. \tag{3.3}$$

From (3.2) and (3.3) we deduce

$$\cosh(u)\sin(\theta) = \sinh(u)\sinh(l), \tag{3.4}$$

and so it follows that

$$\sinh^2(l) = \frac{1}{\csc^2(\theta) - 4}. \tag{3.5}$$

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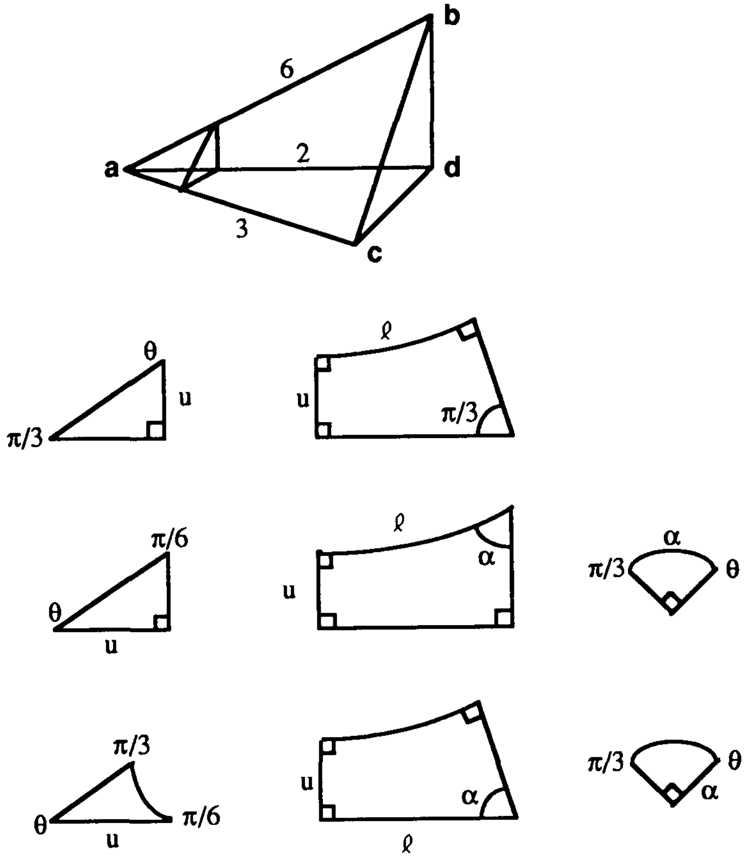

FIGURE 7

Next take the family $\Gamma_{1,3}(\theta)$. Here we consider the quadrilateral $\langle y, z, d, c\rangle$. Again all but one of the the dihedral angles are $\pi/2$. We find the missing angle α using some spherical trigonometry: using the ball model of hyperbolic geometry, move the vertex c back to the origin by an isometry and take a small round ball centered at c. Consider the spherical triangle on the boundary of this ball; it has angles given by the dihedral angles of the faces of the prism at the vertex c. One of these angles is $\pi/2$, another is $\pi/3$, and the third is θ. The angle α that we seek is the angle subtended between the vertices of the spherical triangle that have angles $\pi/3$ and θ. From the laws of spherical geometry

$$\cos(\pi/2) = -\cos(\pi/3)\cos(\theta) + \sin(\pi/3)\sin(\theta)\cos(\alpha), \tag{3.6}$$

and so

$$\cos(\alpha) = \frac{\cos(\theta)}{\sqrt{3}\sin(\theta)}. \tag{3.7}$$

From hyperbolic trigonometry, we deduce

$$\cosh^2(l) = \frac{\sinh^2(u)\sin^2(\alpha)}{\sinh^2(u) + \cos^2(\alpha)}. \tag{3.8}$$

Equation (3.8) simplifies to

$$\sinh(l) = \cot(\alpha)\coth(u). \tag{3.9}$$

Next, consideration of the hyperbolic triangle $\langle x, y, z\rangle$ this time yields

$$\cosh(u) = \frac{\cos(\pi/6)}{\sin(\theta)} = \frac{\sqrt{3}}{2\sin(\theta)}. \tag{3.10}$$

Then (3.9) and (3.10) together yield

$$\sinh^2(l) = \frac{3}{(4\sin^2(\theta) - 1)(4 - \sec^2(\theta))}. \tag{3.11}$$

An entirely analogous procedure yields the result for the groups $\Gamma_{1,2}(\theta)$. In these cases one obtains

$$\sinh^2(l) = \frac{3\cos^2(\theta) + 2\sqrt{3}\cos(\theta) + 1}{(4\sin^2(\theta) - 1)(4\cos^2(\theta) + 2\sqrt{3}\cos(\theta))}. \tag{3.12}$$

To summarise the situation, here then is a table of the relationship between l and θ for all eight infinite families. In an appendix the formulas for the finite families of finite co-volume groups are all given.

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Eight Infinite Families

$$\Gamma_{1,6}(\theta) : \sinh^2(l) = \frac{1}{\csc^2(\theta) - 4}$$

$$\Gamma_{5,3}(\theta) : \sinh^2(l) = \frac{\sin^2(\theta)}{2(\cos(2\theta) + \cos(\theta))}$$

$$\Gamma_{4,6}(\theta) : \sinh^2(l) = \frac{2}{\csc^2(\theta) - 4}$$

$$\Gamma_{7,3}(\theta) : \sinh^2(l) = \frac{\sin^2(\theta)}{\cos(2\theta) + \cos(\theta)}$$

$$\Gamma_{6,6}(\theta) : \sinh^2(l) = \frac{\sqrt{5}+3}{2(\csc^2(\theta) - 4)}$$

$$\Gamma_{9,3}(\theta) : \sinh^2(l) = \frac{\sqrt{5}+3}{4}\frac{\sin^2(\theta)}{\cos(2\theta) + \cos(\theta)}$$

$$\Gamma_{2,4}(\theta) : \sinh^2(l) = \frac{1}{2(\cot^2(\theta) - 1)}$$

$$\Gamma_{5,2}(\theta) : \sinh^2(l) = \frac{1}{4\cot(\theta)(\cot(\theta) + \csc(\theta))}$$

As mentioned earlier, we can now make a volume calculation. We have in each case a formula of the form

$$\sinh^2(l) = F(\theta). \tag{3.13}$$

The change in volume from angle θ_o to θ is given by

$$\triangle V = -\frac{1}{2}\int_{\theta_o}^{\theta} l \, d\theta. \tag{3.14}$$

Together (3.13) and (3.14) yield the following formula for the co-volumes of the discrete orientation preserving groups;

$$\text{co-Vol}(\Gamma_i^{\mathrm{O}}) = 2V_i + \int_{\theta}^{\theta_i} \operatorname{arcsinh}\left(\sqrt{F(\theta)}\right) d\theta \tag{3.15}$$

where V_i is the initial volume of the tetrahedron, θ_i is the initial dihedral angle on the cusp and θ is the new angle at truncation. The factor of 2 reflects the

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index. Thus we have the following formulas for the co-volume of the infinite families of discrete cocompact groups:

Co-volumes of the infinite families.

$$\text{co-Vol}(\Gamma^{o}_{1,6}(q)) = 2V_1 + 2\int_{\pi/q}^{\pi/6} \operatorname{arcsinh}\left(\frac{1}{\sqrt{\csc^2(\theta)-4}}\right) d\theta, \qquad q \geq 7.$$

$$\text{co-Vol}(\Gamma^{o}_{4,6}(q)) = 2V_4 + 2\int_{\pi/q}^{\pi/6} \operatorname{arcsinh}\left(\frac{\sqrt{2}}{\sqrt{\csc^2(\theta)-4}}\right) d\theta, \qquad q \geq 7.$$

$$\text{co-Vol}(\Gamma^{o}_{6,6}(q)) = 2V_6 + 2\int_{\pi/q}^{\pi/6} \operatorname{arcsinh}\left(\sqrt{\frac{\sqrt{5}+3}{2}}\frac{1}{\sqrt{\csc^2(\theta)-4}}\right) d\theta, \qquad q \geq 7.$$

$$\text{co-Vol}(\Gamma^{o}_{5,3}(q)) = 2V_5 + 2\int_{\pi/q}^{\pi/3} \operatorname{arcsinh}\left(\frac{\sin(\theta)}{\sqrt{2\cos(2\theta)+2\cos(\theta)}}\right) d\theta, \qquad q \geq 4.$$

$$\text{co-Vol}(\Gamma^{o}_{7,3}(q)) = 2V_7 + 2\int_{\pi/q}^{\pi/3} \operatorname{arcsinh}\left(\frac{\sin(\theta)}{\sqrt{\cos(2\theta)+\cos(\theta)}}\right) d\theta, \qquad q \geq 4.$$

$$\text{co-Vol}(\Gamma^{o}_{9,3}(q)) = 2V_9 + 2\int_{\pi/q}^{\pi/3} \operatorname{arcsinh}\left(\sqrt{\frac{\sqrt{5}+3}{4}\frac{\sin(\theta)}{\cos(2\theta)+\cos(\theta)}}\right) d\theta, \qquad q \geq 4.$$

$$\text{co-Vol}(\Gamma^{o}_{2,4}(q)) = 2V_2 + 2\int_{\pi/q}^{\pi/4} \operatorname{arcsinh}\left(\frac{1}{\sqrt{2}\sqrt{\cot^2(\theta)-1}}\right) d\theta, \qquad q \geq 5.$$

$$\text{co-Vol}(\Gamma^{o}_{5,2}(q)) = 2V_5 + 2\int_{\pi/q}^{\pi/2} \operatorname{arcsinh}\left(\frac{1}{2\sqrt{\cot(\theta)(\cot(\theta)+\csc(\theta))}}\right) d\theta, \qquad q \geq 3.$$

The convergence of each of these integrals (for the stated values of q) is an easy matter to establish. Below is a table of the first few volumes and the asymptotic volume in each case, accurate to the first four decimal places.

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Table of volumes

q	=	7	8	9	10	11	12	∞
$\Gamma^{o}_{1,6}(q)$	$\approx$	0.17712	0.21442	0.2365	0.25106	0.26130	0.26883	0.30532
$\Gamma^{o}_{4,6}(q)$	$\approx$	0.32593	0.37579	0.40589	0.42597	0.44017	0.45065	0.50191
$\Gamma^{o}_{6,6}(q)$	$\approx$	0.46651	0.52188	0.55565	0.57828	0.59435	0.6062	0.66465

q	=	4	5	6	7	8	9	∞
$\Gamma^{o}_{3,3}(q)$	$\approx$	0.42884	0.50121	0.53766	0.55797	0.57076	0.57936	0.61064
$\Gamma^{o}_{7,3}(q)$	$\approx$	0.75158	0.85195	0.90131	0.92969	0.94763	0.95972	1.00383
$\Gamma^{o}_{9,3}(q)$	$\approx$	1.3727	1.4855	1.5414	1.5737	1.5941	1.6079	1.6582

q	=	5	6	7	8	9	10	∞
$\Gamma^{o}_{2,4}(q)$	$\approx$	0.33192	0.39232	0.42435	0.44382	0.45667	0.46563	0.50191

q	=	3	4	5	6	7	8	∞
$\Gamma^{o}_{5,2}(q)$	$\approx$	0.64848	0.75149	0.79509	0.81789	0.83136	0.84000	0.86767

It is not too difficult to find out in each case above what the limiting polyhedron happens to be. It has five faces and five vertices, one of which is ideal. Normalising so the ideal vertex is $\infty \in \partial\mathbf{H}^3$, the four unbounded faces form an infinite rectangular prism. Labeling the four finite vertices as in Figure 8, we give a table of the dihedral angles at each edge of this limit polyhedron.

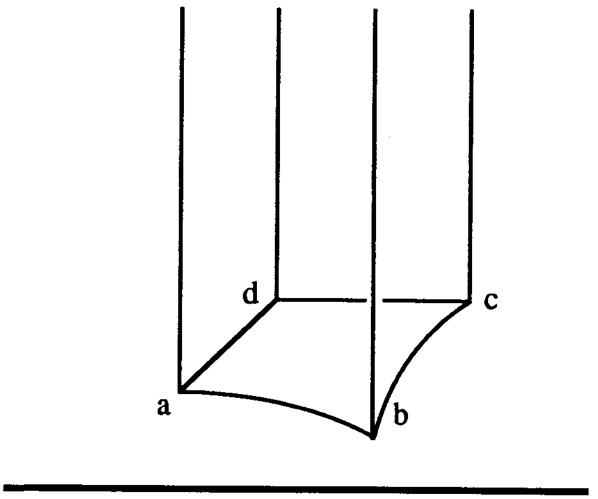

FIGURE 8. The degenerating prism

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Dihedral angles

Group	ab	bc	cd	da
$\Gamma^{o}_{1,6}(\infty)$	$\pi/3$	$\pi/3$	$\pi/2$	$\pi/2$
$\Gamma^{o}_{4,6}(\infty)$	$\pi/3$	$\pi/4$	$\pi/2$	$\pi/2$
$\Gamma^{o}_{6,6}(\infty)$	$\pi/3$	$\pi/5$	$\pi/2$	$\pi/2$
$\Gamma^{o}_{3,3}(\infty)$	$\pi/3$	$\pi/3$	$\pi/3$	$\pi/2$
$\Gamma^{o}_{7,3}(\infty)$	$\pi/3$	$\pi/3$	$\pi/4$	$\pi/2$
$\Gamma^{o}_{9,3}(\infty)$	$\pi/3$	$\pi/3$	$\pi/5$	$\pi/2$
$\Gamma^{o}_{2,4}(\infty)$	$\pi/4$	$\pi/3$	$\pi/2$	$\pi/2$
$\Gamma^{o}_{5,2}(\infty)$	$\pi/3$	$\pi/4$	$\pi/3$	$\pi/2$

The degenerate prisms for $\Gamma^{o}_{2,4}(\infty)$ and $\Gamma^{o}_{4,6}(\infty)$ are the same and these groups have the same geometric limit. The following lemma is easy.

LEMMA 3.1. *For all* $0 \leq \theta < \pi/6$,

$$\frac{1}{2\cot^2(\theta) - 2} < \frac{2}{\csc^2(\theta) - 4}. \tag{3.16}$$

Thus in the formulas for the co-volumes, the integrand for that of $\Gamma^{o}_{2,4}(q)$ is strictly less than the integrand for that of $\Gamma^{o}_{4,6}(q)$. Notice that as $q \to \infty$, these two values have the same limit and that initially, according to our table, $\Gamma^{o}_{2,4}(q) > \Gamma^{o}_{4,6}(q)$. Thus, from elementary calculus we deduce the following theorem.

THEOREM 3.2. *For each* $q \geq 7$,

$$\text{co-Vol}(\Gamma_{4,6}(q)) < \text{co-Vol}(\Gamma_{2,4}(q)).$$

Thus we have a geometric proof of the following.

THEOREM 3.3. *The two sequences of groups* $\Gamma_{2,4}(q)$ *and* $\Gamma_{4,6}(q)$ *have the same geometric and algebraic limit as* $q \to \infty$. *No two of these groups are isomorphic. All groups are factor groups of the limit group* $\Gamma_{4,6}(\infty) \cong \Gamma_{2,4}(\infty)$.

PROOF. It is clear that $\Gamma_{2,4}(p) \cong \Gamma_{4,6}(q)$ implies $p = q$. Since these groups are both are of finite co-volume, since co-volume is an algebraic invariant in such instances, the result follows from Theorem 3.2.

The isomorphism between $\Gamma^{o}_{2,4}(\infty)$ and $\Gamma^{o}_{4,6}(\infty)$ is easily found by interchanging the generators a and c and also the generators d and e in the presentations we have given. On the other hand, it is not so easy to obtain an algebraic proof that $\Gamma^{o}_{2,4}(q)$ and $\Gamma^{o}_{4,6}(q)$ are not isomorphic. One way to do this is by using the Reidemeister-Schreier process to show that the second factors in the derived series of the two groups are finite and infinite respectively.

The next important thing to observe is that the limit prism in the case of the $\Gamma^{o}_{1,6}(q)$ groups is actually made up of two copies of the tetrahedron with Coxeter diagram ∘—∘═∘═∘. The limit volume is the number

$$0.30542\ldots = 4 \times \mathrm{Vol}(\circ\!-\!\circ\!=\!\circ\!=\!\circ)$$

which is the smallest orbifold volume limit, see [2]. Hence $\Gamma_{1,6}(\infty)$ is the smallest volume orbifold which is a limit of other volumes. Also $\Gamma_{1,6}(6)$ is the smallest volume cusped orbifold, see [18]. Thus the family of groups $\Gamma_{1,6}(q)$ give a natural sequence of volumes between these two extremes.

Also, for each $q \geq 6$, let $V(q)$ be the smallest volume orbifold with a component of the singular set having degree q. The sequence $\{V(q)\}$ is presumably strictly increasing. From [8], $V(6) = \text{co-Vol}(\Gamma^{o}_{1,6}(6))$. Also $V(q)$ has an upper bound which is at least $\text{co-Vol}(\Gamma^{o}_{1,6}(\infty))$, and evidently this is the correct upper bound. More facts concerning $\{V(q)\}$, and lower bounds in particular, can be found in [9]. We are content to record the above in the following theorem.

THEOREM 3.4. *Let $V(q)$ denote the smallest volume amongst all orientable hyperbolic orbifolds with a component of the singular set having degree q. Then*

$$V(6) = \text{co-Vol}(\Gamma^{o}_{1,6}(6)) \quad \textit{and} \quad |V(q) - \text{co-Vol}(\Gamma_{1,6}(q))| \to 0 \quad \textit{as} \quad q \to \infty.$$

We have conjectured in [9] that $V(q) = \text{co-Vol}(\Gamma_{1,6}(q))$.

4. Symmetries and subgroups

Various subgroups of the groups we have discussed above exhibit interesting features. It is quite surprising that in most instances there are torsion-free subgroups of relatively low index, indeed close to the minimal index of a torsion-free subgroup of the triangle subgroup of each group. In order to establish the existence of torsion free subgroups of a given index the following lemma is useful (and is well known and holds in somewhat more generality):

LEMMA 4.1. *Let Γ be a discrete group generated by reflections in a 3-dimensional hyperbolic polyhedron Ω. Then every element of finite order in Γ is conjugate to an edge relation in the presentation of Γ.*

PROOF. Let $\gamma \in \Gamma$ be an element of finite order and let A be the axis of fixed points of γ. As Ω is a fundamental domain, $A \cap \text{int}(\Omega) = \phi$. As A is uncountable and as the vertices and edges of Ω are locally finite (in $\mathbf{H}^3$) some translate of Ω contains a subarc of A in its boundary. It is not difficult to see that such an arc cannot lie in a face of any translate of Ω, and the result follows.

The following lemma is clear and shows us that the stabilizer of the hyperplane Π we used to truncate the deformed hyperbolic tetrahedron to produce the groups $\Gamma^{o}_{i,p}(q)$ is actually nonorientable, when restricted to this hyperplane. This happens because although each element of $\Gamma^{o}_{i,p}(q)$ is orientable, the stabilizer of Π is generated by three elements of order two whose axes form a hyperbolic triangle in Π. The group these elements generate is orientable as a subgroup of isometries of $\mathbf{H}^3$, but when restricted to Π the rotation of order two about an axis is the same as a reflection across this axis.

LEMMA 4.2. *Let Π be the hyperplane used to truncate the tetrahedron to produce the group $\Gamma^{o}_{i,p}(q)$. Then the stabilizer of Π in $\Gamma^{o}_{i,p}(q)$ restricted to Π, $\text{Stab}_{\Gamma^{o}_{i,p}(q)}(\Pi)|\Pi$, is a nonorientable hyperbolic triangle reflection group.*

The point of this lemma is that in any torsion-free subgroup H of $\Gamma^{o}_{i,p}(q)$, giving us an oriented hyperbolic 3-manifold M^3, we shall find a totally geodesic surface F^2 corresponding to the intersection $H \cap \Delta$, where Δ is a group generated by reflections in a hyperbolic triangle. Because the restriction to the invariant hyperplane is not orientation preserving however, the surface may not be orientable. In every instance we have investigated, in the orbit space of a smallest index torsion free subgroup, F is nonorientable but M admits a double cover for which the lift of F is oriented (and of course totally geodesic).

We now want to look at the torsion-free subgroups. First consider the finite index torsion-free subgroups of $\Gamma^{o}_{1,6}(q)$, as these groups have smallest co-volume. Let Δ denote the $(2, 3, q)$ triangle subgroup and Δ^* the extended triangle subgroup of Lemma 4.2. As $\Delta < \Gamma$ the index of such a subgroup is bounded below by the index of a torsion free subgroup of a $(2, 3, p)$ triangle group. For the group $\Gamma^{o}_{1,6}(7)$ and the $(2, 3, 7)$ triangle group, this number is 84. Surprisingly there is a torsion-free subgroup of index 84. This subgroup is not normal and is not even normalised by the generator of order 7. Similarly there is a torsion-free subgroup of index 48 in $\Gamma^{o}_{1,6}(8)$ (the same as for $\triangle(2,3,8)$); but the obvious conjecture fails for the group $\Gamma^{o}_{1,6}(9)$ whose smallest torsion free subgroup has index 72 (as compared with 36 for $\triangle(2,3,9)$). Notice that a torsion-free subgroup Γ_{237} of index 84 in $\Gamma^{o}_{1,6}(q)$ covers an oriented hyperbolic 3-manifold $M_{(2,3,7)} = \mathbf{H}^3/\Gamma_{237}$ with a volume of $84 \times 0.17712\ldots \approx 14.88\ldots$. This manifold has a nonoriented totally geodesic surface whose double cover has genus two, corresponding to the index 84 subgroup of the $\mathbb{Z}_2$-extension of the (2,3,7) triangle goup Δ^*. Also $M_{(2,3,7)}$ has a double cover for which the lift of the totally geodesic surface is oriented. Using $\Gamma^{o}_{1,6}(8)$ we are provided with an oriented hyperbolic 3-manifold M_{238} of volume $48 \times 0.21442\ldots \approx 10.29\ldots$ again with a nonorientable totally geodesic surface and admitting a double cover.

But it is the group $\Gamma^{o}_{1,6}(12)$ that provides some of the most interesting examples.

THEOREM 4.3. *Let* $\Gamma = \Gamma^{o}_{1,6}(12)$. *Then there are torsion free subgroups* Γ_0 *and* Γ_1 *of* Γ *with the following properties.*

$$\Gamma_1 \le \Gamma_0, \quad |\Gamma : \Gamma_0| = 24, \quad |\Gamma : \Gamma_1| = 48 \quad \textit{and} \quad |\Gamma_0 : \Gamma_1| = 2.$$

Let $M_0 = \mathbf{H}^3/\Gamma_0$ *and* $M_1 = \mathbf{H}^3/\Gamma_1$. *Then* M_0 *is a smallest volume hyperbolic 3-manifold with a totally geodesic closed surface (it is nonorientable), and* $\mathrm{Vol}(M_0) \approx 3.2260$, *where* M_1 *double covers* M_0 *and is a smallest volume hyperbolic 3-manifold with a separating totally geodesic closed surface, and* $\mathrm{Vol}(M_1) \approx 6.4519$.

Let H_0 *(respectively* H_1*) be the core of* Γ_0 *(respectively* Γ_1*) in* Γ. *Then the factor group* Γ/H_0 *is isomorphic to* $\mathrm{PSL}(2,23)$ *and* Γ/H_1 *is isomorphic to* $\mathrm{PSL}(2,23) \times \mathbb{Z}_2$.

PROOF. (Sketch) The construction of the torsion-free subgroups is similar to that of Theorem 4.4 below and so we leave the matter there. Notice that Γ contains the (2,3,12) triangle group, so that any finite index torsion-free subgroup must contain a surface group. The formula for the volumes of M_0 and M_1 can be found from the table following 3.15. These numbers are the same as given by Kojima and Miyamoto [14] for the minimal volume hyperbolic 3-manifold with totally geodiesic boundary (it takes a little thought to see how to arrange copies of the fundamental polyhedral prism that we constructed into the requisite number of truncated hyperbolic ideal tetrahedra of [14]) . Cutting M_0 open along this surface yields a hyperbolic 3-manifold with totally geodesic boundary (it doesn't give two such manifolds as the surface is not separating).

In M_1 the lift of the surface is separating. Cutting M_1 open along this surface gives two manifolds with totally geodesic boundary; and since one of these must have volume of no more than half the volume of M_0 (and since this is minimal), then both pieces have the same volume as the smallest such manifold (again by [14]) and the result follows.

Next we discuss another exceptionally interesting class of these groups.

THEOREM 4.4. *Whenever q is an odd multiple of 6, say $q = 6m$ where m is odd, the group $\Gamma = \Gamma^{o}_{2,4}(q)$ has a torsion free subgroup H of index $4q$, with core*

$$N = \bigcap_{g \in \Gamma} g^{-1} H g$$

of index $24q$. The images of two elements in Γ which generate the $(2, 4, q)$ triangle group generate in Γ/N a subgroup of order $4q$, isomorphic to the automorphism group of the Accola-Maclachlan surface of genus $(q-2)/2$, which is isometrically and totally geodesically embedded in $\mathbf{H}^3/\Gamma$.

PROOF. Let us use the presentation of the group $\Gamma_{2,4}(q)$ given in the appendix. Set

$$X = eb, Y = ea, U = ec \text{ and } V = ed,$$

so the relations imply

$$X^2 = U^2 = V^2 = (XY)^3 = (UY)^2 = (VY)^2 = (XU)^4 = (XV)^2 = (UV)^q = 1.$$

Now make another change of variables, taking

$$x = VX, y = XU, t = X \text{ and } w = UY.$$

In terms of the earlier generators, $X = t$, $U = X^2U = ty$, $V = VX^2 = xt$ and $Y = X^2U^2Y = tyw$; and the relations become

$$x^2 = y^4 = (xy)^q = t^2 = (xt)^2 = (yt)^2 = w^2 = (xyw)^2 = (yw)^3 = 1.$$

In particular $\langle x, y \rangle$ is isomorphic to the $(2, 4, q)$ triangle group, and $\langle x, y, t \rangle$ to its $\mathbb{Z}_2$-extension. Next let $K = 4q = 24m$, and construct a transitive permutation representation of Γ on the set $\{1, 2, 3 \ldots, , K\}$ by letting x, y, t act as follows.

$$x(k) = \begin{cases} k-3 & \text{if } k \equiv 1 \text{ or } 3 \bmod 4 \\ k+3 & \text{if } k \equiv 2 \text{ or } 4 \bmod 4 \end{cases}$$

$$y(k) = \begin{cases} k+1 & \text{if } k \equiv 1 \text{ or } 2 \text{ or } 3 \bmod 4 \\ k-3 & \text{if } k \equiv 4 \bmod 4 \end{cases}$$

$$t(k) = K+1-k \qquad \text{for all } k$$

$$w(k) = \begin{cases} k-4m+3 & \text{if } k \equiv 1 \bmod 8 \\ k+16m+1 & \text{if } k \equiv 2 \bmod 8 \\ k+8m-1 & \text{if } k \equiv 3 \bmod 8 \\ k+4m-1 & \text{if } k \equiv 4 \bmod 8 \\ k+8m+1 & \text{if } k \equiv 5 \bmod 8 \\ k+16m-1 & \text{if } k \equiv 6 \bmod 8 \\ k-4m+1 & \text{if } k \equiv 7 \bmod 8 \\ k+4m-3 & \text{if } k \equiv 8 \bmod 8 \end{cases}$$

In each case here the right hand side of the definition is considered mod K where necessary. For example in the case $q = 6$ these permutations are

$$x = (1,22)(2,5)(3,24)(4,7)(6,9)(8,11)(10,13)(12,15)(14,17)(16,19)(18,21)(20,23)$$

$$y = (1,2,3,4)(5,6,7,8)(9,10,11,12)(13,14,15,16)(17,18,19,20)(21,22,23,24)$$

$$t = (1,24)(2,23)(3,22)(4,21)(5,20)(6,19)(7,18)(8,17)(9,16)(10,15)(11,14)(12,13)$$

$$w = (1,24)(2,19)(3,10)(4,7)(5,14)(6,21)(8,9)(11,18)(12,15)(13,22)(16,17)(20,23)$$

It is not difficult to check the relations

$$x^2 = y^4 = (xy)^q = t^2 = (xt)^2 = (yt)^2 = w^2 = (xyw)^2 = (yw)^3 = 1$$

are satisfied, and that each of the permutations x, y, xy, t, xt, yt, w, xyw and yw not only has the appropriate order (as indicated by the exponents in the relations), but also is semiregular (having all cycles the same length). In

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particular, every nontrivial power of one of these permutations is fixed point free, and it follows that if H is the stabilizer of the point 1 in this representation, then H is a torsion free subgroup of Γ of index K.

Similarly it is not too difficult to see that x and y^2 commute so that the commutator $[x, y^2]$ is trivial (in fact H is generated by conjugates of $[x, y^2]$ together with xy^2t and the element $wt(xy)^{m-1}$.) Thus $\langle x, y\rangle$ is a central extension of $\langle y^2\rangle$ (of order two) by a dihedral group of order $2q$, and is therefore a factor group of the $(2, 4, q)$ triangle group, of order $4q$. This shows that $\langle x, y\rangle$ is isomorphic to the automorphism group of the Accola-Maclachlan surface of genus $(4q-8)/8 = (q-2)/2$. Next, $\langle x, y, t\rangle$ is a group of twice this order, and finally $\langle x, y, t, w\rangle$ has order $24q$ (with the identity, w and wx in distinct cosets of $\langle x, y, t\rangle$) so that N, the kernel of the representation, has index $24q$ in Γ.

Finally, let Δ denote the $(2, 4, q)$ triangle subgroup of Γ and Δ^* the extended triangle group. The remainder of the theorem follows from the fact that the suface subgroup $\Gamma \cap \Delta$ has index $4q$ in Δ and index $8q$ in Δ^*.

COROLLARY 4.5. *Let N be as in Theorem 4.2 and set $N^3(q) = \mathbf{H}^3/N$, where $q = 6m$, for m odd. Then $N^3(q)$ is an orientable hyperbolic 3-manifold containing a totally geodesic orientable suface of genus $3m-1$. Moreover for large q*

$$\mathrm{Vol}(N^3(q)) \approx 12.05q,$$

and the order of the automorphism group of $N^3(q)$ is exactly $24q$.

PROOF. The only thing that is not clear from what we have proved above is that the order of the automorphism group of $N^3(q)$ is precisely $24q$ for all q sufficiently large. This follows from the fact that

$$\frac{\mathrm{Vol}(N^3(q))}{24q} \to 0.501\ldots \quad \text{as } q \to \infty.$$

The order of the automorphism group is divisible by $24q$. If there were an infinite sequence of q with the order of the automorphism group larger than $24q$, then the limit above would be less than $\frac{1}{2} \times (0.501\ldots) < 0.26$. But as we have earlier pointed out the smallest limit orbifold volume is 0.30532... This contradiction establishes the claim.

Below we have presented a table of what we have found out about the groups $\Gamma^o_{1,6}(q)$ for some small values of q using the University of Sydney's CAYLEY package.

$\Gamma^{o}_{1,6}(7)$ –

168 torsion-free subgroups of index **84**. All associated with the factor group PSL(2,83) of order 285852,
e.g. $< bdca, dacdbecdcb, decabeae >$

$\Gamma^{o}_{1,6}(8)$ –

344 torsion-free subgroups of index **48**:
96 with associated factor group PGL(2,23) of order 12144,
e.g. $< dceb, daebca, cbedcaedcbab >$
48 with associated factor group soluble of order 6144 (= 3×2^{11}),
e.g. $< caeb, baedcd, bcdcdcdc, cbdcedcdcabc >$
48 with associated factor group soluble of order 98304 (=3×2^{15}),
e.g. $< caed, dcdbcdcb, cbedcbab >$
48 with associated factor group of order 20160 (a subdirect product of S_5 and PGL(2,7)),
e.g. $< adcb, bcebcdca, edcaedcbcd >$
48 with associated factor group of order 5643509764108 (a wreath product of A_5 by PGL(2,7)),
e.g. $< adcb, edcaeb, ebdcdcdc, cabcedcdcdbc >$
24 with associated factor group of order 8064 (a direct product of S_4 and PGL(2,7)),
e.g. $< adcb, edcaeb, cdcdebcdcd, dcbcedcdcdbc >$
24 with associated factor group of order 2016 (a direct product of S_3 and PGL(2,7)),
e.g. $< edcdcb, cdcbcdab, cabcdcdb, caeace >$
8 with associated factor group PGL(2,7) of order 336,
e.g. $< edcdcb, bdcdca, cabacdcdbc >$

$\Gamma^{o}_{1,6}(9)$ –

No torsion free subgroups of index 36. There are many of index 72. Some associated with factor group A_{72},
e.g. $< dceb, cbcaedab, cdcbcdcbac, cadcbdcdcdcdebac >$
Some associated with factor group of order $2^{34}\times 36!$ a wreath product of C_2 by A_{36}, e.g. $< dceb, cbcaedab, cdcdcdcbac, cdcbcaedcbcdac, cabcbdcdacbdcdac >$

$\Gamma^{o}_{1,6}(12)$ –

64 torsion-free subgroups of index **24**:
24 with associated factor group PSL(2,23) of order 6072,
e.g. $< dcbe, cdcbac, cbcaedab >$
24 with associated factor group PGL(2,11) of order 1320,
e.g. $< caeb, dcbdcb, bdcaedbc >$
16 with associated factor group soluble of order 648,
e.g. $< dcbe, daebac, bdcaebab >$

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272 torsion-free subgroups of index **48** intersecting $< bc, bd >$ in a subgroup of index 24 and $< bc, bd, be >$ in a subgroup of index 48:
72 with associated factor group PSL(2,23)$\times C_2$ of order 12144,
e.g. $< bdca, eacdbe, bcdcbdcd, bcbdabcdcdcd >$
48 with associated factor group soluble of order $3^7 \times 2^{22}$,
e.g. $< bdca, eacdbe, bcdcbdcd, dcdcdcbacdcd >$
48 with associated factor group of order 73738368 (containing the direct product of two copies of PSL(2,23) as a subgroup of index 2),
e.g. $< bdca, eacdbe, dacdbcbdcb, dcbdcdcdcb >$
24 with associated factor group of order 7308155289600 (containing the direct product of two copies of a wreath product of C_2 by PSL(2,11) as a subgroup of index 4),
e.g. $< cdcabd, beaeab, babcbdcd, bacdcdbc >$
24 with associated factor group of order 1742400 (containing the direct product of two copies of PSL(2,21) as a subgroup of index 4),
e.g. $< cdcabd, beaeab, babcbdcd, bacdcdbc >$
24 with associated factor group PGL(2,11) $\times C_2$ of order 2640,
e.g. $< cdcabd, becabeab, cabacdcd, bacdcdbc >$
16 with associated factor group soluble of order 139968 ($3^7 \times 2^6$),
e.g. $< cdcabd, beaeab, babcbdcd, bcdcbacd >$
16 with associated factor group soluble of order 1296,
e.g. $< cdcabd, becabeca, babcbdcd, bcdcbacd >$

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References

[1] C. C. Adams, 'The noncompact hyperbolic 3-manifold of minimal volume', *Proc. Amer. Math. Soc.* **100** (1987), 601–606.

[2] ——, 'Limit volumes of hyperbolic three-orbifolds', *J. Differential Geom.* **34** (1991), 115–142.

[3] A. Beardon, *The geometry of discrete groups* (Springer-Verlag, 1983).

[4] R. Brooks and P. Matelski, 'Collars in Kleinian groups', *Duke Math. J.* **49** (1982), 163–182.

[5] T. Chinburg and E. Friedman, 'The smallest arithmetic hyperbolic three-orbifold', *Invent. Math.* **86** (1986), 507 – 527.

[6] H. S. M. Coxeter, 'The functions of Schläfli and Lobachevshy', *Quart. J. Math. Ser.* (2) **6** (1935), 13–29.

[7] H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups* (Springer-Verlag, Berlin, 1979).

[8] F. W. Gehring and G. J. Martin, '6-Torsion and hyperbolic volume', *Proc. Amer. Math. Soc.*, to appear.

[9] ———, 'Commutators, collars and the geometry of Möbius groups', preprint.

[10] ———, 'Torsion and volume in hyperbolic 3-folds', preprint.

[11] ———, 'Inequalities for matrices and Möbius transformations', *J. Reine Angew. Math.* **418** (1991), 31–76.

[12] C. Hodgson, *Degeneration and regeneration of hyperbolic structures on three-manifolds* (Ph.D. Thesis, Princeton University, 1986).

[13] T. Jørgensen, 'On discrete groups of Möbius transformations', *Amer. J. Math.* **98** (1976), 739–749.

[14] S. Kojima and Y. Miyamoto, 'The smallest hyperbolic 3-manifolds with totally geodesic boundary', *J. Differential Geom.* **34** (1991), 175–192.

[15] F. Lanner, 'On complexes with transitive groups of automorphisms', *Comm. Sém. Math. Univ. Lund.* **11** (1950), 41.

[16] G. J. Martin, 'The volume of regular tetrahedra and sphere packing in hyperbolic space', *Math. Chronicle* **20** (1991), 127–147.

[17] B. Maskit, *Kleinian groups* (Springer-Verlag, Berlin, 1987).

[18] R. Meyerhoff, 'The cusped hyperbolic 3-orbifold of minimum volume', *Bull. Amer. Math. Soc.* **13** (1985), 154–156.

[19] ———, 'A lower bound for the volume of hyperbolic 3-orbifolds', *Duke Math. J.* **57** (1988), 185–203.

[20] G. D. Mostow, 'Quasiconformal mappings in n-space and the rigidity of hyperbolic space forms', *Inst. Hautes Études Sci. Publ. Math.* **34** (1968), 53–104.

[21] W. P. Thurston, *The geometry and topology of three-manifolds* (Lecture notes, Princeton University, 1977).

[22] E. B. Vinberg, 'Discrete groups generated by reflections in Lobacevski spaces', *Mat. Sb.* **72** (114) (1967), 429–444.

5. Appendix : Presentation of the reflection groups

1. 

$$\begin{aligned}
\Gamma_{1,6}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2 \\
&\qquad =(ab)^3=(ac)^2=(ad)^2=(bc)^3=(bd)^2=(cd)^q=1\rangle \\
\Gamma_{1,3}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2 \\
&\qquad =(ab)^3=(ac)^2=(ad)^2=(bc)^q=(bd)^2=(cd)^6=1\rangle \\
\Gamma_{1,2}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2 \\
&\qquad =(ab)^3=(ac)^2=(ad)^2=(bc)^3=(bd)^q=(cd)^6=1\rangle
\end{aligned}$$

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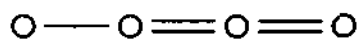
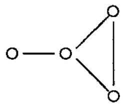
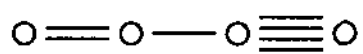
2. 

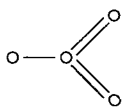
$$\begin{aligned}\Gamma_{2,2}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^3=(ac)^2=(ad)^2=(bc)^4=(bd)^q=(cd)^4=1\rangle\\ \Gamma_{2,4}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^3=(ac)^2=(ad)^2=(bc)^4=(bd)^2=(cd)^q=1\rangle\\ \Gamma'_{2,4}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^3=(ac)^2=(ad)^2=(bc)^q=(bd)^2=(cd)^4=1\rangle\end{aligned}$$

3. 

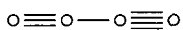
$$\begin{aligned}\Gamma_{3,3}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^3=(ac)^2=(ad)^2=(bc)^3=(bd)^3=(cd)^q=1\rangle\\ \Gamma'_{3,3}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^3=(ac)^2=(ad)^2=(bc)^q=(bd)^3=(cd)^3=1\rangle\end{aligned}$$

4. 

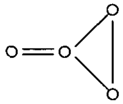
$$\begin{aligned}\Gamma_{4,6}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^4=(ac)^2=(ad)^2=(bc)^3=(bd)^2=(cd)^q=1\rangle\\ \Gamma_{4,3}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^4=(ac)^2=(ad)^2=(bc)^q=(bd)^2=(cd)^6=1\rangle\\ \Gamma_{4,2}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^4=(ac)^2=(ad)^2=(bc)^3=(bd)^q=(cd)^6=1\rangle\end{aligned}$$

5. 

$$\begin{aligned}\Gamma_{5,4}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^3=(ac)^2=(ad)^2=(bc)^q=(bd)^4=(cd)^2=1\rangle\\ \Gamma_{5,2}(q) &= \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^3=(ac)^2=(ad)^2=(bc)^4=(bd)^4=(cd)^q=1\rangle\end{aligned}$$

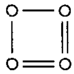
6. 

$$\begin{aligned}\Gamma_{6,6}(q) &\cong \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^5=(ac)^2=(ad)^2=(bc)^3=(bd)^2=(cd)^q=1\rangle\\ \Gamma_{6,3}(q) &\cong \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^5=(ac)^2=(ad)^2=(bc)^q=(bd)^2=(cd)^6=1\rangle\\ \Gamma_{6,2}(q) &\cong \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2\\ &\qquad =(ab)^5=(ac)^2=(ad)^2=(bc)^3=(bd)^q=(cd)^6=1\rangle\end{aligned}$$

7. 

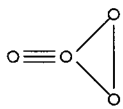
$$\Gamma_{7,3}(q) \cong \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2 = (ab)^4=(ac)^2=(ad)^2=(bc)^3=(bd)^3=(cd)^q=1\rangle$$

$$\Gamma'_{7,3}(q) \cong \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2 = (ab)^4=(ac)^2=(ad)^2=(bc)^q=(bd)^3=(cd)^3=1\rangle$$

8. 

$$\Gamma_{8,4}(q) \cong \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2 = (ab)^3=(ac)^2=(ad)^3=(bc)^4=(bd)^2=(cd)^q=1\rangle$$

$$\Gamma_{8,2}(q) \cong \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2 = (ab)^3=(ac)^2=(ad)^3=(bc)^4=(bd)^q=(cd)^4=1\rangle$$

9. 

$$\Gamma_{9,3}(q) \cong \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2 = (ab)^5=(ac)^2=(ad)^2=(bc)^3=(bd)=(cd)^q=1\rangle$$

$$\Gamma'_{9,3}(q) \cong \langle a,b,c,d,e : a^2=b^2=c^2=d^2=e^2=(eb)^2=(ec)^2=(ed)^2 = (ab)^5=(ac)^2=(ad)^2=(bc)^q=(bd)^3=(cd)^3=1\rangle$$

6. Appendix: co-Volumes of the finite families

For the finite families (of finite co-volume) we have computed the volumes (by numerical integration) and have presented these below.

$$\text{co-Vol}(\Gamma^{\mathrm{o}}_{1,3}(q)) = 2V_1 + \int_{\pi/q}^{\pi/3} \operatorname{arcsinh}\left(\frac{\sqrt{3}}{\sqrt{(3\sec^2(\theta)-4)(4-\sec^2(\theta))}}\right) d\theta$$

$$\text{co-Vol}(\Gamma^{\mathrm{o}}_{1,3}(4)) \approx 0.294192\ldots, \qquad \text{co-Vol}(\Gamma^{\mathrm{o}}_{1,3}(5)) \approx 0.44129\ldots,$$

$$\text{co-Vol}(\Gamma^{\mathrm{o}}_{1,3}(6)) \approx 0.612888\ldots$$

$$\text{co-Vol}(\Gamma^{\mathrm{o}}_{4,3}(3)) = 2V_4 + \int_{\pi/4}^{\pi/3} \operatorname{arcsinh}\left(\frac{\sqrt{3}}{\sqrt{(3\sec^2(\theta)-4)(4-\sec^2(\theta))}}\right) d\theta \approx 0.639249\ldots$$

$$\text{co-Vol}(\Gamma^{\mathrm{o}}_{1,2}(q)) = 2V_1 + \int_{\pi/q}^{\pi/2} \operatorname{arcsinh}\left(\frac{\sqrt{3}\cos(\theta)+1}{\sqrt{(4\sin^2(\theta)-1)(4\cos^2(\theta)+2\sqrt{3}\cos(\theta))}}\right) d\theta$$

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$$\text{co-Vol}(\Gamma^{o}_{1,2}(3)) \approx 0.537619\ldots, \qquad \text{co-Vol}(\Gamma^{o}_{1,2}(4)) \approx 0.748154\ldots$$
$$\text{co-Vol}(\Gamma^{o}_{1,2}(5)) \approx 0.915902\ldots, \qquad \text{co-Vol}(\Gamma^{o}_{1,2}(6)) \approx 1.09947\ldots$$

$$\text{co-Vol}(\Gamma^{o}_{4,2}(q)) = 2V_4 + \int_{\pi/q}^{\pi/2} \operatorname{arcsinh}\left(\frac{\sqrt{3}\cos(\theta)+1}{\sqrt{(2\sin^2(\theta)-1)(4\cos^2(\theta)+2\sqrt{3}\cos(\theta))}}\right) d\theta$$

$$\text{co-Vol}(\Gamma^{o}_{4,2}(3)) \approx 0.901266\ldots, \qquad \text{co-Vol}(\Gamma^{o}_{4,2}(4)) \approx 1.33469\ldots$$

$$\text{co-Vol}(\Gamma^{o}_{6,2}(3)) = 2V_6 + \int_{\pi/3}^{\pi/2} \operatorname{arcsinh}\left(\frac{\sqrt{3}\cos(\theta)+1}{\sqrt{(A\sin^2(\theta)-1)(4\cos^2(\theta)+2\sqrt{3}\cos(\theta))}}\right) d\theta$$
$$\approx 0.940957\ldots, \qquad A = \frac{8}{\sqrt{5}+1}$$

$$\text{co-Vol}(\Gamma^{o\prime}_{3,3}(q)) = 2V_3 + \int_{\pi/q}^{\pi/3} \operatorname{arcsinh}\left(\frac{\cos(\theta)+1}{\sqrt{2(4\sin^2(\theta)-1)(\cos(2\theta+\cos(\theta))}}\right) d\theta$$

$$\text{co-Vol}(\Gamma^{o\prime}_{3,3}(4)) \approx 0.530572\ldots, \qquad \text{co-Vol}(\Gamma^{o\prime}_{3,3}(5)) \approx 0.727099\ldots,$$
$$\text{co-Vol}(\Gamma^{o\prime}_{3,3}(6)) \approx 0.92314\ldots$$

$$\text{co-Vol}(\Gamma^{o\prime}_{7,3}(4)) = 2V_7 + \int_{\pi/4}^{\pi/3} \operatorname{arcsinh}\left(\frac{\cos(\theta)+1}{(4\sin^2(\theta)-2)(\cos(2\theta)+\cos(\theta))}\right) d\theta$$
$$\approx 1.03621\ldots$$

$$\text{co-Vol}(\Gamma^{o\prime}_{2,4}(q)) = 2V_2 + \int_{\pi/q}^{\pi/4} \operatorname{arcsinh}\left(\frac{1}{(4\sin^2(\theta)-1)(1-\tan^2(\theta))}\right) d\theta$$

$$\text{co-Vol}(\Gamma^{o\prime}_{2,4}(5)) \approx 0.425962\ldots, \qquad \text{co-Vol}(\Gamma^{o\prime}_{2,4}(6)) \approx 0.634324\ldots$$

$$\text{co-Vol}(\Gamma^{o}_{2,2}(q)) = 2V_2 + \int_{\pi/q}^{\pi/2} \operatorname{arcsinh}\left(\sqrt{\frac{\cos(\theta)+1}{2\cos(\theta)(1-4\sin^2(\theta))}}\right) d\theta$$

$$\text{co-Vol}(\Gamma^{o}_{2,2}(3)) \approx 0.666637\ldots, \qquad \text{co-Vol}(\Gamma^{o}_{2,2}(4)) \approx 0.887613\ldots$$
$$\text{co-Vol}(\Gamma^{o}_{2,2}(5)) \approx 1.05936\ldots, \qquad \text{co-Vol}(\Gamma^{o}_{2,2}(6)) \approx 1.24493\ldots$$

$$\text{co-Vol}(\Gamma^{o}_{5,4}(q)) = 2V_5 + \int_{\pi/q}^{\pi/4} \operatorname{arcsinh}\left(\frac{1}{(1-4\sin^2(\theta))(1-2\sin^2(\theta))}\right) d\theta$$

$$\text{co-Vol}(\Gamma^{o}_{5,4}(5)) \approx 0.619684\ldots, \qquad \text{co-Vol}(\Gamma^{o}_{5,4}(6)) \approx 0.84577\ldots.$$

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