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SUBSPACES OF REARRANGEMENT-INVARIANT SPACES

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ABSTRACT. We prove a number of results concerning the embedding of a Banach lattice X into an r. i. space Y. For example we show that if Y is an r. i. space on $[0,\infty)$ which is p-convex for some $p > 2$ and has nontrivial concavity then any Banach lattice X which is r-convex for some $r > 2$ and embeds into Y must embed as a sublattice. Similar conclusions can be drawn under a variety of hypotheses on Y; if X is an r. i. space on $[0,1]$ one can replace the hypotheses of r-convexity for some $r > 2$ by $X \neq L_2$.

We also show that if Y is an order-continuous Banach lattice which contains no complemented sublattice lattice-isomorphic to ℓ_2, X is an order-continuous Banach lattice so that ℓ_2 is not complementably lattice finitely representable in X and X is isomorphic to a complemented subspace of Y then X is isomorphic to a complemented sublattice of Y^N for some integer N.

1. **Introduction.** The study of the Banach space geometry of general rearrangement-invariant Banach function spaces may be considered to originate with the work of Bretagnolle and Dacunha-Castelle on subspaces of Orlicz function spaces [3]. A very important development in the theory was the publication of a systematic study of r. i. spaces by Johnson, Maurey, Schechtman and Tzafriri in 1979 [21]. The appearance of this memoir revolutionized the subject. Since then, a number of authors have considered problems of classifying subspaces of certain special r. i. spaces; see [5], [6], [7], [8], [9], [13], [14], [17], [19], [20], [39], [40] for a variety of different results of this type.

In general, most of the literature relates to the problem of embedding a Banach lattice X (either atomic or nonatomic) with additional symmetry conditions into an r. i. space Y, and the techniques used rely heavily on symmetrization. In [27], however, the second author considered the general problem of determining conditions when an order-continuous Banach lattice X could be complementably embedded in an order-continuous Banach lattice Y, minimizing the use of symmetry. The aim was to show that under certain hypotheses on X and Y one could deduce that X (or perhaps only a non-trivial band in X) would be lattice-isomorphic to a complemented sublattice of Y. A number of such results were obtained (we refer for details to [27]); of course, the additional assumption that either X or Y is r. i. could still be used to obtain stronger results of this nature. In the final section of this paper (Section 8, which can be read independently of the remainder) we obtain a significant improvement of one of the results of [27] by showing that if X, Y are order-continuous separable Banach lattices, such that Y contains no complemented

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sublattice which is lattice-isomorphic to ℓ_2 and ℓ_2 is not complementably lattice finitely representable in X, and if X is isomorphic to a complemented subspace of Y then X is lattice-isomorphic to a complemented sublattice of Y^N for some N. Of course if Y is r. i. then X must be a complemented sublattice of Y itself.

The main body of the paper (Sections 3–7) is concerned with similar problems but without assumptions of complementation. We consider an r. i. space Y on $[0,\infty)$ (or $[0,1]$, but there our results are not quite so strong) and consider a generally nonatomic Banach lattice X which is isomorphic to a subspace of Y; we would like to show, under appropriate hypotheses that X is lattice-isomorphic to a sublattice of Y. Of course, there is no hope of such a result in general; the spaces $L_p[0,1]$ for $1 \leq p < 2$ have a very rich subspace structure (*cf.* [39], [40]); in particular L_r embeds into L_p if $p < r \leq 2$. However, there are some suggestive results in the literature which tend to indicate the possibility of strong conclusions if Y is "on the other side of 2."

We first observe that Johnson, Maurey, Schechtman and Tzafriri [21] Theorem 1.8, showed that if X is a Banach lattice which embeds into $L_p[0,1]$ where $p > 2$ and X is r-convex for some $r > 2$ (or, equivalently ℓ_2 is not lattice finitely representable in X) then X is lattice-isomorphic to $L_p(\mu)$ for some measure μ, and so is lattice-isomorphic to a sublattice of L_p. Note that this result requires no symmetry conditions on X. For the case when X is an r. i. space on $[0,1]$ there are some other positive results. In [21] Theorem 7.7 shows that if $Y = L_F[0,\infty)$ is a p-convex Orlicz space, with nontrivial concavity, where $p > 2$ and if X is an r. i. space on $[0,1]$ which embeds into Y, with $X \neq L_2[0,1]$, then X must be lattice-isomorphic to a sublattice of Y. Later Carothers [5] proved the same result for the Lorentz spaces $L_{p,q}$ where $2 < q < p$. These spaces are also strictly 2-convex (*i.e.* r-convex for some $r > 2$). However in [6], Carothers extended his work to the Lorentz spaces $L_{p,q}$ where $1 \leq q \leq 2 < p$. These spaces are not even 2-convex.

Our main results include all these previous theorems. In Theorem 7.2, we show that if Y is a strictly 2-convex r. i. space on $[0,\infty)$ with nontrivial concavity and X is a strictly 2-convex Banach lattice then if X embeds into Y, then X is lattice-isomorphic to a sublattice of Y. The assumption of strict 2-convexity on Y can be relaxed for a special class of r. i. spaces which we term of Orlicz-Lorentz type (this class includes all reflexive Orlicz and Lorentz spaces); if Y is of Orlicz-Lorentz type we need only assume that Y is 2-convex or that its lower Boyd index $p_Y > 2$. In the case when Y is an r. i. space on $[0,1]$ our results are not quite as good; for example if Y is strictly 2-convex and has nontrivial concavity and X is strictly 2-convex we deduce only that some nontrivial band in X is lattice-isomorphic to a sublattice of Y. In the case when X is an r. i. space on $[0,1]$ we give (Corollary 7.4) a very general result which includes the above mentioned results of [5], [6] and [21] for Orlicz and Lorentz spaces. Precisely, suppose Y is an r. i. space on $[0,1]$ or $[0,\infty)$ with nontrivial concavity and suppose that either Y is strictly 2-convex or Y is of Orlicz-Lorentz type with $p_Y > 2$; suppose X is an r. i. space on $[0,1]$ which embeds into Y. Then either $X = L_2[0,1]$ or X is lattice-isomorphic to a sublattice of Y (so that $X = Y_f[0,1]$, for some $f \in Y$).

We also give a result on embedding $L_p[0,1]$ where $p > 2$ into a p-concave r. i. space Y. We show in Theorem 7.7 that this implies that either the Haar basis of L_p is lattice finitely representable in Y or $Y[0,1] = L_p[0,1]$. The former alternative is impossible if Y is of Orlicz-Lorentz type or is strictly 2-convex.

Let us now briefly discuss the method of proof of these results. For reasons discussed below, we consider quasi-Banach lattices and develop a theory of cone-embeddings. If X and Y are quasi-Banach lattices, a cone-embedding $L: X \longrightarrow Y$ is a positive linear operator such that for some $\delta > 0$, $\|Lx\|_Y \geq \delta \|x\|_X$ for every $x \geq 0$. We consider cone-embeddings in Sections 4 and 5. The aim is to produce conditions on X and Y so that one can pass from the existence of a cone-embedding to the existence of a lattice-embedding. Crucial use is made of the theory of random measure representations of positive operators. A typical result is that if X is strictly 1-convex and if Y is an r. i. space on $[0,\infty)$ which is an interpolation space between L_1 and L_∞ then if X cone-embeds into Y it also lattice-embeds. The assumption on Y is satisfied if Y is a Banach r. i. space, by the Calderón-Mityagin theorem, but also holds for certain non-Banach examples, where the lower Boyd index $p_Y > 1$.

The next step carried out in Section 6 is to consider the case when X is a Banach lattice which embeds into an r. i. space Y. The aim here is to put hypotheses on X and Y so that one can induce a cone-embedding $L: X_{1/2} \longrightarrow Y_{1/2}$ where $X_{1/2}, Y_{1/2}$ are the 2-concavifications of X and Y (these spaces may not be locally convex). This can be done if one puts a somewhat technical hypothesis on X and Y (Theorem 6.7). To put this hypothesis in perspective, let us note that if X is an r. i. space on $[0, 1]$ and one aimed simply to guarantee that $L \neq 0$ it would suffice to assume that the Haar basis of X was not equivalent to a disjoint sequence in Y. This is a typical hypothesis in [21] (Theorems 5.1 and 6.1) where the aim is only to draw the weaker conclusion that $X[0,1] \subset Y[0,1]$. In fact some (and perhaps all) of these results can be recovered from our method. However, to obtain X as a sublattice we need L to be a cone-embedding. Fortunately our stronger technical condition is satisfied when Y is strictly 2-convex or of Orlicz-Lorentz type.

Finally one can put these steps together and obtain, under the right hypotheses, that if X embeds into Y then $X_{1/2}$ lattice-embeds into $Y_{1/2}$ and so X lattice-embeds into Y.

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2. **Definitions and notation.** We first recall that a (quasi-)Banach lattice X is said to be *order-continuous* if and only if every order-bounded increasing sequence is norm convergent (see [34] p. 7). A quasi-Banach lattice which does not contain a copy of c_0 is automatically order-continuous but the converse is false. An atom in a Banach lattice is a positive element a so that $0 \leq x \leq a$ implies that $x = \alpha a$ for some $0 \leq \alpha \leq 1$. A Banach lattice is nonatomic if it contains no atoms. The reader is referred to Lindenstrauss-Tzafriri [34] or Meyer-Nieberg [36] as a general reference for Banach lattices.

We will in general use the same notation as in [27]. Let Ω be a Polish space (*i.e.* a separable complete metric space) and let μ be a σ-finite Borel measure on Ω. We refer to the pair (Ω, μ) as a Polish measure space; if μ is a probability measure then we say

(Ω, μ) is a Polish probability space. If E is a Borel set then χ_E denotes its indicator function. We denote by $L_0(\mu)$ the space of all Borel measurable functions on Ω, where we identify functions differing only on a set of measure zero; the natural topology of L_0 is convergence in measure on sets of finite measure. If $0 < p \leq 1$, an admissible p-norm is then a lower-semi-continuous map $f \longrightarrow \|f\|$ from $L_0(\mu)$ to $[0, \infty]$ such that:

(a) $\|\alpha f\| = |\alpha|\,\|f\|$ whenever $\alpha \in \mathbf{R}, f \in L_0$.

(b) $\|f+g\|^p \leq \|f\|^p + \|g\|^p$, for $f, g \in L_0$.

(c) $\|f\| \leq \|g\|$, whenever $|f| \leq |g|$ a.e. (almost everywhere).

(d) $\|f\| < \infty$ for a dense set of $f \in L_0$,

(e) $\|f\| = 0$ if and only if $f = 0$ a.e.

If $p = 1$, we call $\| \;\|$ an admissible norm; an admissible quasinorm is an admissible p-norm for some $0 < p \leq 1$.

A quasi-Köthe function space on (Ω, μ) is defined to be a dense order-ideal X in $L_0(\mu)$ with an associated admissible quasinorm $\| \;\|_X$ such that if $X_{\max} = \{f : \|f\|_X < \infty\}$ then either:

(1) $X = X_{\max}$ (X is *maximal*) or:

(2) X is the closure of the simple functions in $X_{\max}$ (X is *minimal*).

If $\| \;\|_X$ is a norm then X is called a *Köthe function* space. Notice that according to our description we consider $\| \;\|_X$ to be well-defined on L_0. Any order-continuous Köthe function space is minimal. Also any Köthe function space which does not contain a copy of c_0 is both maximal and minimal.

Given any Köthe function space X and $0 < p < \infty$ we define X_p to be the quasi-Köthe space of all f such that $|f|^p \in X$ with the associated admissible quasinorm $\|f\|_{X_p} = \big\| |f|^p \big\|_X^{1/p}$. It is readily verified that $\| \;\|_{X_p}$ is an admissible p-norm when $0 < p < 1$ and an admissible norm when $p > 1$. We will primarily use the case $p = 1/2$ in this paper. We will also use the subscript + to denote the positive cone in a variety of situations, *e.g.* $X_+ = \{f : f \in X, f \geq 0\}$.

If X is an order-continuous Köthe function space then X^* can be identified with the Köthe function space of all f such that:

$$\|f\|_{X^*} = \sup_{\|g\|_X \leq 1} \int |fg| \, d\mu < \infty.$$

X^* is always maximal.

If μ is a probability measure then we say following [21], that a Köthe function space X is *good* if $L_\infty \subset X \subset L_1$ and further for $f \in L_0$, $\|f\|_1 \leq \|f\|_X \leq 2\|f\|_\infty$. It is well-known that any separable order-continuous Banach lattice can be represented as (*i.e.* is isometrically lattice-isomorphic to) a good Köthe function space on some Polish probability space (Ω, μ) (see [21] and [34]).

In the case when X is nonatomic we can require that $\Omega = [0, 1]$ and $\mu = \lambda$ is Lebesgue measure. Alternatively we can take $\Omega = \Delta = \{-1, +1\}^{\mathbf{N}}$ to be the Cantor group and take μ to be normalized Haar measure on Δ which we again denote by λ. We will use this second representation freely and now take the opportunity to introduce some notation from [27].

Thus for $\epsilon_k = \pm 1$, we denote by $\Delta(\epsilon_1, \ldots, \epsilon_n)$ the clopen subset of Δ of all $(d_j)_{j=1}^{\infty}$ such that $d_j = \epsilon_j$ for $1 \leq j \leq n$. For each n let $\mathcal{A}_n$ denote the collection of $\Delta(\epsilon_1, \ldots, \epsilon_n)$. Let CS_n denote the linear span of $\{\chi_E : E \in \mathcal{A}_n\}$. We also define the Haar functions $h_E = \chi_{\Delta(\epsilon_1,\ldots,\epsilon_n,+1)} - \chi_{\Delta(\epsilon_1,\ldots,\epsilon_n,-1)}$ for $E = \Delta(\epsilon_1, \ldots, \epsilon_n)$.

A Köthe function space (or, more generally a quasi-Köthe function space) X is said to be *p-convex* (where $0 < p < \infty$) if there is a constant C such that for any $f_1, \ldots, f_n \in X$ we have

$$\Big\| \Big(\sum_{i=1}^{n} |f_i|^p \Big)^{1/p} \Big\|_X \leq C \Big(\sum_{i=1}^{n} \|f_i\|_X^p \Big)^{1/p}.$$

X is said to have an *upper p*-estimate if for some C and any disjoint $f_1, \ldots, f_n \in X$,

$$\Big\| \sum_{i=1}^{n} f_i \Big\|_X \leq C \Big(\sum_{i=1}^{n} \|f_i\|_X^p \Big)^{1/p}.$$

X is said to be *q-concave* $(0 < q < \infty)$ if for some $c > 0$ and any $f_1, \ldots, f_n \in X$ we have

$$\Big\| \Big(\sum_{i=1}^{n} |f_i|^q \Big)^{1/q} \|_X \geq c \Big(\sum_{i=1}^{n} \|f_i\|_X^q \Big)^{1/q}.$$

X is said to have a *lower q*-estimate if for some $c > 0$ and any disjoint $f_1, \ldots, f_n \in X$,

$$\Big\| \sum_{i=1}^{n} f_i \Big\|_X \geq c \Big(\sum_{i=1}^{n} \|f_i\|_X^q \Big)^{1/q}.$$

Notice that a quasi-Köthe function space which satisfies a lower q-estimate is automatically both maximal and minimal since it cannot contain a copy of c_0. A Köthe function space must, of course, be 1-convex. A quasi-Köthe function space must satisfy an upper p-estimate for some $p > 0$ but need not be p-convex for any $p > 0$; however, if X satisfies a lower q-estimate for some $q < \infty$ then it is p-convex for some $p > 0$. This result is proved in [24] (Theorems 4.1 and 2.2) and a simpler proof is presented in [30] Theorem 3.2. A quasi-Köthe function space which is s-convex for some $s > 0$ and satisfies an upper r-estimate is p-convex for every $0 < p < r$ (see [24]).

A (quasi-)Banach lattice X is p-convex, satisfies an upper p-estimate, is q-concave or satisfies a lower q-estimate according as any concrete representation of X as a Köthe function space has the same property. We shall say that X is *strictly p-convex* if it is r-convex for some $r > p$ and *strictly q-concave* if it is s-concave for some $s < q$.

A Banach space X is said to be of (Rademacher) type p $(1 \leq p \leq 2)$ if there is a constant C so that for any $x_1, \ldots, x_n \in X$,

$$\mathrm{Ave}_{\epsilon_i = \pm 1} \Big\| \sum_{i=1}^{n} \epsilon_i x_i \Big\| \leq C \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{1/p}$$

and X is of cotype q $(2 \leq q < \infty)$ if for some $c > 0$ and any $x_1, \ldots, x_n \in X$ we have

$$\mathrm{Ave}_{\epsilon_i = \pm 1} \Big\| \sum_{i=1}^{n} \epsilon_i x_i \Big\| \geq c \Big(\sum_{i=1}^{n} \|x_i\|^q \Big)^{1/q}.$$

We recall that a (quasi-)Banach lattice has nontrivial cotype (*i.e.* has cotype $q < \infty$ for some q) if and only if it has nontrivial concavity (*i.e.* is q-concave for some $q < \infty$). If X is a Banach lattice which has nontrivial concavity then there is a constant $C = C(X)$ so that for any $x_1, \ldots, x_n \in X$ we have

$$\frac{1}{C}\Big(\operatorname{Ave}_{\epsilon_k=\pm 1}\Big\|\sum_{k=1}^{n}\epsilon_k x_k\Big\|^2\Big)^{1/2} \leq \Big\|\Big(\sum_{k=1}^{n}|x_k|^2\Big)^{1/2}\Big\|_X \leq C\Big(\operatorname{Ave}_{\epsilon_k=\pm 1}\Big\|\sum_{k=1}^{n}\epsilon_k x_k\Big\|^2\Big)^{1/2}.$$

In fact we will need the same conclusion for quasi-Banach lattices; as far as we know this has never been explicitly stated although it is probably well-known. We therefore state it formally as a proposition.

PROPOSITION 2.1. *Let X be a quasi-Banach lattice with nontrivial concavity (equivalently nontrivial cotype). Then there is a constant $C = C(X)$ so that for any $x_1, \ldots, x_n \in X$ we have*

$$\frac{1}{C}\Big(\operatorname{Ave}_{\epsilon_k=\pm 1}\Big\|\sum_{k=1}^{n}\epsilon_k x_k\Big\|^2\Big)^{1/2} \leq \Big\|\Big(\sum_{k=1}^{n}|x_k|^2\Big)^{1/2}\Big\|_X \leq C\Big(\operatorname{Ave}_{\epsilon_k=\pm 1}\Big\|\sum_{k=1}^{n}\epsilon_k x_k\Big\|^2\Big)^{1/2}.$$

PROOF. We have that X is q-concave for some $q < \infty$. As remarked above it is also p-convex for some $p > 0$. It is now easy to adapt the standard argument based on Khintchine's inequality as in [34] Theorem 1.d.6, p. 49. ∎

REMARK. In fact we will only apply this proposition in situations when the p-convexity of X for some $p > 0$ is automatic (*i.e.* X is the concavification of some Köthe function space).

Let us now turn to rearrangement-invariant spaces (*cf.* [21], [34]). For any $f \in L_0(\Omega, \mu)$ we define its decreasing rearrangement $f^* \in L_0\big[0, \mu(\Omega)\big)$ by $f^*(t) = \inf\{x : \mu(|f| > x) \leq t\}$. Now let X be a quasi-Köthe function space on either $[0, \infty)$ or $[0, 1]$ with Lebesgue measure. We say that X is a quasi-Banach *rearrangement-invariant* (r. i.) space if $\|f\|_X = \|f^*\|_X$ for all $f \in L_0$, and if $\|\chi_{[0,1]}\|_X = 1$. We use the term r. i. space for a Banach r. i. space. If X is a quasi-Banach r. i. space on $[0, \infty)$ (respectively, $[0, 1]$) and (Ω, μ) is a Polish measure space (respectively, with $\mu(\Omega) \leq 1$,) then we define $X(\Omega, \mu)$ to be the set of $f \in L_0(\mu)$ such that $f^* \in X$ with $\|f\|_X = \|f^*\|_X$. For example, it will be of some advantage to consider $X(\Delta, \lambda)$ in place of $X[0, 1]$. Let us remark that if X is a quasi-Banach r. i. space on $[0, 1]$ then it is always possible to write $X = Y[0, 1]$ where Y is some quasi-Banach r. i. space on $[0, \infty)$. We will only be interested in separable (or order-continuous) r. i. spaces, which are necessarily minimal.

On any quasi-Banach r. i. space X on $[0, \infty)$ (resp. $[0, 1]$) we define the dilation operators D_s for $0 < s < \infty$ by

$$D_s f(t) = f(t/s)$$

for all t (resp. whenever $0 \leq t \leq \min(1, s)$ and $D_s f(t) = 0$ otherwise). The Boyd indices p_X and q_X are defined by

$$p_X = \lim_{s \to \infty} \frac{\log s}{\log \|D_s\|}$$

$$q_X = \lim_{s\to 0} \frac{\log s}{\log \|D_s\|}.$$

In general $0 < p_X \leq q_X \leq \infty$; if X is a Banach r. i. space (*i.e.* is 1-convex) then $1 \leq p_X$. If X is an order-continuous Banach r. i. space, then X has an unconditional basis if and only if $1 < p_X \leq q_X < \infty$; in this case the Haar basis of X is an unconditional basis (see [34] p. 157–161).

Recall that if $f \in L_0(\Omega, \mu)$ then $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$, for $t > 0$. We say that a quasi-Banach r. i. space X on $[0,1]$ or $[0,\infty)$ has property (d) if there exists C so that if $f \in X$ and $g \in L_0$ satisfy $g^{**} \leq f^{**}$ then $g \in X$ with $\|g\|_X \leq C\|f\|_X$. It is well-known that every Banach r. i. space satisfies property (d) (*cf.* [34] p. 125) with $C = 1$. However there are non-locally convex examples; any quasi-Banach r. i. space X with $p_X > 1$ satisfies property (d) (see [26]). A quasi-Banach r. i. space with property (d) is an interpolation space for the pair (L_1, L_∞); this is a mild generalization of the classical Calderón-Mityagin theorem ([4], [35]) which follows from considerations of the K-functional (see, for example Bennett-Sharpley [2], Chapters 3 and 5; this treats only the normed case, but the modifications are trivial).

We also recall a definition from [29]. If X is an r. i. space on $[0,\infty)$ (resp. $[0,1]$) we define E_X to be the closed subspace of X spanned by the functions $e_n = \chi_{[2^n,2^{n+1})}$ for $n \in \mathbf{Z}$ (resp. $n \in \mathbf{Z}_- = \{n : n < 0\}$). If X is separable then (e_n) forms an unconditional basis for E_X and E_X can be regarded as a sequence space modelled on $\mathbf{J} = \mathbf{Z}$ or $\mathbf{Z}_-$. We shall say that X is of *Orlicz-Lorentz type* if E_X is naturally isomorphic to a modular sequence space, *i.e.* there exist Orlicz functions $(F_n)_{n\in\mathbf{J}}$ so that $E_X = \ell_{(F_n)}(\mathbf{J})$ (see [33] pp. 168ff). This is a convenient definition to specify a class of spaces X which includes the standard Orlicz spaces and Lorentz spaces, and a variety of "mixed" spaces.

To illustrate these ideas consider the following method of defining an r. i. space on $[0,\infty)$. Let Y be a Köthe function space on $[0,\infty)$ with the property that the dilation operators $D_t\colon Y \longrightarrow Y$ are all bounded. Then we can define p_Y, q_Y as in the rearrangement-invariant case. Assume that $1 < p_Y \leq q_Y < \infty$. Now let $\tilde{Y}$ be the space defined by $f \in \tilde{Y}$ if and only if $f^* \in Y$ and define $\|f\|_{\tilde{Y}} = \|f^*\|_Y$. The inequality $(f+g)^* \leq 2D_2 f^* + 2D_2 g^*$ shows that $\|\;\|_{\tilde{Y}}$ is a quasinorm and that $\tilde{Y}$ is an order-ideal. In fact, we also have:

PROPOSITION 2.2. *There exists a constant C so that if $f \in L_0$ then*

$$\|f\|_{\tilde{Y}} \leq \Big\|\sum_{n\in\mathbf{Z}} f^{**}(2^n)e_n\Big\|_Y \leq C\|f\|_{\tilde{Y}}.$$

PROOF (DUE TO S. MONTGOMERY-SMITH). Clearly $f^* \leq \sum_{n\in\mathbf{Z}} f^{**}(2^n)e_n$. However $f^{**}(2^n) \leq \sum_{k=1}^{\infty} 2^{-k} f^*(2^{n-k})$. Hence $\sum_{n\in\mathbf{Z}} f^{**}(2^n)e_n \leq \sum_{k=1}^{\infty} 2^{-k} D_{2^{k+1}} f^*$. But now since $p_Y > 1$ it follows that $\sum_{k=1}^{\infty} 2^{-k}\|D_{2^{k+1}}\|_Y < \infty$ and the result follows. ∎

The proof above only uses the hypothesis that $p_Y > 1$, and not that $q_Y < \infty$. Proposition 2.2 shows that $\tilde{Y}$ is a Banach r. i. space by providing an equivalent norm. It is now immediate that $p_Y \leq p_{\tilde{Y}}$. We next show that $E_{\tilde{Y}}$ coincides with E_Y. This implies that if Y is an Orlicz-Musielak space or generalized Orlicz space (*cf.* [37]) then the associated

r. i. space $\tilde{Y}$ is of Orlicz-Lorentz type as defined above. In particular, if we take Y to be a weighted L_p-space (with, of course the conditions $1 < p_Y \leq q_Y < \infty$) we obtain the usual Lorentz spaces as examples of spaces of Orlicz-Lorentz type.

PROPOSITION 2.3. *We have $E_{\tilde{Y}} = E_Y$ (and the norms are equivalent).*

PROOF. In fact suppose $f = \sum_{n\in\mathbf{Z}} a_n e_n$ where $a_n \geq 0$ is finitely nonzero. Let $g = \sum_{n=0}^{\infty} D_{2^{-n}} f$. The assumption $q_Y < \infty$ and the fact that $q_{\tilde{Y}} \leq q_Y$ is sufficient to establish that $\|g\|_Y \leq C\|f\|_Y$ and $\|g\|_{\tilde{Y}} \leq C\|f\|_{\tilde{Y}}$ for a suitable constant C. Note that $\|g\|_Y = \|g\|_{\tilde{Y}}$ since g is decreasing. The result follows immediately. ■

3. **Remarks on sublattices.** In this section, we collect together some elementary remarks on the structure of sublattices of r. i. spaces.

LEMMA 3.1. *Suppose X is a quasi-Köthe function space on (Ω, μ) and that Y is a quasi-Banach* r. i. *space on $[0,\infty)$. Suppose $I = [0,1]$ or $[0,\infty)$ and that $U: X \longrightarrow Y(I)$ is a lattice homomorphism. Then there is a lattice homomorphism $V: X \longrightarrow Y\big(\Omega \times [0,\infty)\big)$ so that for any $x \in X$ and $\alpha > 0$ we have*

$$\frac{1}{2}\lambda(U|x| > 2\alpha) \leq (\mu \times \lambda)(V|x| > \alpha) \leq \lambda(U|x| > \alpha),$$

and such that V can be represented as $Vx(\omega,t) = a(\omega,t)x(\omega)$ where a is a nonnegative Borel function on $\Omega \times [0,\infty)$ of the form

$$a(\omega,t) = \sum_{k\in\mathbf{Z}} 2^{m(k,\omega)} e_k(t)$$

with $m: \mathbf{Z} \times \Omega \longrightarrow \mathbf{Z} \cup -\infty$ is a Borel map with $k \longrightarrow m(k,\omega)$ decreasing for each ω. Furthermore if $I = [0,1]$ then a is supported on a set of measure one in the product space.

PROOF. It will suffice to consider the case when X contains L_∞. We suppose the existence of a lattice embedding $Ux = bx \circ \sigma$ where b is a nonnegative Borel function and $\sigma: I \longrightarrow \Omega$ is a Borel map. First pick b' with $\frac{1}{2}b \leq b' \leq b$ so that $b' = \sum_{n\in\mathbf{Z}} 2^n \chi_{E_n}$ where E_n are disjoint Borel sets. Let $U'x = b'x \circ \sigma$.

Now for each n define the measure $\nu_n(B) = \lambda(\bigcup_{k\geq n} E_k \cap \sigma^{-1}B)$. Since $U'\chi_\Omega \in Y$ it is clear that each ν_n is a finite measure. Furthermore, if $\mu B = 0$ then $U\chi_B = 0$ a.e. and hence $\nu_n(B) = 0$. Hence we can find nonnegative Borel functions w_n on Ω so that $\nu_n(B) = \int_B w_n \, d\mu$, and we may suppose that $w_n(\omega)$ is decreasing for each fixed ω. Notice that $\int_\Omega w_n \, d\mu = \nu_n(\Omega) \leq \lambda(I)$, so that if $I = [0,1]$ then $\int_\Omega w_n \, d\mu \leq 1$ for all n,

For any fixed $n \in \mathbf{Z}$, we define $A_n = \{(\omega,t) : t \leq w_n(\omega)\}$ and let $a' = \sum_{n\in\mathbf{Z}} 2^n(\chi_{A_n} - \chi_{A_{n+1}})$. Define $V': X \longrightarrow Y\big(\Omega \times (0,\infty)\big)$ by $V'x(\omega,t) = a'(\omega,t)x(t)$. Finally define a a Borel function on $\Omega \times (0,\infty)$ by setting $a(\omega,t) = 2^m$ if $a'(\omega,2^{k+1}) = 2^m$ where $2^k \leq t < 2^{k+1}$ and $k,m \in \mathbf{Z}$. We set $a(\omega,t) = 0$ if $a'(\omega,2^{k+1}) = 0$. Notice that $(\mu \times \lambda)\{a > 0\} \leq (\mu \times \lambda)\{a' > 0\} \leq 1$ if $I = [0,1]$. Define $Vx(\omega,t) = a(\omega,t)x(t)$.

Now suppose $x \geq 0, x \in X$. Then $0 \leq Vx \leq V'x$. Furthermore for fixed ω,

$$\lambda\{t : V'x(\omega, t) > \alpha\} \leq 2\lambda\{t : Vx(\omega, t) > \alpha\}$$

so that

$$(\mu \times \lambda)(Vx > \alpha) \geq \frac{1}{2}(\mu \times \lambda)(V'x > \alpha).$$

Now again for fixed α, let $F_n = \{2^n x < \alpha \leq 2^{n+1}x\}$. We note that

$$\begin{aligned}(\mu \times \lambda)(V'x > \alpha) &= \sum_{n \in \mathbf{Z}} \int_{F_n} w_n \, d\mu \\ &= \sum_{n \in \mathbf{Z}} \nu_n(F_n) \\ &= \sum_{n \in \mathbf{Z}} \lambda\Big(\bigcup_{k \geq n} E_k \cap \sigma_n^{-1} F_n\Big) \\ &= \lambda(U'x > \alpha).\end{aligned}$$

Hence

$$\frac{1}{2}\lambda(U'x > \alpha) \leq (\mu \times \lambda)(Vx > \alpha) \leq \lambda(U'x > \alpha).$$

Since $\frac{1}{2}Ux \leq U'x \leq Ux$ the result follows. ∎

We next state the immediate conclusion for lattice embeddings.

PROPOSITION 3.2. *Let X be a quasi-Köthe function space on (Ω, μ). Suppose Y is a quasi-Banach* r. i. *space on $[0, \infty)$, and suppose that X is lattice-isomorphic to a sublattice of $Y(I)$, where $I = [0, 1]$ or $[0, \infty)$. Then there is a lattice embedding $V: X \longrightarrow Y\big(\Omega \times [0, \infty)\big)$ of the form $Vx(\omega, t) = a(\omega, t)x(\omega)$ where a is a nonnegative Borel function on $\Omega \times [0, \infty)$ of the form:*

$$a(\omega, t) = \sum_{k \in \mathbf{Z}} 2^{m(k,\omega)} e_k(t)$$

where $m: \mathbf{Z} \times \Omega \longrightarrow \mathbf{Z} \cup \{-\infty\}$ is a Borel map such that $k \longrightarrow m(k, \omega)$ is decreasing for each ω. Furthermore if $I = [0, 1]$ then a is supported on a set of finite measure.

If Y is an r. i. space on I=$[0, 1]$ or $[0, \infty)$ and $f \in Y_+ \setminus \{0\}$ then we define Y_f to be the r. i. space on I defined by $y \in Y_f$ if and only if $y \otimes f \in Y(I \times I)$ where $y \otimes f(s, t) = y(s)f(t)$. The norm on Y_f is given by $\|y\|_{Y_f} = \|y \otimes f\|_Y$. Notice that since f dominates a function of the form $\alpha\chi_E$ where $\alpha > 0$ and $\lambda(E) > 0$ there exists a constant C depending on f so that $\|y\|_Y \leq C\|y\|_{Y_f}$.

PROPOSITION 3.3. *Suppose Y is an order-continuous quasi-Banach* r. i. *space on $[0, \infty)$ and that X is an order-continuous quasi-Banach* r. i. *space on $[0, 1]$. Let $U: X \longrightarrow Y$ be a lattice homomorphism and let $U\chi_{[0,1]} = f \neq 0$. Then:*

(1) There exists C so that if $x \in X$ then $\|x\|_{Y_f} \leq C\|x\|_X$.
(2) If U is a lattice embedding then $X = Y_f[0, 1]$.

REMARK. If U is a lattice embedding of X into $Y[0, 1]$ then the above proposition gives $X = Y_f[0, 1]$ where $f \in Y[0, 1]$.

PROOF. We use Lemma 3.1 to construct the lattice homomorphism $V: X \longrightarrow Y([0,1]\times[0,\infty))$. Notice that if $g \in Y[0,\infty)$ has the same distribution as $V\chi_{[0,1]}$ then $Y_f[0,1] = Y_g[0,1]$ with equivalent norms.

Let u be any nonnegative simple function on $[0,1]$ of the form $u = \sum_{j=1}^n \alpha_j \chi_{B_j}$ where $\{B_1,\ldots,B_n\}$ is a Borel partition of $[0,1]$. For any N let

$$a_N(s,t) = \sum_{|k|\le N} \sum_{|m(k,s)|\le N} 2^{m(k,s)} e_k(t)$$

and let $b_N = a - a_N$. We can partition $g = g_N + h_N$ where g_N has the same distribution as a_N and h_N has the same distribution as b_N.

Now $a_N = \sum_{|k|\le N}\sum_{|l|\le N} 2^l \chi_{A_{kl}}(s) e_k(t)$ where $(A_{kl})_{k,l}$ are Borel subsets of $[0,1]$. We can therefore use Liapunoff's theorem to find Borel sets $B_1',\ldots,B_n'$ so that $\lambda(B_j') = \lambda(B_j)$ for all j and $\lambda(B_j'\cap A_{kl}) = \lambda(B_j)\lambda(A_{kl})$ whenever $1 \le j \le n$ and $-N \le k,l \le N$. Let $u' = \sum_{j=1}^n \alpha_j \chi_{B_j'}$. Then $a_N(s,t)u'(t)$ has the same distribution as $u \otimes g_N$. Hence

$$\|u\otimes g_N\|_Y \le \|Vu'\|_Y \le \|u\otimes g_N\|_Y + \|u\|_\infty \|\chi_{[0,1]}\otimes h_N\|_Y.$$

For case (1) we let $N \longrightarrow \infty$ and deduce that $\|u\|_{Y_g} \le \|U\|\,\|u\|_X$.

For case (2) we observe that, since Y is order-continuous,

$$\lim_{N\to\infty} \|\chi_{[0,1]}\otimes h_N\|_Y = 0.$$

Since V is an embedding there exists $c > 0$ so that we have a lower-estimate $\|Vu'\|_Y \ge c\|u\|_X$. Hence $\|u\|_X \le c^{-1}\|u\|_{Y_g}$.

If X lattice embeds into $Y[0,1]$ then a has support of measure at most one and hence so has f so that we can assume that $f \in Y[0,1]$. ∎

COROLLARY 3.4. *Suppose Y is an order-continuous quasi-Banach* r. i. *space on $[0,\infty)$ and that X is an order-continuous quasi-Banach* r. i. *space on $[0,1]$. Let $U: X \longrightarrow Y$ be a lattice homomorphism. If $U \ne 0$ then there is a constant C so that $\|x\|_Y \le C\|x\|_X$ for $x \in X[0,1]$.*

PROOF. This follows from (1) of the preceding proposition combined with the remarks before it. ∎

REMARK. This corollary is well-known (see Abramovich [1] and remarks in the introduction to [27]).

For our final result of this section, we will need the following factorization theorem, which is essentially due to Krivine [31] ([34], Theorem 1.d.11 and Corollary 1.d.12, pp. 57–59); we will, however, prove the form of the theorem required here.

PROPOSITION 3.5. *Suppose $0 < p < \infty$. Suppose Y is an p-concave quasi-Köthe function space on (Ω,μ) and suppose that either (a) $P: L_p(\Delta,\lambda) \longrightarrow Y$ is a lattice homomorphism or (b) $p \ge 1$ and $P: L_p(\Delta,\lambda) \longrightarrow Y$ is a positive operator. Then there is a Borel function $w \in L_0(\mu)$ with $w > 0$ a.e. so that*

$$\|f\|_Y \le \|fw\|_p$$

for $f \in L_0(\mu)$ *and*

$$\|w(Pf)\|_p \le \|P\| \, \|f\|_p$$

for $f \in L_p(\Delta)$.

PROOF. We can suppose $P \neq 0$. We require the following property of P which is valid in cases (a) or (b): if $f_1, \ldots, f_n \ge 0$ in L_p then $P\big((\sum_{i=1}^n f_i^p)^{1/p}\big) \ge (\sum_{i=1}^n (Pf_i)^p)^{1/p}$ (see [34] p. 55). Let u be any strictly positive function in Y. Now consider the subsets E and F of L_∞ defined by $E = \{f : f \ge 0, \|uf^{1/p}\|_Y > \|P\|\}$ and $F = \{f : \exists 0 \le x \in L_p, \|x\|_p \le 1, u^p f \le (Px)^p\}$.

It is clear that E is convex. We argue that $\operatorname{co} F$ does not meet E. Indeed suppose $f_1, \ldots, f_n \in F$ and $c_1, \ldots, c_n \ge 0$ with $\sum_{j=1}^n c_j = 1$. Suppose $u^p f_j \le (Px_j)^p$ where $x_j \ge 0$ and $\|x_j\|_p \le 1$. Then $u^p(\sum_{j=1}^n c_j f_j) \le \sum_{j=1}^n c_j (Px_j)^p \le (Py)^p$ where $y = (\sum_{j=1}^n c_j x_j^p)^{1/p}$ so that $\|y\|_p \le 1$ (see [34] Proposition 1.d.9). Since F includes the negative cone it has non-empty interior. Now, by the Hahn-Banach theorem, there exists $\Phi \in L_\infty^*$ so that $\Phi(f - g) > 0$ if $f \in E$ and $g \in F$. Clearly $\Phi \ge 0$, and $\Phi(f) > 0$ if $f \ge 0$ and $f \neq 0$; hence since P is not zero we have $\inf_{g \in E} \Phi(g) > 0$. By normalizing we can suppose $\inf_{g \in E} \Phi(g) = 1$. Let us write $\Phi(f) = \int f \phi \, d\mu + \Phi_0(f)$ where $\phi \in L_1(\mu)$, and Φ_0 is singular with respect to μ. If $f \in E$ we may find $0 \le f_n \uparrow f$ a.e. so that $\Phi(f_n) \uparrow \int f \phi \, d\mu$. However by order continuity $f_n \in E$ for large enough n and so $\int f \phi \, d\mu \ge 1$ for $f \in E$.

Now it is clear that if $y \in Y_+$ with $\|y\|_Y = 1$. Then for $\epsilon > 0$ we have that $(\|P\| + \epsilon)^p y^p u^{-p} \in E$ and so $\|yu^{-1}\phi^{1/p}\|_p \ge \|P\|^{-1}$. Thus if $y \in Y$ then $\|y\|_Y \le \|yw\|_p$ where $w = \|P\| \phi^{1/p} u^{-1}$. If $f \in L_p(\Delta, \lambda)$ with $\|f\|_p = 1$ then $\big(P(|f|)\big)^p u^{-p} \in F$ and so

$$\int \big(P(|f|)\big)^p \phi u^{-p} \, d\mu \le 1,$$

so that $\|wP(|f|)\|_p \le \|P\|$ which implies the theorem. ∎

THEOREM 3.6. *Suppose* $0 < p < \infty$ *and* Y *is a* p*-concave quasi-Banach* r. i. *space on* $[0,1]$ *or* $[0,\infty)$. *Suppose* L_p *is lattice-isomorphic to a sublattice of* Y. *Then* $Y[0,1] = L_p[0,1]$.

PROOF. It suffices to consider the case when $Y = Y[0,\infty)$. By Proposition 3.3 there exists $f \in Y$ so that $Y_f[0,1] = L_p[0,1]$. Thus there is a lattice embedding $V : L_p \to Y\big([0,1] \times [0,\infty)\big)$ of the form $x \to x \otimes f$. We assume $\|x\|_p \le \|x \otimes f\|_Y \le C\|x\|_p$. Applying Proposition 3.5, there is a nonnegative weight function w on $[0,1] \times [0,\infty)$ so that $\|y\|_Y \le \|yw\|_p$ for $y \in Y$ and $\|x\|_p \le \|x \otimes f\|_Y \le \|w(x \otimes f)\|_p \le C\|x\|_p$ for $x \in L_p$.

Now let $v(t) = (\int_0^1 w(s,t)^p \, ds)^{1/p}$. It follows from a symmetrization argument that if $y \in Y$ then

$$\|y\|_Y \le \Big(\int_0^1 \int_0^\infty v(t)^p |y(s,t)|^p \, ds \, dt\Big)^{1/p},$$

and that

$$\int_0^\infty f(t)^p v(t)^p \, dt \le C^p.$$

Let u be the increasing rearrangement of v so that $u(t) = \inf_{\lambda(E)=t} \sup_{s\in E} v(s)$. Then if as usual y^* is the decreasing rearrangement of $|y|$, the first equation yields that if $y \in Y[0,\infty)$ then

$$\|y\|_Y \leq \Big(\int_0^\infty y^*(t)^p u(t)^p \, dt\Big)^{1/p}.$$

In particular for $0 < s < 1$,

$$s \leq \|D_s f^*\|_Y^p \leq \int_0^\infty f^*(t/s)^p u(t)^p \, dt.$$

This in turn implies that

$$\int_0^\infty f^*(t)^p u(st)^p \, dt \geq 1.$$

Now $\int_0^\infty f^*(t)^p u(t)^p \, dt \leq C^p$. Letting $s \longrightarrow 0$ we obtain from the Dominated Convergence Theorem that $\lim_{t\to 0} u(t) = c > 0$ and $\int_0^\infty f(t)^p \, dt \leq C^p c^{-p}$.

Pick $0 < \tau < \infty$ so that $\|f^*\chi_{[\tau,\infty)}\|_Y \leq 1/2$. It follows from p-concavity that

$$\|D_s(f^*\chi_{[\tau,\infty)})\|_Y \leq s^{1/p}/2.$$

On the other hand $\|D_s f^*\|_Y = \|\chi_{[0,s]} \otimes f\|_Y \geq s^{1/p}$. Hence $\|D_s(f^*\chi_{[0,\tau]})\|_Y \geq s^{1/p}/2$. From this and p-concavity we also obtain easily that

$$\Big(\frac{1}{s\tau}\int_0^{s\tau} f^*(t/s)^p \, dt\Big)^{1/p} \|\chi_{[0,s\tau)}\|_Y \geq \frac{1}{2} s^{1/p}.$$

Hence $\|\chi_{[0,t]}\|_Y \geq c_1 t^{1/p}$ when $0 \leq t \leq 1$ for a suitable constant c_1. This in turn implies, by p-concavity, that if $y \in Y[0,1]$ then $\|y\|_Y \geq c_1\|y\|_p$ and this is enough to show that $Y[0,1] = L_p[0,1]$. ∎

4. **Cone-embeddings.** Let X and Y be quasi-Banach lattices. We will say that a positive operator $L: X \longrightarrow Y$ is a *cone-embedding* if L satisfies a lower bound for positive elements, *i.e.* there exists $\delta > 0$ so that $\|Lx\|_Y \geq \delta\|x\|_X$ for $x \geq 0$. We will say that L is a *strong cone-embedding* if it additionally satisfies the condition that for some $C > 0$ and every $x_1, \ldots, x_n \geq 0$ we have $\|\max_{1\leq k\leq n} x_k\|_X \leq C\|\max_{1\leq k\leq n} Lx_k\|_Y$. This is trivially equivalent to requiring the same inequality for $x_1, \ldots, x_n$ mutually disjoint.

Our first results demonstrate conditions under which every cone-embedding is a strong cone-embedding.

LEMMA 4.1. *Suppose $s, \delta > 0$, and $1 < p, q < \infty$. Then there is a constant $C = C(s,p,q,\delta)$ so that if X is a p-convex Köthe function space, Y is an s-convex, q-concave quasi-Köthe function space (where each constant of convexity and concavity is one) and if $L: X \longrightarrow Y$ is a cone-embedding satisfying $\delta\|x\|_X \leq \|Lx\|_Y \leq \|x\|_X$ for $x \geq 0$ then if $x_1, \ldots, x_n \geq 0$ are disjoint,*

$$\Big\|\sum_{j=1}^n x_j\Big\|_X \leq C\|\max_{1\leq j\leq n} Lx_j\|_Y.$$

PROOF. We pick $m = m(p,\delta)$ so that $2^{m(1-1/p)}\delta > 2$.

First notice that if $x_1,\dots,x_n$ are disjoint,

$$\Big(\mathrm{Ave}_{\epsilon_{ij}=\pm 1}\sum_{j=1}^n\prod_{i=1}^m(1+\epsilon_{ij})^p x_j^p\Big)^{1/p} = 2^{m(1-1/p)}\sum_{j=1}^n x_j.$$

Thus by p-convexity

$$2^{m(1-1/p)}\Big\|\sum_{j=1}^n x_j\Big\|_X \le \Big(\mathrm{Ave}_{\epsilon_{ij}=\pm 1}\Big\|\sum_{j=1}^n\prod_{i=1}^m(1+\epsilon_{ij})x_j\Big\|_X^p\Big)^{1/p}.$$

Now it follows that

$$\begin{aligned}2^{m(1-1/p)}\Big\|\sum_{j=1}^n x_j\Big\|_X &\le \delta^{-1}\Big(\mathrm{Ave}_{\epsilon_{ij}=\pm 1}\Big\|\sum_{j=1}^n\prod_{i=1}^m(1+\epsilon_{ij})Lx_j\Big\|_Y^p\Big)^{1/p}\\ &\le \delta^{-1}\sum_{I\subset[m]}\Big(\mathrm{Ave}_{\epsilon_{ij}=\pm 1}\Big\|\sum_{j=1}^n\prod_{i\in I}\epsilon_{ij}Lx_j\Big\|_Y^p\Big)^{1/p}\\ &\le \delta^{-1}\Big(\Big\|\sum_{j=1}^n Lx_j\Big\|_Y + C_1(2^m-1)\Big\|\Big(\sum_{j=1}^n|Lx_j|^2\Big)^{1/2}\Big\|_Y\Big),\end{aligned}$$

where $C_1 = C_1(q,s)$, using Theorem 1.d.6 of [34].

Reorganizing we have, since $2^{m(1-1/p)}\delta - 1 > 1$, and $\|L\|\le 1$,

$$\begin{aligned}\delta^{-1}\Big\|\sum_{j=1}^n x_j\Big\|_X &\le C_1 2^m\Big\|\Big(\sum_{j=1}^n|Lx_j|^2\Big)^{1/2}\Big\|_Y\\ &\le C_1 2^m\Big\|\sum_{j=1}^n Lx_j\Big\|_Y^{1/2}\|\max_{1\le j\le n} Lx_j\|_Y^{1/2}\end{aligned}$$

and this in turn implies, since Y is s-convex for some $s > 0$,

$$\Big\|\sum_{j=1}^n x_j\Big\|_X \le C_1^2 2^m\delta^2\|\max_{1\le j\le n} Lx_j\|_Y. \qquad \blacksquare$$

Let us give a simple application.

THEOREM 4.2. *Suppose Y is an r-convex Banach lattice where $r > 2$ which is q-concave for some $q < \infty$. Suppose that X is a p-convex Banach lattice, where $p > 2$, which is isomorphic to a subspace of Y. Then X is r-convex.*

REMARKS. This result is well-known for $1 < r \le 2$ (*cf.* [34], p. 51). The hypothesis on X is equivalent to the statement that ℓ_2^n is not lattice finitely representable in X (note that X must be of type 2, and apply Lemma 2.4 of [21]). In [21] there are two results closely related to Theorem 4.2. Theorem 2.3 of [21] is the analogous result for upper r-estimates in place of r-convexity, while Theorem 2.6 (or Proposition 2.e.10 of [34]) implies the above theorem for the special case when X is an r. i. space on $[0,1]$. In this latter case one can replace the hypothesis that X is strictly 2-convex by the weaker hypothesis that $X \neq L_2[0,1]$.

PROOF. It suffices to consider the case when the r-convexity, q-concavity constants of Y are both one and the p-convexity constant of X is one. We may also suppose that X and Y are Köthe function spaces. We will suppose that there is a bounded linear operator $S: X \longrightarrow Y$ with $\delta\|x\|_X \leq \|Sx\|_Y \leq \|x\|_X$. It will also suffice to prove the result when X is finite-dimensional, *i.e.* $\Omega = \{1,2,\ldots,n\}$ and thus has a 1-unconditional basis $(e_k)_{k=1}^n$ consisting of atoms, provided we establish a uniform bound on the r-convexity constant $M^r(X)$ in terms of (p,q,r,δ).

To this end we define a map $L: X_{1/2} \longrightarrow Y_{1/2}$ by $Le_k = |Se_k|^2$. It follows from Krivine's theorem ([34] Theorem 1.f.4 p. 93) that if $x = \sum_{k=1}^n \xi_k e_k \geq 0$ then

$$\begin{aligned}\|Lx\|_{Y_{1/2}} &= \Big\|\Big(\sum_{k=1}^n \xi_k |Se_k|^2\Big)^{1/2}\Big\|_Y^2 \\ &\leq K_G^2 \Big\|\Big(\sum_{k=1}^n \xi_k e_k\Big)^{1/2}\Big\|_X^2 \\ &\leq K_G^2 \|x\|_{X_{1/2}}.\end{aligned}$$

Also since Y is q-concave, there exists $C_0 = C_0(q)$ so that

$$\begin{aligned}\|Lx\|_{Y_{1/2}} &\geq C_0^{-2}\Big(\operatorname{Ave}_{\epsilon_k=\pm 1}\Big\|\sum_{k=1}^n \epsilon_k \xi_k^{1/2} Se_k\Big\|_Y\Big)^2 \\ &\geq C_0^{-2}\delta^{-2}\|x\|_{X_{1/2}}.\end{aligned}$$

Now by Lemma 4.1 applied to $K_G^{-2}L$, using the fact that $X_{1/2}$ is $p/2$-convex and $Y_{1/2}$ is $r/2$-convex and $q/2$-concave we obtain the existence of $C_1 = C_1(p,q,r,\delta)$ so that for $x = \sum_{k=1}^n \xi_k e_k \geq 0$,

$$\|x\|_{X_{1/2}} \leq C_1 \|\max_{1\leq k\leq n} \xi_k Le_k\|_{Y_{1/2}}$$

which in turn implies that if $x \in X$, with $x = \sum \xi_k e_k$,

$$\|x\|_X \leq C_2 \Big\|\max_{1\leq k\leq n} |\xi_k|\,|Se_k|\,\Big\|_Y$$

where $C_2^2 = C_1$. Now suppose $x_1, \ldots, x_m \in X$ with $x_j = \sum_{k=1}^n \xi_{jk} e_k$. Then

$$\begin{aligned}
\Big\|\Big(\sum_{j=1}^m |x_j|^r\Big)^{1/r}\Big\|_X &\le C_2 \Big\| \max_{1\le k\le n}\Big(\sum_{j=1}^m |\xi_{jk}|^r\Big)^{1/r} |Se_k|\Big\|_Y \\
&\le C_2 \Big\|\Big(\sum_{k=1}^n \sum_{j=1}^m |\xi_{jk}|^r |Se_k|^r\Big)^{1/r}\Big\|_Y \\
&= C_2 \Big\|\Big(\sum_{j=1}^m \Big(\sum_{k=1}^n |\xi_{jk}|^r |Se_k|^r\Big)\Big)^{1/r}\Big\|_Y \\
&\le C_2 \Big(\sum_{j=1}^m \Big\|\Big(\sum_{k=1}^n |\xi_{jk}|^r |Se_k|^r\Big)^{1/r}\Big\|_Y^r\Big)^{1/r} \\
&\le C_2 \Big(\sum_{j=1}^m \Big\|\Big(\sum_{k=1}^n |\xi_{jk}|^2 |Se_k|^2\Big)^{1/2}\Big\|_Y^r\Big)^{1/r} \\
&\le K_G C_2 \Big(\sum_{j=1}^m \Big\|\Big(\sum_{k=1}^n |\xi_{jk}|^2 |e_k|^2\Big)^{1/2}\Big\|_X^r\Big)^{1/r} \\
&\le K_G C_2 \Big(\sum_{j=1}^m \|x_j\|_X^r\Big)^{1/r}.
\end{aligned}$$

This completes the proof. ∎

We now give a second criterion for a cone-embedding to be a strong cone-embedding.

LEMMA 4.3. *Suppose* $0 < q, s < \infty$ *and that* X *is an* s*-convex quasi-Banach* r. i. *space on* $[0,1]$ *or* $[0,\infty)$ *with* $p_X > 1$. *Suppose* Y *is an* s*-convex* q*-concave quasi-Köthe function space and* $L\colon X \longrightarrow Y$ *is a cone-embedding. Then there is a constant* C *so that if* $x_1, \ldots, x_n \ge 0$ *are disjoint,*

$$\Big\|\sum_{j=1}^n x_j\Big\|_X \le C \| \max_{1\le j\le n} Lx_j\|_Y.$$

PROOF. We suppose that $\|L\| \le 1$ and that $\delta > 0$ is such that if $x \ge 0$ then $\delta\|x\|_X \le \|Lx\|_Y \le \|x\|_X$. We may also suppose that for some $p > 1$ and some constant C_0 we have $\|D_t\|_X \le C_0 t^{1/p}$ for $t \ge 1$.

We select first an integer m so that $2^{m(p-1)} \ge 2^{p+1} C_0^p \delta^{-p}$. Let $\theta = 2^{-m}$.

Now suppose $x_1, \ldots, x_n \ge 0$ are given; it will suffice to consider the case when each x_i is a countably simple function (*i.e.* takes only a countable set of values) and $\| \sum_{i=1}^n x_i\|_X = 1$. Suppose N is an integer with $N > 4(2^m n)$. Then for each $1 \le i \le n$ we can write $x_i = \sum_{j=1}^N x_{ij}$ as a disjoint sum where $x_{ij}^* = D_{(1/N)} x_i^*$.

Let $\epsilon_{ijk} = \pm 1$ be a choice of signs for $1 \le i \le n$, $1 \le j \le N$ and $1 \le k \le m$ and denote by ϵ the array (ϵ_{ijk}). We define

$$u(\epsilon) = \sum_{i=1}^n \sum_{j=1}^N \prod_{k=1}^m (1 + \epsilon_{ijk}) x_{ij}.$$

Let $\xi_i(\epsilon)$ be the number of j such that $\epsilon_{ijk} = 1$ for $1 \leq k \leq m$. As functions on the natural finite probability space of all choices of signs ϵ, the functions ξ_i for $1 \leq i \leq n$ are independent and identically distributed with binomial distributions corresponding to a sample size N and probability for an individual trial of θ. They each have mean $\alpha = N\theta$ and variance $N\theta(1-\theta) < \alpha$. Notice that by choice of N we have $\alpha > 4n$.

We thus have

$$\mathrm{Ave}_\epsilon \max_{1\leq i\leq n} |\xi_i - \alpha|^2 \leq \sum_{i=1}^{n} \mathrm{Ave}_\epsilon |\xi_i - \alpha|^2 < n\alpha.$$

Let $\zeta(\epsilon) = \min_{1\leq i\leq n} \xi_i(\epsilon)$. Then

$$\mathrm{Ave}_\epsilon |\alpha - \zeta|^2 < n\alpha$$

and so

$$\mathrm{Ave}_\epsilon \zeta > \alpha - (n\alpha)^{1/2} > \frac{1}{2}\alpha.$$

We next turn to estimating $\mathrm{Ave}\,\|u(\epsilon)\|_X^p$. In fact we have that for each ϵ,

$$\Big\|\sum_{i=1}^{n} x_i\Big\|_X \leq 2^{-m}\|D_{N/\zeta}u(\epsilon)\|_X.$$

Thus we have an estimate that

$$1 \leq C_0\theta N^{1/p}\zeta(\epsilon)^{-1/p}\|u(\epsilon)\|_X.$$

Reorganizing and averaging gives

$$\mathrm{Ave}_\epsilon \|u(\epsilon)\|_X^p \geq C_0^{-p}\theta^{-p}N^{-1}\,\mathrm{Ave}_\epsilon \zeta \geq \frac{1}{2}C_0^{-p}\theta^{1-p}.$$

The original choice of m now gives the estimate

$$\mathrm{Ave}_\epsilon \|u(\epsilon)\|_X^p \geq 2^p\delta^{-p}$$

which implies

$$\Big(\mathrm{Ave}_\epsilon \Big\|L\big(u(\epsilon)\big)\Big\|_Y^p\Big)^{1/p} \geq 2.$$

We now proceed as in Lemma 4.1, expanding out and concluding that for some constant C_1 depending only on Y,

$$\Big(\mathrm{Ave}_\epsilon \Big\|L\big(u(\epsilon)\big)\Big\|_Y^p\Big)^{1/p} \leq \Big\|\sum_{i=1}^{n}\sum_{j=1}^{N} Lx_{ij}\Big\|_Y + 2^m C_1 \Big\|\Big(\sum_{i=1}^{n}\sum_{j=1}^{N} |Lx_{ij}|^2\Big)^{1/2}\Big\|_Y.$$

Since $\|\sum_{i=1}^{n}\sum_{j=1}^{N} Lx_{ij}\|_Y \leq 1$ we can conclude that

$$\Big\|\Big(\sum_{i=1}^{n}\sum_{j=1}^{N} |Lx_{ij}|^2\Big)^{1/2}\Big\|_Y \geq C_1^{-1}\theta.$$

Again this implies that

$$\|\max_{1\leq i\leq n} Lx_i\|_Y \geq \|\max_{i,j} Lx_{ij}\|_Y \geq C_1^{-2}\theta^{-2}.$$

The result now follows. ∎

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PROPOSITION 4.4. *Let X be an order-continuous Köthe function space on (Δ, λ), which contains L_∞. Let Y be a quasi-Köthe function space on (Ω, μ) which is s-convex for some $s > 0$ and q-concave for some $q < \infty$. Suppose $L: X \longrightarrow Y$ is a strong cone-embedding. Then, for $n \geq 1$, there exist Borel maps $a_n: \Omega \longrightarrow [0, \infty)$ and $\sigma_n: \Omega \longrightarrow \Delta$ so that $a_1 \geq a_2 \geq \cdots \geq 0$ and $\sigma_m(\omega) \neq \sigma_n(\omega)$ if $m \neq n$, and for some $C > 0$ we have for any $x \in X$ with $x \geq 0$,*

$$C^{-1}\|x\|_X \leq \|\max_n a_n x \circ \sigma_n\|_Y \leq \Big\|\sum_{n=1}^{\infty} a_n x \circ \sigma_n\Big\|_Y \leq C\|x\|_X.$$

PROOF. We use the random measure representation of positive operators (see [25], [41], [42]). There exists a Borel map $\omega \longrightarrow \nu_\omega$ from Ω to $\mathcal{M}(\Delta)$, endowed with the weak* topology, so that for any $x \in X$ we have

$$Lx(\omega) = \int x(t)\, d\nu_\omega \quad \mu - \text{a.e.}$$

Further we can write

$$\nu_\omega = \sum_{n=1}^{\infty} a_n(\omega)\delta_{\sigma_n(\omega)} + \nu'_\omega \quad \mu - \text{a.e}$$

where $a_n: \Omega \longrightarrow [0, \infty)$ and $\sigma_n: \Omega \longrightarrow \Delta$ are Borel maps satisfying the assumptions above, and ν'_ω is a continuous measure.

Since L is a strong cone-embedding there exists a constant C so that $\|L\| \leq C$ and whenever $x_1, \ldots, x_n$ are disjoint and positive in X then

$$\Big\|\sum_{j=1}^{n} x_j\Big\|_X \leq C\|\max_{1\leq j\leq n} Lx_j\|_Y.$$

Now suppose $x \geq 0$. Then for each m,

$$\|x\|_X \leq C\|\max_{E\in\mathcal{A}_m} L(x\chi_E)\|_Y.$$

For the definition of $\mathcal{A}_m$ see Section 2. Now $\max_{E\in\mathcal{A}_m} L(x\chi_E)$ is monotone decreasing to $\max_n a_n x \circ \sigma_n$ so that, by the order-continuity of Y,

$$C^{-1}\|x\|_X \leq \|\max_n a_n x \circ \sigma_n\|_Y \leq \Big\|\sum_{n=1}^{\infty} a_n x \circ \sigma_n\Big\|_Y \leq \|Lx\|_Y \leq C\|x\|_X. \qquad \blacksquare$$

REMARK. Of course there is no special significance in modelling X on (Δ, λ) here; we clearly have the same result for any Polish measure space (K, ν). Note also that in the above argument the pointwise maximum $\max_n a_n x \circ \sigma_n$ exists μ-a.e. for $x \in X$.

PROPOSITION 4.5. *Suppose Y is an order-continuous quasi-Banach* r. i. *space on $[0, \infty)$ with property (d). Suppose that either X is an order-continuous atomic quasi-Banach lattice or that X is an order-continuous quasi-Köthe function space on (Δ, λ) and* *that $L: X \longrightarrow Y$ is a strong cone-embedding. Then X is lattice-isomorphic to a sublattice of Y.*

REMARK. We recall that Y has property (d) if there is a constant C so that, given $f \in Y$ and $g \in L_0$ with $g^{**} \leq f^{**}$ then $g \in Y$ and $\|g\|_Y \leq C\|f\|_Y$.

PROOF. Let C be a constant greater than the property (d) constant of Y and the constant in the definition of the strong cone-embedding. Let us prove this first for the case when X is atomic. Then we regard X as a sequence space (a quasi-Köthe space modelled on $\mathbf{N}$). Let $(e_n)_{n\in\mathbf{N}}$ be the basis vectors and let $u_n = Le_n$. We define a map $V: X \longrightarrow L_0\big(\mathbf{N} \times [0,\infty)\big)$ by $Ve_n = v_n$ where $v_n(k,t) = 0$ if $k \neq n$ and $v_n(n,t) = u_n(t)$. If $a_1, \dots, a_n \geq 0$ then it is easy to see that

$$\Big(\sum_{k=1}^{n} a_k v_k\Big)^{**} \leq \Big(\sum_{k=1}^{n} a_k u_k\Big)^{**}$$

and so by property (d) we have that V is bounded and $\|V\| \leq C\|L\|$. However since L is a strong cone-embedding

$$\Big\|\sum_{k=1}^{n} a_k e_k\Big\|_X \leq C\|\max_{1\leq k\leq n} a_k u_k\|_Y \leq C\Big\|\sum_{k=1}^{n} a_k v_k\Big\|_Y$$

so that V is an isomorphism onto its range.

The nonatomic case is similar. We can suppose that X is a quasi-Köthe function space on (Δ, λ) containing L_∞ and that L is of the form

$$Lx = \sum_{n=1}^{\infty} a_n x \circ \sigma_n$$

where for some constant C_1 we have

$$C_1^{-1}\|x\|_X \leq \|\max_n a_n x \circ \sigma_n\|_Y \leq \Big\|\sum_{n=1}^{\infty} a_n x \circ \sigma_n\Big\|_Y \leq C_1\|x\|_X.$$

Define $V: X \longrightarrow L_0\big(\mathbf{N} \times [0,\infty)\big)$ by the formula $Vx(n,t) = a_n(t)x\big(\sigma_n(t)\big)$. Then if $x \geq 0$ we have $(Vx)^{**} \leq (Lx)^{**}$ so that V is bounded, while

$$C_1^{-1}\|x\|_X \leq \|\max_n a_n x \circ \sigma_n\|_Y \leq \|Vx\|_Y,$$

so that V is also an isomorphism. ∎

PROPOSITION 4.6. *Suppose Y is an order-continuous quasi-Banach* r.i. *space on* $[0,1]$ *with property (d). Suppose for some $p > 1$, $Y_{1/p}$ has property (d). Suppose X is an order-continuous quasi-Köthe function space on* $[0,1]$, *and that $L: X \longrightarrow Y$ is a strong cone-embedding. Then there is a Borel subset E of* $[0,1]$ *with $\lambda(E) > 0$ so that $X(E)$ is lattice-isomorphic to a sublattice of Y.*

PROOF. We again may suppose that X is a quasi-Köthe function space containing L_∞. Note first that we must have $Y_{1/p} \subset L_1$ and hence $Y \subset L_p$. We may extend Y to be a quasi-Banach r. i. space on $[0,\infty)$ in several different ways. Precisely we define W to be the space of $f \in L_0[0,\infty)$ so that $f^*\chi_{[0,1]} \in Y$ and $f \in L_1[0,\infty)$ with the associated quasi-norm $\|f\|_W = \max(\|f^*\chi_{[0,1]}\|_Y, \|f\|_1)$. We define Z to be the space of $f \in L_0[0,\infty)$ so that $f^*\chi_{[0,1]} \in Y$ and $f \in L_p[0,\infty)$ with the associated quasi-norm $\|f\|_Z = \max(\|f^*\chi_{[0,1]}\|_Y, \|f\|_p)$. Then both W and Z have property (d). Note that both $W[0,1]$ and $Z[0,1]$ coincide with Y and hence L may be regarded as mapping into either W or Z. Note also that $W \subset Z$ with continuous inclusion.

Appealing to the preceding Proposition, we can find a lattice embedding $U: X \longrightarrow W[0,\infty)$ in such a way that for some constant C we have $C^{-1}\|x\|_X \leq \|Ux\|_Z$ and $\|Ux\|_W \leq C\|x\|_X$ for $x \geq 0$.

It now follows by Lemma 3.1 and Proposition 3.2 that we can find a a nonnegative Borel function a on $[0,1] \times [0,\infty)$ with $a(t,s))$ decreasing in s for each fixed t so that the map $Vx(t,s) = a(t,s)x(t)$ defines a lattice embedding of X into $Z_1\big([0,1] \times [0,\infty)\big)$ and such that for some C_1 we have $C_1^{-1}\|x\|_X \leq \|Vx\|_Z$ and $\|Vx\|_W \leq C_1\|x\|_X$ for $x \geq 0$.

Notice in particular that

$$\int_0^1 \int_0^\infty a(t,s)^r\, ds\, dt < \infty$$

for $r = 1$ and $r = p$. We therefore can find constants $0 < c < M < \infty$ so that there is a Borel subset E of $[0,1]$ of positive measure such that if $t \in E$ then $\int_0^\infty a(t,s)\, dt \leq M$ and $\int_0^\infty a(t,s)^p \geq c^p$. If $t \in E$ then $a(t,s) \leq Ms^{-1}$ and so we also have $\int_{s_0}^\infty a(t,s)^p\, ds \leq M^p s_0^{1-p}$.

Recall that $Y_{1/p}$ has property (d) and therefore $Z_{1/p}$ also has property (d) and is an interpolation space between L_1 and L_∞ with some constant $\gamma^p \geq 1$. Pick $u > 1$ so that $cu^{1-1/p} > 4\gamma M$. We then modify V to form $V_0: X \longrightarrow Z$ by setting $V_0x = Vx\chi_{E\times[0,u]}$. We will show that V_0 is a lattice embedding of $X(E)$ into $Z\big([0,1] \times [0,\infty)\big)$.

Let P be the positive operator defined on $L_1\big([0,1] \times [0,\infty)\big)$ and $L_\infty\big([0,1] \times [0,\infty)\big)$ by

$$Pg(t,s) = \Big(\frac{1}{u}\int_0^u g(t,v)\, dv\Big) a(t,s)^p \chi_E(t)\chi_{[u,\infty)}(s).$$

It is easy to calculate that $\|Pg\|_\infty \leq M^p u^{-p}\|g\|_\infty$. Similarly $\|Pg\|_1 \leq M^p u^{1-p}$. It follows that $\|P\|_{Z_{1/p}} \leq M^p u^{1-p}\gamma^p$.

It follows that if $f \in Z$ then

$$\Big\|\big(P(|f|^p)\big)^{1/p}\Big\|_Z \leq \gamma M u^{\frac{1}{p}-1}\|f\|_Z \leq \frac{12}{c}\|f\|_Z.$$

Suppose in particular $x \in X(E)$ and $x \geq 0$. Let $f = Vx$. Then

$$P(|f|^p)(t,s) = x(t)\Big(\frac{1}{u}\int_0^u a(t,v)^p, dv\Big)^{1/p} a(t,s)\chi_E(t)\chi_{[u,\infty)}(s).$$

However for $t \in E$

$$\int_0^u a(t,v)^p\, dv \geq c^p - M^p u^{1-p} \geq \frac{1}{2}c^p \geq (c/2)^p.$$

Hence

$$\|Vx - V_0x\|_Z \leq 2\gamma M u^{\frac{1}{p}-1} c^{-1}\|Vx\|_Z \leq \frac{1}{2}\|Vx\|_Z.$$

It follows that V_0 maps $X(E)$ isomorphically into $Z([0,1] \times [0,u])$ which is lattice isomorphic to Y. ∎

COROLLARY 4.7. *Suppose Y is an order-continuous quasi-Banach* r. i. *space on* $[0,1]$ *with property (d). Suppose for some $p > 1$, $Y_{1/p}$ has property (d). Suppose X is an order-continuous quasi-Banach* r. i. *space on* $[0,1]$, *and that $L\colon X \longrightarrow Y$ is a strong cone-embedding. Then there exists $f \in Y_+ \setminus \{0\}$ such that $X = Y_f$.*

PROOF. This follows from Proposition 3.3. ∎

5. Cone-embeddings of r. i. spaces.

PROPOSITION 5.1. *Suppose $0 < s < q < \infty$ and that X is an s-convex, q-concave quasi-Köthe function space on (Ω, μ). Suppose $m \geq q$ is a natural number. Then there is a constant $C = C(X)$ so that if $x_1, \ldots, x_n \in X_+$ and $b_1, \ldots, b_n \geq 0$ then*

$$\Big(\mathrm{Ave}_{\pi\in\Pi_n}\Big\|\sum_{i=1}^{n} b_{\pi(i)}x_i\Big\|^m\Big)^{1/m} \leq C\max\Big((\mathrm{Ave}_{\pi\in\Pi_n}\|\max_{1\leq i\leq n} b_{\pi(i)}x_i\|^m)^{1/m}, \frac{1}{n}\Big(\sum_{i=1}^{n} b_i\Big)\Big\|\sum_{i=1}^{n} x_i\Big\|\Big).$$

PROOF. This is a somewhat disguised form of the so-called *Classification Formula* (Theorem 2.1 of [21] or Theorem 2.e.5 of [34]). It can be derived from this formula; we indicate the direct proof. We assume that X has q-concavity constant one. Then $Z = L_m(\Pi_n\colon X)$ has m-concavity constant one where Π_n is given its natural probability measure. Now there is a constant C_0 depending only on m so that if $f_1, \ldots, f_n \in Z_+$

$$\Big(\sum_{i=1}^{n} f_i\Big)^m \leq C_0\Big(\sum_{|A|=m}\Big(\prod_{i\in A} f_i\Big) + \Big(\sum_{i=1}^{n} f_i^2\Big)\Big(\sum_{i=1}^{n} f_i\Big)^{m-2}\Big).$$

Hence for $C_1 = C_1(s,m)$

$$\Big\|\sum_{i=1}^{n} f_i\Big\|_Z \leq C_1\max(\|S_1\|, \|S_2\|)$$

where $S_1 = \big(\sum_{|A|=m}(\prod_{i\in A} f_i)\big)^{1/m}$ and $S_2 = (\max f_i)^{1/m}(\sum f_i)^{1-1/m}$. Now as Z is s-convex we can estimate:

$$\begin{aligned}\|S_2\| &\leq (\|\max f_i\|)^{1/m}\Big(\Big\|\sum f_i\Big\|\Big)^{1-1/m}\\ &\leq \frac{1}{m}\|\max f_i\| + \Big(1 - \frac{1}{m}\Big)\Big\|\sum f_i\Big\|\end{aligned}$$

It then follows that

$$\Big\|\sum f_i\Big\| \leq \max(m\|S_1\|, \|\max f_i\|).$$

Now let $f_i = b_i\xi_i$ where $\xi_i(\pi) = x_{\pi(i)}$. Then if $n \geq 2m$, we use m-concavity:

$$\begin{aligned}\|S_1\| &= \Big(\mathrm{Ave}_\pi\Big\|\Big(\sum_{|A|=m}\prod_{i\in A} b_i x_{\pi(i)}\Big)^{1/m}\Big\|^m\Big)^{1/m}\\ &\leq \Big\|(\mathrm{Ave}_\pi \sum_{|A|=m}\prod_{i\in A} b_i x_{\pi(i)})^{1/m}\Big\|\\ &\leq \Big(\frac{(n-m)!}{n!}\Big)^{1/m}\Big(\sum_{i=1}^n b_i\Big)\Big\|\sum_{i=1}^n x_i\Big\|\\ &\leq \frac{2}{n}\Big(\sum_{i=1}^n b_i\Big)\Big\|\sum_{i=1}^n x_i\Big\|.\end{aligned}$$

The proposition now follows easily. ∎

PROPOSITION 5.2. *Let X, Y be order-continuous quasi-Banach* r. i. *spaces on* $[0,1]$. *Suppose that $p_Y > 1$, Y is q-concave for some $q < \infty$ and that there is a cone-embedding $L: X \longrightarrow Y$. Then either $X = L_1[0,1]$ or X is lattice-isomorphic to a sublattice of Y and so $X = Y_f$ for some $f \in Y_+$.*

PROOF. For ease of notation we regard X as modelled on (Δ, λ).

Let us first note that the proof is trivial if we assume $p_X > 1$. Indeed in this case L is a strong cone-embedding (Lemma 4.3) and $Y_{1/r}$ has property (d) as long as $1 < r < p_Y$. So Corollary 4.7 applies. We therefore need only to prove that if $X \neq L_1$ then $p_X > 1$.

Assume then $X \neq L_1$. Note first that $Y \subset L_r$ if $1 < r < p_Y$.

We can assume that, for some $\delta > 0$ and every $x \in X$, $\delta\|x\|_X \leq \|Lx\|_Y \leq \|x\|_X$ for $x \geq 0$. Let us consider the random measure representation of L *i.e.*

$$Lx(s) = \int x\, d\mu_s$$

where $s \longrightarrow \mu_s$ is a weak*-Borel map from $[0,1]$ to $\mathcal{M}(\Delta)$. We can as usual write

$$\mu_s = \sum_{n=1}^{\infty} a_n(s)\delta_{\sigma_n(s)} + \nu_s$$

where $a_n: [0,1] \longrightarrow [0,\infty)$ and $\sigma_n: [0,1] \longrightarrow \Delta$ are Borel maps and $\sigma_m(s) \neq \sigma_n(s)$ if $m \neq n$, and ν_s is for each s nonatomic.

Since Y has nontrivial concavity there is a constant C_0 and an integer m so that if $y_1, \ldots, y_n \in Y_+$ and $b_1, \ldots, b_n \geq 0$,

$$\begin{aligned}&\Big(\mathrm{Ave}_{\pi\in\Pi_n}\Big\|\sum_{i=1}^n b_{\pi(i)}y_i\Big\|_Y\Big)^{1/m}\\ (*)\qquad &\leq C_0 \max\Big((\mathrm{Ave}_{\pi\in\Pi_n}\|\max_{1\leq i\leq n} b_{\pi(i)}y_i\|_Y^m)^{1/m}, \frac{1}{n}\Big(\sum_{i=1}^n b_i\Big)\Big\|\sum_{i=1}^n y_i\Big\|_Y\Big).\end{aligned}$$

Let us introduce the functional on X defined by

$$\Gamma(x) = \sup\{\|\max_n a_n u \circ \sigma_n\|_Y : u^* = x^*\}.$$

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Consider a nonnegative simple function $x \in CS_{n_0}(\Delta)$. For each $n \geq n_0$ we can write $x = \sum_{E \in \mathcal{A}_n} \xi_E \chi_E$. For each permutation π of $\mathcal{A}_n$ let $x_\pi = \sum_{E \in \mathcal{A}_n} \xi_{\pi(E)} \chi_E$. Let $y_E = L\chi_E \in Y$.

We also define for each n, and each $s \in [0,1]$ $\tau_n(s)$ to be the least integer τ so that $\big(\sigma_i(s)\big)_{i=1}^{\tau}$ belong to distinct members of $\mathcal{A}_n$. Note that $\lim_{n\to\infty} \tau_n(s) = \infty$ for all s.

Note that

$$\max_{E \in \mathcal{A}_n} \xi_{\pi(E)} y_E(s) \leq \max_{1 \leq k \leq \tau_n} a_k x_\pi \circ \sigma_k + \|x\|_\infty \Big(\sum_{k > \tau_n} a_k(s) + \max_{E \in \mathcal{A}_n} \nu_s(E) \Big),$$

so that

$$\|\max_{E \in \mathcal{A}_n} \xi_{\pi(E)} y_E\|_Y \leq \Gamma(x) + \eta_n \|x\|_\infty$$

where $\lim_{n\to\infty} \eta_n = 0$. Now appealing to $(*)$ gives that

$$\delta \|x\|_X \leq C_0 \max\big(\Gamma(x), \|x\|_1 \|L\chi_\Delta\|_Y\big) + C_0 \eta_n \|x\|_\infty$$

which gives us

$$(**) \qquad \delta \|x\|_X \leq C_0 \max\big(\Gamma(x), \|x\|_1 \|L\chi_\Delta\|_Y\big).$$

First suppose a_1 vanishes a.e. so that $\Gamma(x) = 0$ for all $x \geq 0$. Then

$$\|x\|_X \leq C_0 \delta^{-1} \|L\chi_\Delta\|_Y \|x\|_1$$

for all $x \in X$ so that $L_1 \subset X$. Since $X \neq L_1$ we must have that $X^* = \{0\}$ and Theorem 4.4 of [25] shows that L must vanish (we remark that in the preparatory Lemma 4.3 of [25] the hypothesis $X^* = \{0\}$ has been omitted in the statement). This is impossible so we must have that $a_1 > 0$ on a set of positive measure. Hence if we set $Sx = a_1 x \circ \sigma_1$ then S is a nontrivial lattice homomorphism of X into Y and Corollary 3.3 will yield that $X \subset Y$. Hence $X \subset L_r$ where $1 < r < p_Y$.

We next show that in fact $\|x\|_X \leq C_1 \Gamma(x)$. If not, there is a sequence x_n with $\|x_n\|_X = 1$, $x_n \geq 0$ and $\Gamma(x_n) \longrightarrow 0$. But, if this happens we must have $x_n \longrightarrow 0$ in measure and $\|x_n\|_r$ bounded. Hence $\lim_{n\to\infty} \|x_n\|_1 = 0$ and $(**)$ yields that $\lim_{n\to\infty} \|x_n\|_X = 0$. This contradiction establishes the claim.

Fix any simple $f \in X[0,1]$ with $\|f\|_X = 1$. Then there exists $0 \leq x \in X(\Delta)$ with $x^* = f^*$ so that

$$\|\max_n a_n x \circ \sigma_n\|_X \geq C_1^{-1}.$$

Let $x = \sum_{j=1}^M \xi_j \chi_{H_j}$ where $H_1, \ldots, H_M$ are disjoint Borel sets. For each s let $k(s)$ be the first index such that $a_k(s) x\big(\sigma_k(s)\big) = \max_{1 \leq n < \infty} a_n(s) x\big(\sigma_n(s)\big)$. Then let $b'(s) = a_{k(s)}(s)$ and $\rho(s) = \sigma_{k(s)}(s)$. The operator $V: X \longrightarrow Y$ given by $Vz = b'z \circ \rho$ is then a lattice homomorphism form X into Y with $\|V\| \leq 1$. For $n \in \mathbf{Z}$ let $F_n = (b')^{-1}(2^n, 2^{n+1}]$ and let $b = \sum_{n \in \mathbf{Z}} 2^n \chi_{F_n}$ so that $\frac{1}{2} b' \leq b \leq b'$. For each n the measures $B \longrightarrow \lambda(\rho^{-1} B \cap F_n)$ are absolutely continuous. Then for any N, we can use Liapunoff's theorem to find sets

$H_j^\alpha \subset H_j$ with $\lambda(H_j^\alpha) = \alpha^{-1}\lambda(H_j)$ for $1 \leq j \leq M$ and $\lambda(\rho^{-1}H_j^\alpha \cap F_n) = \alpha^{-1}\lambda(\rho^{-1}H_j \cap F_n)$ for $|n| \leq N$. Let $G_N = \bigcup_{|n|\leq N} F_n$. Then

$$\Big\|V\Big(\sum_{j=1}^{M} \xi_j \chi_{H_j}^\alpha\Big)\Big\|_Y \geq \frac{1}{2}\|D_{1/\alpha}(\chi_{G_N}Vx)\|_Y \geq \frac{1}{2}\|D_\alpha\|_Y^{-1}\|\chi_{G_N}Vx\|_Y.$$

Letting $N \longrightarrow \infty$ we have

$$\|Vx\|_Y \leq 2\|D_\alpha\|_Y \|D_{\alpha^{-1}}x\|_X.$$

Hence

$$\|f\|_X \leq 2C_1\|D_\alpha\|_Y\|D_\alpha^{-1}f\|_X.$$

As this inequality holds for all simple $f \geq 0$ we obtain

$$\|D_\alpha\|_X \leq 2C_1\|D_\alpha\|_Y$$

for $\alpha > 1$ so that $p_X \geq p_Y > 1$. As observed in the introductory remarks, this is sufficient to prove the theorem. ∎

The following proposition is trivially false in the case when $p = 1$ since the map $x \longrightarrow \left(\int_0^1 x(s)\,ds\right)\chi_{[0,1]}$ is a cone embedding of $L_1[0,1]$ into $L_p[0,1]$ when $p < 1$.

PROPOSITION 5.3. *Suppose* $1 < p < \infty$. *Suppose* Y *is a* p*-concave quasi-Banach* r. i. *space on* $[0,1]$ *or* $[0,\infty)$ *and that there is a cone-embedding of* $L_p(\Delta,\lambda)$ *into* Y. *Then* $Y[0,1] = L_p[0,1]$.

PROOF. We assume that Y is s-normed. Let (Ω,μ) represent either $[0,1]$ or $[0,\infty)$ with associated Lebesgue measure. We apply Lemma 4.1 and Proposition 4.4. There exists a constant C and Borel maps $a_n\colon \Omega \longrightarrow [0,\infty)$ and $\sigma_n\colon \Omega \longrightarrow \Delta$ so that $\sigma_m(\omega) \neq \sigma_n(\omega)$ if $m \neq n$ and so that for $0 \leq x \in L_p$,

$$C^{-1}\|x\|_p \leq \|\max_n a_n x \circ \sigma_n\|_Y \leq \Big\|\sum_{n=1}^{\infty} a_n x \circ \sigma_n\Big\|_Y \leq C\|x\|_p.$$

Let $Lx = \sum a_n x \circ \sigma_n$; then $L\colon L_p \longrightarrow Y$ is a positive operator. We can also apply Proposition 3.5: there is a weight function $w > 0$ on Ω so that $\|y\|_Y \leq \|wy\|_p$ for $y \in Y$ and $\|w(Lx)\|_p \leq C\|x\|_p$ for $x \in L_p$.

At this point we define measures ν_n on Δ by $\nu_n(B) = \int_{\sigma_n^{-1}B} w^p a_n^p\,d\mu$. It is easy to see that each ν_n is a finite Borel measure absolutely continuous with respect to λ. Hence we can find derivatives $v_n = d\nu_n/d\lambda$. Now if $0 \leq x \in L_p(\Delta)$ then

$$\int_\Omega w^p \sum_{n=1}^{\infty} a_n^p (x \circ \sigma_n)^p\,d\mu = \int_\Delta x^p\Big(\sum_{n=1}^{\infty} v_n\Big)\,d\lambda$$

and so it follows that

$$\sum_{n=1}^{\infty} v_n(t) \leq C^p$$

almost everywhere. By an application of Egoroff's theorem we can find a Borel set $E \subset \Delta$ of positive measure and N so that

$$\sum_{n=N+1}^{\infty} v_n(t) \leq (2^{1/s}C)^{-p}$$

for $t \in E$.

Now, observe that if $0 \leq x \in L_p(E)$ then

$$\begin{aligned} \|\max_{n\geq N+1} a_n x \circ \sigma_n\|_Y &\leq \Big\| w\Big(\sum_{n=N+1}^{\infty} a_n^p (x\circ\sigma_n)^p\Big)^{1/p}\Big\|_p \\ &\leq \Big(\int_E x^p\Big(\sum_{n=N+1}^{\infty} v_n\Big)\, d\lambda\Big)^{1/p} \\ &\leq \frac{1}{2^{1/s}C}\|x\|_p. \end{aligned}$$

Hence

$$\|\max_{1\leq n\leq N} a_n x\circ\sigma_n\|_Y \geq \Big(\|Lx\|_Y^s - \frac{1}{2C^s}\|x\|_p^s\Big)^{1/s} \geq \frac{1}{2^{1/s}C}\|x\|_p.$$

This implies that L_p is isomorphic to a sublattice of Y^N and hence to a sublattice of Y. Finally we can apply Theorem 3.6 to deduce that $Y[0,1] = L_p[0,1]$. ∎

6. The main construction.

LEMMA 6.1. *Let X be a q-concave Köthe function space on some Polish measure space (Ω,μ), where $q < \infty$. Then there is a constant C depending only on X so that if $f_1,\ldots,f_n \in X$, and $h = (\sum_{i=1}^n f_i^2)^{1/2}$, then for any $M > 1$ we have:*

$$(\mathrm{Ave}_{\epsilon_i=\pm1}\|g_\epsilon \chi_{H_\epsilon}\|^q)^{1/q} \leq CM^{-1}\|h\|$$

where $g_\epsilon = \sum_{i=1}^n \epsilon_i f_i$ and $H_\epsilon = \{|g_\epsilon| \leq M^{-1}h\} \cup \{|g_\epsilon| \geq Mh\}$.

PROOF. Note first that

$$\mathrm{Ave}_{\epsilon_i=\pm1}\|g_\epsilon\chi_{(|g_\epsilon|\leq M^{-1}h)}\|^q \leq M^{-q}\|h\|^q.$$

On the other hand, if C_0 is the q-concavity constant of X,

$$(\mathrm{Ave}_{\epsilon_i=\pm1}\|g_\epsilon\chi_{(|g_\epsilon|\geq Mh)}\|^q)^{1/q} \leq C_0^{-1}\|\phi\|$$

where

$$\phi(s) = \Big(\int_{g_\epsilon(s)\geq Mh(s)} |\sum_{i=1}^m \epsilon_i f_i(s)|^q\, d\epsilon\Big)^{1/q}.$$

We can estimate (assuming $h(s) > 0$)

$$\phi(s)^q \leq M^{-q}h(s)^{-q}\int\Big|\sum_{i=1}^m \epsilon_i f_i(s)\Big|^{2q} d\epsilon \leq C_1^q M^{-q} h(s)^q$$

where C_1 is a constant determined by the constant in Khintchine's inequality for $2q$. Combining we have

$$(\mathrm{Ave}_{\epsilon_i=\pm 1}\|g_\epsilon \chi_{(|g_\epsilon|\geq Mh)}\|^q)^{1/q} \leq C_0^{-1}C_1M^{-1}\|h\|.$$

The result now follows. ∎

We now introduce some notation. If $[a,b]$ is a closed interval with $1 < a$ we write $\Gamma(a,b)$ for the collection of measurable functions f on $[0,\infty)$ which satisfy that almost everywhere, either $f(s) = 0$ or $a \leq |f(s)| \leq b$ or $b^{-1} \leq |f(s)| \leq a^{-1}$. Let $[a_n, b_n]_{n=0}^\infty$ be a sequence of intervals with $a_0 = 1$. If $(\eta_n)_{n=0}^\infty$ is a sequence with $0 < \eta_n < 1$ then $[a_n, b_n]$ is (η_n)-*separated* if $b_n \leq \eta_n a_{n+1}$ for all n.

LEMMA 6.2. *Let X be an* r. i. *space on* $[0,\infty)$. *Suppose* $0 < \delta < 1$, *and that* $\sigma > 0$ *is such that* $2^5\sigma < \delta$. *Suppose* $(\eta_n)_{n=1}^\infty$ *is any sequence satisfying* $\sum \eta_n < \sigma$ *and that* $[a_n, b_n]_{n=0}^\infty$ *are* (η_n)-*separated then for any* $f_0, \ldots, f_N \in X$ *such that*

(1) $\delta \leq \|f_j\| \leq 1$ *for* $0 \leq j \leq N$

(2) $f_j \in \Gamma(a_j, b_j)$

we have that $(f_j)_{j=0}^N$ *is 2-equivalent to a disjointly supported sequence* $(g_j)_{j=0}^N$ *with* $g_j \in \Gamma(a_j, b_j)$ *for* $0 \leq j \leq N$.

PROOF. Let $E_k = \{s : |f_k(s)| = \max_{0\leq j\leq N}|f_j(s)|, |f_k(s)| > |f_j(s)| \text{ if } j < k\}$. Let $g_k = f_k\chi_{E_k}$. Then (with appropriate modifications if $k = 0$ or $k = N$)

$$\begin{aligned}
\|f_k - g_k\| &\leq \sum_{j<k}\|f_k\chi_{(|f_k|\leq|f_j|)}\| + \sum_{j>k}\|f_k\chi_{(|f_k|<|f_j|)}\| \\
&\leq \sum_{j<k} a_k^{-1}b_j\|f_j\| + \sum_{j>k} b_k a_j^{-1}\|f_j\| \\
&\leq \sum_{j<k}\prod_{i=j}^{k-1}\eta_i + \sum_{j>k}\prod_{i=k}^{j-1}\eta_i \\
&\leq (\eta_{k-1} + \eta_k)\prod_{i=1}^{\infty}(1+\eta_i) \\
&\leq e^\sigma(\eta_{k-1}+\eta_k) \leq 4(\eta_k + \eta_{k-1}).
\end{aligned}$$

Hence $\|g_k\| \geq \delta - 8\sigma \geq \frac{\delta}{2}$. We also have $\sum \|f_k - g_k\| \leq 8\sigma \leq \frac{\delta}{4}$. Since (g_k) is a disjoint sequence it follows from standard perturbation theory that (f_k) is 2-equivalent to (g_k). ∎

LEMMA 6.3. *Let X be an* r. i. *space on* $[0,\infty)$ *or* $[0,1]$. *Suppose* $0 < \delta < \frac{1}{2}$ *and* $[a_n, b_n]_{n=0}^\infty$ *are* $(2^{-(n+6)}\delta)$-*separated. Then for any positive disjoint* $f_0, f_1, \ldots, f_N \in X$ *such that* $\delta \leq \|f_j\| \leq 1$ *and* $f_j \in \Gamma(a_j, b_j)$ *for* $0 \leq j \leq N$ *we have that* $(f_j)_{j=0}^N$ *is 6-equivalent to a disjointly supported sequence in* E_X.

PROOF. We suppose at first that X is an r. i. space on $[0,\infty)$. For $0 \leq j \leq N$ we choose $m_j \in \mathbf{Z}$ so that

$$\frac{\delta}{2^{j+4}b_j} < \|\chi_{[0,2^{m_j}]}\| \leq 2\frac{\delta}{2^{j+4}b_j}.$$

Similarly we choose $n_j \in \mathbf{Z}$ for $1 \leq j \leq N$ so that

$$\frac{a_j\delta}{2^{j+4}} < \|\chi_{[0,2^{n_j}]}\| \leq 2\frac{a_j\delta}{2^{j+4}}.$$

It is clear that $m_N \leq m_{N-1} \leq \cdots \leq m_0 \leq n_1 \leq \cdots \leq n_N$.

Let $u_j = \lambda(a_j \leq f_j \leq b_j)$ and $v_j = \lambda(b_j^{-1} \leq f_j \leq a_j^{-1})$. Then if $1 \leq j \leq N$,

$$\begin{aligned}\|\chi_{[0,2^{m_j}+u_j]}\| &\leq \|\chi_{[0,2^{m_j}]}\| + \|\chi_{[0,u_j]}\| \\ &\leq 2^{-(j+3)}b_j^{-1}\delta + a_j^{-1} \\ &\leq 2a_j^{-1} \\ &\leq 2^{-(j+4)}\delta b_{j-1}^{-1} \\ &< \|\chi_{[0,2^{m_{j-1}}]}\|\end{aligned}$$

so that $2^{m_j} + u_j \leq 2^{m_{j-1}}$. Similarly, if $1 \leq j \leq N-1$,

$$\|\chi_{[0,2^{n_j}+v_j]}\| \leq 2^{-(j+3)}a_j\delta + b_j < 2b_j$$

so that

$$\|\chi_{[0,2^{n_j}+v_j]}\| \leq 2^{-(j+5)}\delta a_{j+1} < \|\chi_{[0,2^{n_{j+1}}]}\|$$

and $2^{n_j} + v_j < 2^{n_{j+1}}$. Finally

$$\|\chi_{[0,2^{m_0}+u_0+v_0]}\| \leq 2^{-3}\delta b_0^{-1} + b_0 < 2b_0 \leq \frac{1}{2}\delta a_1.$$

Hence $2^{m_0} + u_0 + v_0 < 2^{n_1}$.

It now follows that we can rearrange $f_0, \ldots, f_N$ in the following manner. We can suppose that f_0 is supported and decreasing on $[2^{m_0}, 2^{n_1})$. Let $f_0' = f_0\chi_{[2^{m_0},1]}$ and $f_0'' = f_0 - f_0'$. For $1 \leq j \leq N$, we let $f_j' = f_j\chi_{(a_j \leq f_j \leq b_j)}$ and $f_j'' = f_j\chi_{(b_j^{-1} \leq f_j \leq a_j^{-1})}$. We can then suppose that for $1 \leq j \leq N$, f_j' is supported and decreasing on $[2^{m_j}, 2^{m_{j-1}})$ and f_j'' is supported and decreasing on $[2^{n_j}, 2^{n_{j+1}})$ where we adopt the convention $n_{N+1} = \infty$.

Now if $e_k = \chi_{[2^k,2^{k+1}]}$ let

$$x_0' = \sum_{k=m_0}^{-1} f_0'(2^{k+1})e_k$$

and

$$x_0'' = \sum_{k=0}^{n_1-1} f_0''(2^{k+1})e_k.$$

and for $1 \leq j \leq N$, let

$$x_j' = \sum_{k=m_j}^{m_{j-1}-1} f_j'(2^{k+1})e_k$$

and

$$x_j'' = \sum_{k=n_j}^{n_{j+1}-1} f_j''(2^{k+1})e_k.$$

We set $x_j = x_j' + x_j''$ for $0 \le j \le N$.

Then $0 \le x_j \le f_j$ for $0 \le j \le N$. However if $D_2g(t) = g(t/2)$ we have $f_j \le D_2x_j + z_j$ where $z_j = b_je_{m_j} + a_j^{-1}e_{n_j}$ for $1 \le j \le N$ and $z_0 \le b_0e_{m_0}$. Thus $\|z_j\|_X \le 2^{-(j+2)}\delta$ for $0 \le j \le N$. Hence if $\alpha_0, \ldots, \alpha_N \ge 0$,

$$\begin{aligned}\Big\|\sum_{j=0}^N \alpha_jf_j\Big\| &\le 2\Big\|\sum_{j=0}^N \alpha_jx_j\Big\| + \sum_{j=0}^N 2^{-(j+2)}\delta\alpha_j \\ &\le 2\Big\|\sum_{j=0}^N \alpha_jx_j\Big\| + \frac{\delta}{2}\max|\alpha_j| \\ &\le 2\Big\|\sum_{j=0}^N \alpha_jx_j\Big\| + \frac{1}{2}\Big\|\sum_{j=0}^N \alpha_jf_j\Big\|\end{aligned}$$

so that $(f_j)_{j=0}^N$ is 4-equivalent to a disjoint sequence in E_X. This completes the proof when X is modelled on $[0,\infty)$.

For the case $X = X[0,1]$, we may regard X as being defined on $[0,\infty)$ and proceed as before, but with each f_j having support of measure at most one. In this case, we have $x_0'' = 0$ while for $1 \le j \le N$, we have $x_j'' \le f_j'' \le a_j^{-1} \le 2^{-6j}\delta$. Hence if $\alpha_j \ge 0$ for $0 \le j \le N$,

$$\Big\|\sum_{j=0}^N \alpha_jx_j''\Big\| \le 2^{-5}\delta\max|\alpha_j| \le \frac{1}{32}\Big\|\sum_{j=0}^N \alpha_jf_j\Big\|.$$

Hence since $\frac{1}{2} + \frac{1}{32} < \frac{2}{3}$,

$$\begin{aligned}\Big\|\sum_{j=0}^N \alpha_jf_j\Big\| &\le 2\Big\|\sum_{j=0}^N \alpha_jx_j\Big\| + \frac{1}{2}\Big\|\sum_{j=0}^N \alpha_jf_j\Big\| \\ &\le 2\Big\|\sum_{j=0}^N \alpha_jx_j'\Big\| + \frac{2}{3}\Big\|\sum_{j=0}^N \alpha_jf_j\Big\|\end{aligned}$$

and $(f_j)_{j=0}^N$ is 6-equivalent to $(x_j')_{j=0}^N$ which is a sequence in $E_{X[0,1]}$. ∎

We now consider a situation which will remain fixed for Lemmas 6.4–6.6. We suppose now that X is a good Köthe function space on (Δ, λ) which is q-concave with constant one where $q < \infty$. We further suppose that Y is an r. i. space on $[0,\infty)$ which is also q-concave with constant one. We will assume that X is isomorphic to a subspace of Y. Let us therefore suppose that $T\colon X \longrightarrow Y$ is a bounded linear operator satisfying $\delta\|x\|_X \le \|Tx\|_Y \le \|x\|_X$ where $\delta > 0$.

For convenience we recall the notation introduced in Section 2. For $\epsilon_k = \pm 1$, we denote by $\Delta(\epsilon_1, \ldots, \epsilon_n)$ the clopen subset of Δ of all $(d_j)_{j=1}^\infty$ such that $d_j = \epsilon_j$ for $1 \le j \le n$. For each n let $\mathcal{A}_n$ denote the collection of $\Delta(\epsilon_1, \ldots, \epsilon_n)$. Let $\mathcal{C}_n$ be the algebra generated by the atoms $\mathcal{A}_n$. We let CS_n denote the linear span of $\{\chi_E : E \in \mathcal{A}_n\}$. We also define the Haar functions $h_E = \chi_{\Delta(\epsilon_1,\ldots,\epsilon_n,+1)} - \chi_{\Delta(\epsilon_1,\ldots,\epsilon_n,-1)}$ for $E = \Delta(\epsilon_1, \ldots, \epsilon_n)$. Let CS be the union of the spaces CS_n.

We define $Q_n\colon CS_n \longrightarrow L_0[0,\infty)$ to be the linear map such that $Q_n(\chi_E) = |Th_E|^2$ where $E \in \mathcal{A}_n$.

LEMMA 6.4. *If $x \in CS_n$ then $\|(Q_n x^2)^{1/2}\|_Y \leq K_G \|x\|_X$, where K_G is the Grothendieck constant.*

PROOF. If $x = \sum_{E \in \mathcal{A}_n} \alpha_E \chi_E$ then, by Krivine's theorem [31],

$$\left\| \left(\sum |\alpha_E|^2 |Th_E|^2 \right)^{1/2} \right\|_Y \leq K_G \left\| \left(\sum |\alpha_E|^2 |h_E|^2 \right)^{1/2} \right\|_X = K_G \|x\|_X. \qquad \blacksquare$$

For any measurable function $f \in L_0[0,\infty)$ and $a \geq 1$ we define $\tau_a f = f \chi_{(a^{-1} \leq |f| \leq a)}$. We then define for $x \in CS_+$,

$$\Psi(x) = \sup_a \liminf_{m \to \infty} \|\tau_a \big((Q_m(x^2))^{1/2} \big) \|_Y.$$

LEMMA 6.5. *There exists a constant $C = C(X,Y)$ so that if $\eta > 0$ and $b \geq 1$, then whenever $x \geq 0$, $x \in CS_n$ with $\Psi(x) < \eta$ then there exists a clopen set D independent of C_n such that $\lambda(D) = \frac{1}{2}$, $\max(\|x\chi_D\|_X, \|x - x\chi_D\|_X) \leq (3/4)^{1/q} \|x\|_X$ and*

$$\left\| \tau_b \big(T(x - 2\chi_D x) \big) \right\| \leq C\eta.$$

PROOF. Let $\eta = \Psi(x)$. We first pick $w \in L_0(\Delta)_+$ so that $\|x\|_X = \|xw^{-1}\|_q$ and $\|\xi\|_X \leq \|\xi w^{-1}\|_q$ for all $\xi \in X$. We write $x = \sum_{E \in \mathcal{A}_n} \alpha_E \chi_E$.

Suppose $m \geq n$. For a choice of signs $\epsilon_F = \pm 1$ we write

$$x_\epsilon = \sum_{E \in \mathcal{A}_n} \alpha_E \sum_{\substack{F \in \mathcal{A}_m \\ F \subset E}} \epsilon_F h_F.$$

We also let $y_\epsilon = Tx_\epsilon \in Y$.

Let $x_{\epsilon,+} = \max(x_\epsilon, 0)$ and $x_{\epsilon,-} = \max(-x_\epsilon, 0)$. We first estimate

$$\|x_{\epsilon,+}\|_X^q \leq \frac{1}{2} \int |x|^q w^{-q} \sum_{F \in \mathcal{A}_m} (\chi_F + \epsilon_F h_F) \, d\lambda.$$

This gives

$$\|x_{\epsilon,+}\|_X^q - \frac{1}{2} \|x\|_X^q \leq \left| \sum_{F \in \mathcal{A}_m} \epsilon_F \int_F |x|^q w^{-q} h_F \, d\lambda \right|.$$

Switching signs we get a similar estimate for $\|x_{\epsilon,-}\|_X^q$ and hence

$$\mathrm{Ave}_{\epsilon_i = \pm 1} \max(\|x_{\epsilon,+}\|_X^q, \|x_{\epsilon,-}\|_X^q) \leq \frac{1}{2} \|x\|_X^q + \left(\sum_{F \in \mathcal{A}_m} \left(\int_F |x|^q w^{-q} \, d\lambda \right)^2 \right)^{1/2}$$

by Khintchine's inequality.

The second term here can be estimated by

$$\max_{F \in \mathcal{A}_m} \left(\int_F |x|^q w^{-q} \, d\lambda \right)^{1/2} \|x\|_X^{q/2}.$$

It follows that for large enough m we have

$$\mathrm{Ave}_{\epsilon_i=\pm 1}\max(\|x_{\epsilon,+}\|_X^q, \|x_{\epsilon,-}\|_X^q) \leq \frac{5}{8}\|x\|_X^q.$$

For such m we have

$$\Pr\Big(\max(\|x_{\epsilon,+}\|_X^q, \|x_{\epsilon,-}\|_X^q) \leq \frac{3}{4}\|x\|^q\Big) \geq \frac{1}{6}.$$

We will now choose m subject to this restriction and such that

$$\|\tau_a(Q_m x^2)^{1/2}\|_Y \leq \eta$$

where $a = b\|x\|_X/\eta$. Let $G = \{a^{-1} \leq (Q_m x^2)^{1/2} \leq a\}$. Then since Y has cotype q, for a suitable constant $C_0 = C_0(Y)$,

$$\begin{aligned}(\mathrm{Ave}_{\epsilon_i=\pm 1}\|y_\epsilon \chi_G\|_Y^q)^{1/q} &\leq C_0\Big\|\chi_G\Big(\sum_{E\in\mathcal{A}_n}\sum_{\substack{F\in\mathcal{A}_m\\ F\subset E}}|\alpha_E|^2|Th_E|^2\Big)^{1/2}\Big\|_Y\\ &= C_0\|\chi_G(Q_m x^2)^{1/2}\|_Y\\ &\leq C_0\eta.\end{aligned}$$

On the other hand, if H is the complement of G and $B_\epsilon = \{b^{-1} \leq |y_\epsilon| \leq b\}$ then $B_\epsilon \cap H \subset \{|y_\epsilon| \leq \eta\|x\|_X^{-1}(Q_m x^2)^{1/2}\} \cup \{|y_\epsilon| \geq \|x\|_X\eta^{-1}(Q_m x^2)^{1/2}\}$. It thus follows from Lemmas 6.1 and 6.4 that

$$(\mathrm{Ave}_{\epsilon_i=\pm 1}\|\tau_b y_\epsilon \chi_H\|_Y^q)^{1/q} \leq C_1\eta\|x\|_X^{-1}\|(Q_m x^2)^{1/2}\|_Y \leq K_G C_1\eta.$$

Hence

$$(\mathrm{Ave}_{\epsilon_i=\pm 1}\|\tau_b y_\epsilon\|_Y^q)^{1/q} \leq C_2\eta$$

where C_2 depends only on X, Y.

Finally it follows there must exist a choice of ϵ_F so that $\max(\|x_{\epsilon,+}\|^q, \|x_{\epsilon,-}\|^q) \leq \frac{3}{4}\|x\|_X^q$ and $\|\tau_b y_\epsilon\|_Y \leq 6C_2\eta$. We conclude by writing $\sum \epsilon_F h_F = 2\chi_D - \chi_\Delta$ and then D satisfies our hypotheses. ∎

LEMMA 6.6. *Suppose* $\inf\{\Psi(x) : \|x\|_X = 1, x \in CS_+\} = 0$. *Then there is a nonatomic Banach lattice Z which is lattice-finitely representable in X so that Z has an unconditional basis which is lattice-finitely representable in E_Y.*

PROOF. Suppose N is a natural number. Let $\gamma = (\frac{3}{4})^{1/q}$. Let C be the constant determined in the previous lemma. We will select $\eta > 0$ so that

$$\eta < \min\Big(\frac{1}{2}\gamma^N\delta, \frac{(1-\gamma)^N\delta}{10^2 2^{N+1}(C+1)}\Big).$$

We pick $x \in CS_+$ so that $\|x\|_X = 1$ and $\Psi(x) < \eta$. Suppose $x \in CS_n$. We construct by induction a sequence of clopen sets $(F_k)_{k=1}^{2^{N+1}-1}$, sequences $(a_k, b_k)_{k=0}^{2^N-1}$, and functions $y_k \in Y$ for $0 \leq k \leq 2^{N+1} - 1$ so that:

(1) $a_0 = 1$ and $F_1 = \Delta$.

(2) Each F_k is independent of $\mathcal{C}_n$.

(3) $F_k = F_{2k} \cup F_{2k+1}$ and $\lambda(F_{2k}) = \lambda(F_{2k+1}) = \frac{1}{2}\lambda(F_k)$ for $1 \leq k \leq 2^N - 1$.

(4) For $1 \leq k \leq 2^N - 1$ we have $(1-\gamma)\|x\chi_{F_k}\|_X \leq \|x\chi_{F_{2k}}\|_X, \|x\chi_{F_{2k+1}}\|_X \leq \gamma\|x\chi_{F_k}\|_X$.

(5) $a_k \leq b_k$ $(1 \leq k \leq 2^{N+1} - 1)$ and $b_k \leq (1-\gamma)^N \delta 2^{-(k+7)} a_{k+1}$ for $1 \leq k \leq 2^{N+1} - 2$.

(6) If $h_0 = x$ and then $h_k = x(2\chi_{F_{2k}} - \chi_{F_k})$ for $1 \leq k \leq 2^N - 1$ then $\|Th_k - y_k\| < (C+1)\eta$.

(7) $y_k \in \Gamma(a_k, b_k)$.

We start the induction as stated with $a_0 = 1, F_1 = \Delta, h_0 = x$. We then select b_0 large enough so that $\|Th_0 - \tau_{b_0} Th_0\|_Y < \eta$ and set $y_0 = \tau_{b_0} Th_0$.

Now suppose $1 \leq k \leq 2^N - 1$ and that $(a_j)_{j=0}^{k-1}$, $(b_j)_{j=0}^{k-1}$, $(y_j)_{j=0}^{k-1}$ and $(F_j)_{j=1}^{2k-1}$ have been determined. We first pick a_k so that $b_{k-1} \leq (1-\gamma)^N \delta 2^{-(k+6)} a_k$ so that (5) holds. Now $\Psi(x\chi_{F_k}) \leq \Psi(x) < \eta$. Hence we are able to apply Lemma 6.5 to find a clopen set D independent of the algebra generated by the sets $\mathcal{C}_n$ and $\{F_1, \ldots, F_{k-1}\}$ so that $\lambda(D) = \frac{1}{2}$, $\max(\|x\chi_{F_k \cap D}\|_X, \|x\chi_{F_k \setminus D}\|_X) \leq \gamma\|x\chi_{F_k}\|_X$, and

$$\left\|\tau_{a_k}\big(T(x\chi_{F_k} - 2x\chi_{F_k \cap D})\big)\right\|_Y \leq C\eta$$

where C is the constant of the previous lemma.

We now let $F_{2k} = F_k \cap D$ and $F_{2k+1} = F_k \setminus D$. Conditions (2) and (3) are immediately satisfied. Condition (4) follows from the triangle law. If we define h_k by (6) we have $\|\tau_{a_k} Th_k\|_Y \leq C\eta$. Therefore we can pick $b_k > a_k$ so large that if G is the set where $b_k^{-1} \leq |Th_k| \leq a_k^{-1}$ or $a_k \leq |Th_k| \leq b_k$ then $\|Th_k - \chi_G Th_k\|_Y \leq (C+1)\eta$. Let $y_k = \chi_G Th_k$. Then (6) and (7) follow.

This completes the inductive construction. We now observe that for every $2^N \leq k \leq 2^{N+1} - 1$ we have $(1-\gamma)^N \leq \|x\chi_{F_k}\| \leq \gamma^N$. In particular $\|h_k\|_X \geq (1-\gamma)^N$ for $0 \leq k \leq 2^N - 1$. Thus $\|Th_k\|_Y \leq (1-\gamma)^N\delta$. By choice of η this implies that $\frac{1}{2}(1-\gamma)^N\delta \leq \|y_k\|_Y \leq 1$. Now we can appeal to Lemma 6.3 to deduce that $(y_k)_{k=0}^{2^N-1}$ is 12-equivalent to a disjoint sequence in E_Y. In particular it is 12-unconditional. Since $\|Th_k - y_k\|\,\|y_k\|^{-1} \leq 2(C+1)\eta(1-\gamma)^{-N}\delta^{-1}$ we have

$$\sum_{k=0}^{2^N} \|Th_k - y_k\|\,\|y_k\|^{-1} \leq 2^{N+1}(C+1)(1-\gamma)^{-N}\delta^{-1}\eta < 10^{-2}.$$

Hence $(Th_k)_{k=0}^{2^N-1}$ is 24-equivalent to a disjoint sequence in E_Y and hence $(h_k)_{k=0}^{2^N-1}$ is $24\delta^{-1}$-equivalent to a disjoint sequence in E_Y.

We can define a linear map $L_N \colon CS_N \longrightarrow X$ by $L_N(\chi_{\Delta(\epsilon_1,\ldots\epsilon_N)}) = x\chi_{F_k}$ where $k = 2^N + \frac{1}{2}\sum_{j=1}^N (1-\epsilon_j)2^{N-j}$. Then we can induce a lattice norm on CS_N by $\|f\|_N = \|L_N f\|_X$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbf{N}$. We define for $f \in CS$,

$$\|f\|_Z = \lim_{\mathcal{U}} \|f\|_N.$$

Then $\|\ \|_Z$ is a lattice norm on CS with the property that if $E \in \mathcal{A}_N$ then $(1-\gamma)^N \leq \|\chi_E\|_Z \leq \gamma^N$. Thus the completion Z of this space is a nonatomic Banach lattice which is finitely representable in X. Also the Haar system is clearly an unconditional basis of Z which is $25\delta^{-1}$-lattice finitely representable in E_Y. ■

Before proving the next theorem, which is the main result of the section, we make some definitions. Let us denote by $[0,\infty]$ the one-point compactification of $[0,\infty)$. Suppose $(\Omega_n,\mu_n)_{n=0}^{\infty}$ is a sequence of Polish spaces with associated σ-finite measures and let $f_n:\Omega_n \longrightarrow [0,\infty]$ be Borel functions such that for each $a>0$ we have $\mu_n(f_n>a)<\infty$. We will say that $(f_n,\mu_n)_{n=1}^{\infty}$ converges to (f_0,μ_0) *in law* if and only if for every continuous function $\phi:[0,\infty]\longrightarrow \mathbf{R}$ so that ϕ vanishes on a neighborhood of 0 we have

$$\lim_{n\longrightarrow\infty}\int_{\Omega_n}\phi\circ f_n\,d\mu_n=\int_{\Omega_0}\phi\circ f_0\,d\mu_0.$$

If f_n converges to f_0 in law then it is not difficult to see that

$$\mu_0(f_0>a)\le\liminf_{n\longrightarrow\infty}\mu_n(f_n>a)\le\limsup_{n\longrightarrow\infty}\mu_n(f_n>a)\le\mu_0(f_0\ge a).$$

Hence we can deduce that $f_n^*\longrightarrow f_0^*$ a.e. on $[0,\infty)$ and for any r. i. space Y this implies that $\|f_0\|_{Y(\Omega_0,\mu_0)}\le\liminf_{n\to\infty}\|f_n\|_{Y(\Omega_n,\mu_n)}$.

THEOREM 6.7. *Suppose Y is an* r. i. *space on* $[0,1]$ *or* $[0,\infty)$ *with nontrivial concavity. Suppose X is a good Köthe function space on* (Δ,λ) *which is isomorphic to a subspace of Y. Then either:*

(1) There is a nonatomic Banach lattice Z which is lattice-finitely representable in X and such that Z has an unconditional basis, which is lattice finitely representable in E_Y, or:

(2) There is a cone-embedding of $X_{1/2}$ into $Y_{1/2}$.

PROOF. Let $\mathcal{C}$ be the (countable) algebra of clopen subsets of Δ. We define a compact space $\Omega=[0,\infty]^{\mathcal{C}}$. We denote the co-ordinate maps on Ω by ξ_E for $E\in\mathcal{C}$.

Let us suppose first that $Y=Y[0,\infty)$; we will describe the minor modifications for the case $[0,1]$ afterwards. We suppose that Y is q-concave with constant one where $q<\infty$. Suppose $p>2q$ is fixed. Let $T:X\longrightarrow Y$ be a linear map satisfying for some $\delta>0$, $\delta\|x\|_X\le\|Tx\|_Y\le\|x\|_X$ for $x\in X$, and define $Q_n: CS_n\longrightarrow L_0[0,\infty)$ as above.

We make first the observation that, as Y is q-concave, we have an estimate $\|\chi_{[0,t]}\|_Y\ge t^{1/q}$ for $t\ge 1$ and hence if $y\in Y$ then $y^*(t)^q\le t^{-1}\|y\|_Y^q$ for $t\ge1$. It follows that if $y\in Y$ then

$$\int_0^\infty\min(1,|y|^{p/2})\,dt\le 1+\|y\|_Y^{p/2q}\int_1^\infty t^{-p/2q}\,dt\le 1+C_0\|y\|_Y^{p/2q}$$

for a suitable constant $C_0=C_0(q,p)$.

Let us define $\kappa_n:[0,\infty)\longrightarrow\Omega$ by $\xi_E\circ\kappa_n=Q_n(\chi_E)$ if $E\in\mathcal{C}_n$ and $\xi_E\circ\kappa_n=0$ otherwise. Let w be the weight function on Ω defined by $w=\min(1,\xi_\Delta^p)$. We will define a Borel measure ν_n on Ω by

$$\nu_n(B)=\int_{\kappa_n^{-1}B}\min\bigl(1,Q_n(\chi_\Delta)^p\bigr)\,d\lambda.$$

Let us first note that

$$\nu_n(\Omega)=\int\min\bigl(1,(Q_n\chi_\Delta)^p\bigr)\,d\lambda\le 1+C_0K_G^{p/q}$$

so that the sequence of Borel measures (ν_n) is bounded in $\mathcal{M}(\Omega)$. It follows that (ν_n) has a weak*-limit point ν. Let us define $\mu_n = w^{-1}\nu_n$ and $\mu = w^{-1}\nu$; these measures are σ-finite.

Note first that if U is an open subset of Ω then $\nu(U) \leq \limsup \nu_n(U)$. We use this first to argue that $\xi_E < \infty$, μ-a.e. for every $E \in \mathcal{C}$. In fact if $a > 0$ then in $E \in \mathcal{C}_n$, we have $\nu_n(\xi_E > a) \leq \lambda\big(Q_n(\chi_E) > a\big)$ and by Lemma 6.4, $a^{1/2}\min\Big(1, \lambda\big(Q_n(\chi_E) > a\big)\Big) \leq K_G\|\chi_E\|_X$. Hence $\lim_{a\to\infty}\nu(\xi_E > a) = 0$ and so $\mu(\xi_E = \infty) = 0$.

Next we argue that if $E, F \in \mathcal{C}$ are disjoint then $\xi_{E\cup F} = \xi_E + \xi_F$ a.e. for μ. In fact, if $\varepsilon > 0$, let U be the set of $\omega \in \Omega$ such that $\xi_E(\omega), \xi_F(\omega), \xi_{E\cup F}(\omega) < \infty$ and $|\xi_E(\omega) + \xi_F(\omega) - \xi_{E\cup F}(\omega)| > \varepsilon$. Then if $E, F \in \mathcal{C}_n$, we have $\nu_n(U) = 0$. Hence $\nu(U) = 0$ and $\mu(U) = 0$. Thus $\xi_E + \xi_F = \xi_{E\cup F}$ a.e. It follows that we can define a linear map $S_0 : CS \longrightarrow L_0(\mu)$ by $S_0(\chi_E) = \xi_E$.

Now suppose $f \in CS_+$. Let $f = \sum_{k=1}^N \alpha_k\chi_{E_k}$ where $E_1, \ldots, E_N$ are clopen sets in Δ, and $\alpha_k \geq 0$ for $1 \leq k \leq N$. Let $g = \sum_{k=1}^N \alpha_k\xi_{E_k}$ so that $g = S_0 f$ a.e. for μ. Let $M = \sum_{k=1}^N \alpha_k$. Then $f \leq M\chi_\Delta$ and $g \leq M\xi_\Delta$, a.e. for μ.

For any $a > 0$, let φ_a be a continuous function on $[0, \infty]$ such that $\varphi_a(t) = 0$ if $0 \leq t \leq 1/(2Ma)$ and $\varphi_a(t) = 1$ if $t \geq 1/(Ma)$. Then let $g_a = (\varphi_a \circ \xi_\Delta)\min(a, g)$. Then $\tau_a g \leq g_a \leq g$, μ-a.e.

For fixed $a > 0$, g_a is continuous on Ω. Furthermore for each n, $\mu_n(g_a > 0) \leq \mu_n(\xi_\Delta > (Ma)^{-1}) \leq \lambda(Q_n(\chi_\Delta) > (Ma)^{-1})$ is uniformly bounded. If $\nu_{n(k)}$ converges weak* to ν then for any continuous function φ on $[0, \infty]$ which vanishes in a neighborhood of the origin, we have

$$\begin{aligned}
\lim_{k\to\infty}\int_\Omega g_a\, d\mu_{n_k} &= \lim_{k\to\infty}\int_\Omega g_a w^{-1}\, d\nu_{n_k} \\
&= \lim_{k\to\infty}\int_\Omega (\varphi_a \circ \xi_\Delta)\max(1, \xi_\Delta^{-p})\min(g, a)\, d\nu_{n_k} \\
&= \int_\Omega (\varphi_a \circ \xi_\Delta)\max(1, \xi_\Delta^{-p})\min(g, a)\, d\nu \\
&= \int_\Omega g_a\, d\mu.
\end{aligned}$$

Thus (g_a, μ_{n_k}) converges in law to (g_a, μ). Since Y is order-continuous, g_a is bounded and the measures of the supports are uniformly bounded, this implies that

$$\lim_{k\to\infty}\|g_a^{1/2}\|_{Y(\mu_{n_k})} = \|g_a^{1/2}\|_{Y(\mu)}.$$

If $E_1, \ldots, E_N \in \mathcal{C}_n$ then we have $g \geq g_a \geq \tau_a g$ a.e. for μ_n. It follows that we have $\|\tau_a g\|_{Y(\mu_n)} \leq \|g_a\|_{Y(\mu_n)} \leq \|g\|_{Y(\mu_n)}$.

Note however that (g, μ_n) coincides in law with $(Q_n f, \lambda)$ for if B is a Borel subset of $(0, \infty)$ then $\mu_n(g^{-1}B) = \int_{g^{-1}B} w^{-1}\, d\nu_n = \lambda(\kappa_n^{-1}g^{-1}B) = \lambda\big((Q_n f)^{-1}B\big)$.

Hence we obtain the estimate

$$\lim_{a\to\infty}\liminf_{n\to\infty}\|\tau_a(Q_n f)^{1/2}\|_Y \leq \|g^{1/2}\|_{Y(\mu)} \leq \limsup_{n\to\infty}\|(Q_n f)^{1/2}\|_Y.$$

We conclude that $f \in CS_+$ we have $\Psi(f^{1/2})^2 \leq \|S_0 f\|_{Y_{1/2}(\mu)} \leq K_G^2 \|f\|_{X_{1/2}}$. Thus S_0 extends to a bounded positive operator $S: X_{1/2} \longrightarrow Y_{1/2}$. If alternative (1) of the theorem is false then, by Lemma 6.6, S has a lower estimate and it is clear that S is a cone-embedding, as required.

In the case when $Y = Y[0,1]$ we can regard Y as being embedded in a space modelled on $[0,\infty)$ and need only observe that in the above proof, the measures μ_n and μ have total mass at most one. ■

THEOREM 6.8. *Suppose Y is an* r. i. *space on* $[0,1]$ *or* $[0,\infty)$ *with nontrivial concavity, which is either strictly* 2*-convex or of Orlicz-Lorentz type. Suppose X is a good Köthe function space on* (Δ,λ) *which is isomorphic to a subspace of Y. Then there is a cone-embedding of $X_{1/2}$ into $Y_{1/2}$.*

PROOF. It is enough to show that the existence of Z in Theorem 6.7 leads to a contradiction. Suppose first that Y is strictly 2-convex; then E_Y is also strictly 2-convex. This implies that the unconditional basis of Z is strictly 2-convex, and hence Z can contain no copy of ℓ_2; however Z must have nontrivial cotype and this contradicts Lemma 2.4 of [11].

If Y is of Orlicz-Lorentz type then E_Y is lattice-isomorphic to a modular sequence space which has nontrivial cotype. Now the unconditional basis of Z is lattice finitely representable in E_Y. This implies that Z also is isomorphic to a modular sequence space, also with nontrivial cotype. This can be established directly without difficulty, but is also a special case of more general results on ultraproducts of Orlicz spaces and Orlicz-Musielak spaces, for which we refer to [12], [18] and [43]. This now contradicts Theorem 4.3 and Corollary 4.4 in [28] (which in turn extends an earlier result of Lindenstrauss and Tzafriri [32]). ■

7. **The main results.** Before proving our main results for embeddings of nonatomic Banach lattices into r. i. spaces, we first give an illustrative theorem for atomic Banach lattices. Compare this result with those of Johnson and Schechtman [22] and Carothers and Dilworth [8].

THEOREM 7.1. *Suppose Y is an* r. i. *space on* $[0,\infty)$ *with nontrivial cotype, and suppose that either (a) Y is* 2*-convex or (b)* $p_Y > 2$*. Suppose* (u_n) *is a strictly* 2*-convex unconditional basic sequence in Y. Then* (u_n) *is equivalent to a disjoint sequence. Equivalently, if X is a strictly* 2*-convex atomic Banach lattice which is isomorphic to a subspace of Y then X is lattice-isomorphic to a sublattice of Y.*

REMARK. We do not know if this theorem holds when Y is an r. i. space on $[0,1]$. ■

PROOF. Let us suppose that X is an atomic Banach lattice represented as a function space of $\mathbf{N}$ with canonical basis vectors e_n and that $S: X \longrightarrow Y$ is an embedding with $Se_n = u_n$. Then by Theorem 1.d.6 of [34] we can define a cone-embedding $L: X_{1/2} \longrightarrow Y_{1/2}$ by $Le_n = |u_n|^2$. The result is now obtained by putting together the facts previously established on cone-embeddings. Since $X_{1/2}$ is strictly 1-convex L is a strong cone-embedding, by Lemma 4.1; then since $Y_{1/2}$ has property (d) under either conditions (a) or (b), Proposition 4.5 shows that $X_{1/2}$ is lattice-isomorphic to a sublattice of $Y_{1/2}$. But this implies the result. ∎

We now prove the nonatomic version of the above theorem.

THEOREM 7.2. *Suppose Y be an* r. i. *space on* $[0,\infty)$ *with nontrivial concavity and either*

(a) Y is strictly 2-convex, or

(b) Y is 2-convex and of Orlicz-Lorentz type, or

(c) $p_Y > 2$ and Y is of Orlicz-Lorentz type. Suppose X be a strictly 2-convex nonatomic Banach lattice. If X is isomorphic to a subspace of Y, then X is isomorphic to a sublattice of Y.

PROOF. We can of course assume that X is a good Köthe function space on (Δ,λ). We first apply Theorem 6.8 to deduce the existence of a cone-embedding of $X_{1/2}$ into $Y_{1/2}$. Now the proof proceeds as in Theorem 7.1. ∎

REMARKS. Let us first note that if Y is 2-convex then X must also be 2-convex at least; the hypothesis that X is strictly 2-convex is then equivalent to the hypothesis that ℓ_2 is not lattice finitely representable in X (*cf.* [21] Lemma 2.4). This result was previously known in the special case $Y = L_p[0,\infty)$ [21], Theorem 1.8 (the atomic case is proved in [16].)

We now turn to the case when Y is an r. i. space on $[0,1]$; here our result is not quite as strong (exactly as in the atomic case: see discussion after Theorem 7.1).

THEOREM 7.3. *Let Y be an* r. i. *space on* $[0,1]$ *with nontrivial concavity and suppose either (a) Y is strictly 2-convex or (b) $p_Y > 2$ and Y is of Orlicz-Lorentz type. Suppose X is a nonatomic strictly 2-convex Banach lattice which is isomorphic to a subspace of Y. Then X contains a nontrivial band X_0 which is lattice-isomorphic to a sublattice of Y.*

PROOF. We will consider X as a good Köthe function space on $[0,1]$. Then there is, by Theorem 6.8, a cone-embedding $L: X_{1/2} \longrightarrow Y_{1/2}$. Furthermore $X_{1/2}$ is s-convex for some $s > 1$ and there exists in either case $r > 2$ so that $Y_{1/r}$ has property (d). Proposition 4.6 then implies that for some Borel set E with $\lambda(E) > 0$ the band $X_{1/2}(E)$ is lattice-isomorphic to a sublattice of $Y_{1/2}$. The result then follows. ∎

We now turn our attention to the case when X is known to be an r. i. space.

COROLLARY 7.4. *Let Y be an* r. i. *space on* $I = [0,1]$ *or* $[0,\infty)$ *with nontrivial concavity. Suppose either*

(a) Y is strictly 2-convex or

(b) Y is of Orlicz-Lorentz type and $p_Y > 2$.

Suppose X is an r. i. *space on* $I = [0,1]$, *with* $X \neq L_2[0,1]$. *Assume that X is isomorphic to a subspace of Y. Then X is isomorphic to a sublattice of Y and there exists $f \in Y$ so that* $X = Y_f[0,1]$.

PROOF. Consider first case (a). By Proposition 2.e.10 of [28] or Section 2 of [21] either $X = L_2$ or X is strictly 2-convex. The result then follows by the preceding Theorems 7.2 and 7.3.

Case (b) is slightly different. In this case Theorem 6.8 implies that there is a cone-embedding of $X_{1/2}$ into $Y_{1/2}$. By Proposition 5.2, either $X_{1/2} = L_1$ (*i.e.* $X = L_2$) or $X_{1/2}$ is isomorphic to a sublattice of $Y_{1/2}$ and the result follows. ∎

REMARKS. Some special cases of Corollary 7.4 are known. In [21] Theorem 7.7 the corollary is proved when Y is a strictly 2-convex Orlicz function space. Later, Carothers [5] and [6] proves the same theorem for Lorentz spaces $L_{p,q}$ where $p > \max(q, 2)$. Carothers considers first the strictly 2-convex case ($2 \leq q < p$) and later modifies the proof to the case $1 \leq q \leq 2 < p$. Note that in these cases and in more general Lorentz spaces considered by Carothers one has the additional information that every $Y_f[0,1]$ coincides with $Y[0,1]$. This is equivalent to an inequality of the form $\|f \otimes g\|_Y \leq K\|f\|_Y\|g\|_Y$ for $f, g \in Y[0,1]$. This additional information is actually used in the proof.

For reference let us state one additional case which follows from Theorem 7.2 and Proposition 3.3.

COROLLARY 7.5. *Let Y be an* r. i. *space on* $[0,\infty)$ *with nontrivial concavity which is* 2*-convex and of Orlicz-Lorentz type. Let X be a strictly* 2*-convex* r. i. *space on* $[0,1]$ *which is isomorphic to a subspace of Y. Then there exists $f \in Y$ so that $X = Y_f[0,1]$.*

Let us note the following special case.

COROLLARY 7.6. *Suppose* $2 < p < \infty$ *and Y is a p-convex* r. i. *space on* $[0,1]$ *or* $[0,\infty)$ *with nontrivial concavity. Suppose L_p is isomorphic to a subspace of Y. Then* $Y[0,1] = L_p[0,1]$.

PROOF. It follows from Corollary 7.3 that $L_p[0,1] = Y_f[0,1] \subset Y[0,1]$ but $Y[0,1] \subset L_p[0,1]$ since Y is p-convex. ∎

REMARKS. The condition that Y is p-convex cannot be relaxed here (*cf.* [19]). We remark that analogues of Corollary 7.6 for $1 \leq p < 2$ have been proved in several places in the literature. In the case $p = 1$, then L_1 embeds into a separable r. i. space $Y[0,1]$ if and only if $Y[0,1] = L_1[0,1]$. This is proved under the additional hypothesis that Y has nontrivial cotype in [21] (*cf.* [34] Corollary 2.e.4); it is proved under the hypothesis that Y does not contain c_0 in [23]. The result with no additional hypothesis follows from Theorem 10.7 and Theorem 7.3 of [27]. For the case $1 < p < 2$ a similar result holds when Y is separable and p-convex provided one eliminates the possibility that Y contains a disjoint sequence equivalent to the Haar basis of $L_p[0,1]$ (see Theorems 7.3 and 10.7 of [27].)

In our final result we consider the case when instead Y is p-concave for some $p > 2$ and L_p embeds into X.

THEOREM 7.7. *Suppose* $2 < p < \infty$ *and that* Y *is a* p*-concave* r. i. *space on* $[0,1]$ *or* $[0,\infty)$. *Suppose that* L_p *is isomorphic to a subspace of* Y. *Then, either:*

(a) The Haar basis of L_p *is lattice finitely-representable in* E_Y *or*

(b) $Y[0,1] = L_p[0,1]$.

In particular, if Y *is strictly* 2*-convex or of Orlicz-Lorentz type, then* $Y[0,1] = L_p[0,1]$.

PROOF. We will apply Theorem 6.7. First suppose that Z is a nonatomic Banach lattice which is lattice finitely representable in L_p, which has an unconditional basis lattice finitely representable in E_Y. Then of course $Z = L_p$. It follows from the reproducibility of the Haar basis (Theorem 2.c.8 of [34]) that the Haar basis is also lattice finitely representable in E_Y, contrary to hypothesis.

We conclude that $L_{p/2}$ can be cone-embedded into $Y_{1/2}$. Now the result follows immediately from Proposition 5.3. ∎

REMARKS. Here, the condition that Y is p-concave cannot be relaxed ([19]). We give a simple application. Suppose $1 \leq r < 2 < p$ and $Y = (L_r + L_p)[0,\infty)$. It follows from the above theorem that L_p is not isomorphic to a subspace of Y which answers a question raised in [17].

8. **Complemented subspaces of** r. i. **spaces.** The following result is quickly deduced from the methods of [27].

THEOREM 8.1. *Let* Y *be a separable order-continuous Banach lattice, which contains no complemented sublattice isomorphic to* ℓ_2. *Suppose* X *is a Banach lattice which is isomorphic to a complemented subspace of* Y. *Then either:*

(a) There is a constant C *so that, for every* n, ℓ_2^n *is* C*-lattice-isomorphic to a complemented sublattice of* X, *or:*

(b) There exists N *so that* X *is lattice isomorphic to a complemented sublattice of* $Y^N = Y \oplus \cdots \oplus Y$.

PROOF. We will prove under the assumption that X is nonatomic. (An exposition of the atomic case, which is proved by the same techniques, will be given in [10].) In this case we may suppose that both X and Y are good Köthe function spaces on (Δ, λ) and that X has the "strong density property." By combining Theorems 6.1 and 6.3 of [27] it is possible to find a sequence of Borel maps $\sigma_n \colon \Delta \longrightarrow \Delta$ and three sequences $(a_n^P), (a_n^Q), (a_n^R)$ of nonnegative Borel functions on Δ so that $a_n^P(s)^2 \leq a_n^Q(s) a_n^R(s)$ and if:

$$Pf = \sum_{n=1}^{\infty} a_n^P f \circ \sigma_n$$

$$Qf = \sum_{n=1}^{\infty} a_n^Q f \circ \sigma_n$$

$$Rf = \sum_{n=1}^{\infty} a_n^R f \circ \sigma_n$$

for $f \in (L_0)_+$ then we have for a suitable constant C_1 that $\|Pf\|_1 \leq C_1 \|f\|_1$, $\|Qf\|_{Y_{1/2}} \leq C_1 \|f\|_{X_{1/2}}$, and $\|Rf\|_{Y^*_{1/2}} \leq C_1 \|f\|_{X^*_{1/2}}$. Note here that Q need only map into $Y_{\max,1/2}$ and not necessarily into $Y_{1/2}$. Now by Theorem 6.4 of [27] it can be seen that if the first alternative fails then there is a constant $c > 0$ so that

$$\int \sup_n a_n^P f \circ \sigma_n \, d\lambda \geq c \int f \, d\lambda$$

for $f \geq 0$. We now use an argument due to Dor [15]. Consider the map $T: L_1 \longrightarrow L_1(c_0)$ defined by $Tf(s) = \big(a_n^P(s) f(\sigma_n(s))\big)$. Then $\|T\| \leq C_1$ and $\|Tf\| \geq c\|f\|$. Note that since c_0 has separable dual, $L_1(c_0)^*$ can be identified with $L_\infty(\ell_1)$. By the Hahn Banach theorem there exist $\phi_n \in L_\infty$ so that $\left\|\sum_{n=1}^\infty |\phi_n|\right\|_\infty \leq C_1 c^{-1}$ and

$$\sum_{n=1}^\infty \int \phi_n a_n^P f \circ \sigma_n \, d\lambda = \int f \, d\lambda$$

for $f \in L_1(\lambda)$.

Now for each n define $E_n = \{s : \phi_n(s) > (2C_1)^{-1}\}$. Then for $f \geq 0$,

$$\sum_{n=1}^\infty \int_{\Delta \setminus E_n} \phi_n a_n^P f \circ \sigma_n \leq \frac{1}{2} \int f \, d\lambda.$$

Hence

$$\sum_{n=1}^\infty \int_{E_n} a_n^P f \circ \sigma_n \, d\lambda \geq \frac{c}{2C_1} \int f \, d\lambda.$$

Notice that $\sum_{n=1}^\infty \chi_{E_n} \leq 2C_1 \sum_{n=1}^\infty |\phi_n| \leq 2C_1^2 c^{-1}$ almost everywhere. Let N be the least integer greater than $2C_1^2 c^{-1}$. Consider the operators P', Q' and R' defined by

$$P'f = \sum_{n=1}^\infty a_n^P \chi_{E_n} f \circ \sigma_n$$

$$Q'f = \sum_{n=1}^\infty a_n^Q \chi_{E_n} f \circ \sigma_n$$

$$R'f = \sum_{n=1}^\infty a_n^R \chi_{E_n} f \circ \sigma_n.$$

Then these operators can each be rewritten in the form,

$$P'f = \sum_{n=1}^N b_n^P f \circ \pi_n$$

$$Q'f = \sum_{n=1}^N b_n^Q f \circ \pi_n$$

$$R'f = \sum_{n=1}^N b_n^R f \circ \pi_n.$$

for suitable nonnegative Borel functions b_n^P, b_n^Q, b_n^R, for $1 \leq n \leq N$, which also satisfy $(b_n^P)^2 \leq b_n^Q b_n^R$ a.e., and for suitable Borel maps $\pi_n: \Delta \longrightarrow \Delta$.

Now define $U: X \longrightarrow Y^N_{\max}$, $V: X^* \longrightarrow (Y^*)^N$, by

$$Uf(s,n) = \big(b_n^Q(s)\big)^{1/2} f\big(\pi_n(s)\big)$$
$$Vf(s,n) = \big(b_n^R(s)\big)^{1/2} f\big(\pi_n(s)\big).$$

It is easy to see that U is bounded for

$$\begin{aligned}\max_{1\le n\le N} \|Uf(\cdot,n)\|_Y &\le \Big\| \Big(\sum_{n=1}^N b_n^Q (f\circ \pi_n)^2\Big)^{1/2}\Big\|_Y \\ &= \|Q'f^2\|_{Y_{1/2}}^{1/2} \\ &\le \|Qf^2\|_{Y_{1/2}}^{1/2} \\ &\le C_1^{1/2}\|f\|_X.\end{aligned}$$

Similarly V is bounded.

The proof is completed by Proposition 2.3 of [27], for if F is a Borel subset of Δ then

$$\begin{aligned}\sum_{n=1}^N \int_{\sigma_n^{-1}F} b_n^Q(s)^{1/2} b_n^R(s)^{1/2}\, d\lambda &\ge \sum_{n=1}^N \int_{\sigma_n^{-1}F} b_n^P(s)\, d\lambda \\ &= \int_\Delta P'\chi_F(s)\, d\lambda \\ &\ge \frac{c}{2C_1}\lambda(F).\end{aligned}$$ ■

This theorem has immediate consequences if Y is an r. i. space.

THEOREM 8.2. *Let Y be a separable* r. i. *space on* $[0,1]$ *or* $[0,\infty)$, *which contains no complemented sublattice isomorphic to ℓ_2. Suppose X is a strictly 2-convex or strictly 2-concave Banach lattice which is isomorphic to a complemented subspace of Y. Then X is lattice-isomorphic to a complemented sublattice of Y.*

We remark that Theorem 8.2 is closely related to Theorem 8.1 of [27], and could be used to simplify some of the arguments in the proof of that theorem somewhat.

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