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RESEARCH ARTICLE

Ulrich modules and weakly lim Ulrich sequences do not always exist

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Abstract

The existence of Ulrich modules for (complete) local domains has been a difficult and elusive open question. For over thirty years, it was unknown whether complete local domains always have Ulrich modules. In this paper, we answer the question of existence for both Ulrich modules and weakly lim Ulrich sequences – a weaker notion recently introduced by Ma – in the negative. We construct many local domains in all dimensions $d \geq 2$ that do not have any Ulrich modules. Moreover, we show that when $d = 2$, these local domains do not have weakly lim Ulrich sequences.

Contents

1. Overview

The question of whether Ulrich modules exist for complete local domains has been open for over three decades. In the first half of the paper, we construct the first known counterexamples to the existence of Ulrich modules for (complete) local domains. In fact, the construction gives many counterexamples in all dimensions $d \geq 2$ and with minimal modification; it can also be used to give counterexamples that are essentially of finite type over fields.

In the second half of the paper, we show that our counterexamples for the existence of Ulrich modules in dimension 2 are counterexamples to the existence of weakly lim Ulrich sequences for (complete) local domains. This is a rather surprising turn of events, given that 1) lim Cohen–Macaulay sequences exist for complete local domains of positive characteristic with F-finite residue fields [BHM] and 2) weakly lim Ulrich sequences (and, in fact, lim Ulrich sequences) exist for standard graded rings over infinite F-finite fields of positive characteristic [M]. Note that our counterexamples are fairly simple affine nonstandard graded algebras.

2. Introduction

Ulrich modules were introduced by Bernd Ulrich in 1984 as a means to study the Gorenstein property of Cohen-Macaulay rings [U]. Since then, the theory of Ulrich modules has become a very active area of research in both commutative algebra and algebraic geometry. Ulrich modules have many powerful and broad applications, ranging from the original purpose of giving a criteria for when a local Cohen-Macaulay ring is Gorenstein [U] to new methods of finding Chow forms of a variety [ES] to longstanding open conjectures in multiplicity theory.

One of the major applications of Ulrich modules is Lech's conjecture on Hilbert–Samuel multiplicities, where the existence of Ulrich modules or Ulrich-like objects has been the main approach for the majority of established cases. More specifically, the existence of Ulrich modules for complete local domains implies the following conjecture:

Conjecture 2.1 (Lech's Conjecture [L1]). *Let $\varphi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a flat local map between local rings. Then $e_{\mathfrak{m}}(R) \leq e_{\mathfrak{n}}(S)$.*

Historically, the existence of Ulrich modules has been a challenging question. The existing literature is sparse and has mainly explored positive existence results (i.e., classes of rings for which Ulrich modules exist). The major existence results are that Ulrich modules exist for the following classes of rings:

1. strict complete intersection rings [HUB]
2. generic determinantal rings [BRW]
3. projective 1-dimensional schemes over arbitrary fields [ES]
    - The subcase of two-dimensional, standard graded Cohen-Macaulay domains was proved earlier in [BHU].
4. some Veronese subrings of degree d of a polynomial ring in n variables:
    - $n = 3$ with d arbitrary and $n = 4$ with $d = 2^{\ell}$ in arbitrary characteristic [Ha99]
    - arbitrary n and d in characteristic 0 [ES]
    - arbitrary n and d for characteristic $p \geq (d-1)n + (n+1)$ [Sa]

Beyond these results, there has been limited progress. In particular, for over 30 years, it has been unknown whether complete local domains always have Ulrich modules. In the first half of the paper, we resolve the question of existence of Ulrich modules in the negative.

Theorem A. *Ulrich modules do not always exist for complete local domains.*

We prove Theorem A by constructing complete local domains R of all dimensions $d \geq 2$ whose S_2-ification S is a regular local ring. The key ingredient to proving that our counterexamples do not have Ulrich modules is Lemma 4.1, which states that any MCM module over R is an MCM module over S. This yields the following intermediary theorem:

Theorem B. *Let $(R, \mathfrak{m}, k)$ be a local domain. Suppose R has an S_2-ification S such that S is a regular local ring. Then every MCM module of R has the form $S^{\oplus h}$. Consequently, R has Ulrich modules if and only if S is an Ulrich module of R.*

In the second half of the paper, we consider the existence of (weakly) lim Ulrich sequences. To give some context, MCM modules (or small Cohen–Macaulay modules) and Ulrich modules (a special type of MCM module) are central to the main approaches to several major conjectures in multiplicity theory such as Serre's positivity and Lech's conjecture, respectively. However, existence results for MCM modules and Ulrich modules have been sparse and difficult to obtain. For example, the existence of MCM modules is open in dimension 3. As such, the weaker notions of lim Cohen–Macaulay sequences by Bhatt, Hochster and Ma [BHM] and (weakly) lim Ulrich sequences by Ma [M] were introduced as

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suitable replacements for small Cohen–Macaulay modules and Ulrich modules in the current approaches to Serre's positivity and Lech's conjecture respectively. Lim Cohen–Macaulay sequences and (weakly) lim Ulrich sequences have been shown to exist in much more general contexts.

In an earlier draft of [M] (prior to the results of this paper), Ma posed the following question:

Question 2.2 [M]. Does every complete local domain of characteristic $p > 0$ with an F-finite residue field admit a lim Ulrich sequence, or at least a weakly lim Ulrich sequence?

Given that lim Cohen–Macaulay sequences exist for completely local domains of positive characteristic with F-finite residue fields [BHM, M], it is reasonable to ask if (weakly) lim Ulrich sequences exist for these rings. However, we answer the question in the negative in this paper.

Theorem C. *Weakly lim Ulrich sequences do not always exist for complete local domains.*

To prove Theorem C, we establish important characterizations of (weakly) lim Cohen–Macaulay sequences and (weakly) lim Ulrich sequences for local domains of dimension 2.

3. Preliminaries

In this section, we review the relevant definitions and properties of Ulrich modules and (weakly) lim Ulrich sequences. Throughout this paper, all rings are commutative with a multiplicative identity. All local rings $(R, \mathfrak{m}, k)$ include the Noetherian condition. For simplicity, we assume that k is infinite unless explicitly stated otherwise. In particular, we assume the existence of a minimal reduction generated by d elements for the maximal ideal $\mathfrak{m}$ of a local ring R of dimension d.

Notation

Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Let M be a finitely generated module over R. Throughout the paper, we use the following notation:

- $\underline{x} = x_1, \ldots, x_d$
- $\ell_R(M)$ is the length of M as a module over R. We write $\ell(M)$ when it is clear from the context which R is being used.
- $H_i(\underline{x}; M)$ is the i-th Koszul homology of the module M with respect to $\underline{x}$.
- $h_i^R(\underline{x}; M) = \ell_R(H_i(\underline{x}; M))$
- $\chi(\underline{x}; M) = \sum_{i=0}^d (-1)^i \ell(H_i(\underline{x}; M)) = \sum_{i=0}^d (-1)^i h_i(\underline{x}; M)$.
- $\chi_1(\underline{x}; M) = \sum_{i=1}^d (-1)^{i-1} \ell(H_i(\underline{x}; M)) = \sum_{i=1}^d (-1)^{i-1} h_i(\underline{x}; M)$.
- $\nu_R(M)$ is the minimal number of generators of M.
- $e_R(M)$ is the multiplicity of M with respect to the maximal ideal $\mathfrak{m}$. When $M = R$, we write $e(R)$.

3.1. Ulrich modules

Definition 3.1 (Hilbert-Samuel Multiplicity). Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Let M be a finitely generated module over R. The *Hilbert-Samuel multiplicity* of M with respect to $\mathfrak{m}$ is

$$e_R(M) := d! \lim_{n\to\infty} \frac{\ell(M/\mathfrak{m}^n M)}{n^d}.$$

Definition 3.2 (MCM). Let M be a finitely generated module over (R, m, k). Then M is *maximal Cohen-Macaulay* (or MCM) module of R if $\operatorname{depth}_R(M) = \dim(R)$.

We review some useful facts.

Proposition 3.3. *Let $(R, \mathfrak{m}, k)$ be a local domain of dimension d. Let M be an MCM module over R. Then we have*

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(a) $e_R(M) = rank_R(M) \cdot e(R)$,
(b) $e_R(M) = \ell(M/IM)$, *where* $I \subseteq \mathfrak{m}$ *is a minimal reduction of* $\mathfrak{m}$, *and*
(c) $e_R(M) \geq \nu_R(M)$.

The statements in Proposition 3.3 are standard in the literature. For example, proofs can be found in [Ha99] and [U].

Definition 3.4. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. Let M be an MCM module over R. Then M is an *Ulrich module* if $e_R(M) = \nu_R(M)$. Equivalently, M is an Ulrich module if $\mathfrak{m}M = IM$ for any minimal reduction $I \subseteq \mathfrak{m}$.

Lemma 3.5. *Let* $(R, \mathfrak{m}, k)$ *be a local domain containing* k*. Let* L *be a finite algebraic extension of* k*. Then* $S = L \otimes_k R$ *is a local ring with maximal ideal* $\mathfrak{m}S$*, and* S *has an Ulrich module if and only if* R *has an Ulrich module.*

The proof of Lemma 3.5 is standard. We include it below for completeness.

Proof. Observe that S is a free module-finite extension of R and that $\mathfrak{m}S$ is the maximal ideal of S. So any system of parameters for R is a system of parameters for S.

Now suppose N is an Ulrich module over S. It is clear that any MCM module over S is an MCM module over R. We have $e(R) = e(S)$ because the length of $S/(\mathfrak{m}S)^t = L \otimes_R (R/\mathfrak{m}^t)$ over S is the same as the length of $R/\mathfrak{m}^t$ over R. Let $[L : k]$ be the degree of the field extension. Then $\nu_R(N) = [L : k]\nu_S(N)$ and we have

$$e_R(N) = \mathrm{rank}_R(N)e(R) = [L : k]\mathrm{rank}_S(N)e(R) = [L : k]\mathrm{rank}_S(N)e(S) = [L : k]e_S(N).$$

Then

$$\frac{e_R(N)}{\nu_R(N)} = \frac{e_S(N)}{\nu_S(N)} = 1.$$

So N is an Ulrich module of R.

Conversely, if M is an MCM module of R, then $S \otimes_R M$ is an MCM module of S, and we have $e_R(M) = e_S(S \otimes_R M)$ and $\nu_R(M) = \nu_S(S \otimes_R M)$. So if M is an Ulrich module of R, then $S \otimes_R M$ is an Ulrich module of S. □

3.2. (Weakly) lim Cohen–Macaulay sequences and (weakly) lim Ulrich sequences

We review the definitions and relevant properties of (weakly) lim Cohen-Macaulay sequences and (weakly) lim Ulrich sequences. (Weakly) lim Cohen-Macaulay sequences were introduced by Bhatt, Hochster and Ma [BHM]. See [Hoc17]. (Weakly) lim Ulrich sequences were introduced by Ma [M].

Definition 3.6. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. A sequence of finitely generated nonzero R-modules $\{M_n\}$ of dimension d is called *lim Cohen–Macaulay* if there exists a system of parameters $\underline{x}$ such that for all $i \geq 1$, we have

$$\lim_{n\to\infty} \frac{h_i(\underline{x}; M_n)}{\nu_R(M_n)} = 0.$$

A sequence of finitely generated R-modules $\{M_n\}$ of dimension d is called *weaky lim Cohen–Macaulay* if there exists a system of parameters $\underline{x}$ such that

$$\lim_{n\to\infty} \frac{\chi_1(\underline{x}; M_n)}{\nu_R(M_n)} = 0.$$

Definition 3.7. Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d. A sequence of finitely generated nonzero R-modules $\{M_n\}$ of dimension d is called *lim Ulrich* (respectively, *weakly lim Ulrich*) if $\{M_n\}$ is lim Cohen–Macaulay (respectively, weakly lim Cohen–Macaulay) and

$$\lim_{n\to\infty} \frac{e_R(M_n)}{\nu_R(M_n)} = 1.$$

Throughout the paper, we use the following proposition due to [BHM] and [M].

Proposition 3.8 [BHM][M]. *Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d.*

(a) *[BHM](See [IMW, Lemma 5.7]) If $\{M_n\}$ is a lim Cohen–Macaulay sequence of R, then for every system of parameters $\underline{x} = x_1, \ldots, x_d$, we have*

$$\lim_{n\to\infty} \frac{h_i(\underline{x}; M_n)}{\nu_R(M_n)} = 0,$$

where $i \geq 1$.

(b) *[M, Prop. 2.6] If $\{M_n\}$ is a weakly lim Cohen–Macaulay sequence of R, then for every system of parameters $\underline{x} = x_1, \ldots, x_d$, we have*

$$\lim_{n\to\infty} \frac{\chi_1(\underline{x}; M_n)}{\nu_R(M_n)} = 0.$$

4. Ulrich modules do not always exist for local domains

Lemma 4.1. *Let $(R, \mathfrak{m}, k)$ be a local domain. If R has an S_2-ification S that is a local ring, then any MCM module M of R is an MCM module of S.*

Proof. Let M be an MCM module over R. We want to show that for any $f \in S - R$ and any $m \in M$, there is a well-defined element $f \cdot m \in M$. Let $W = R - \{0\}$. Since M is MCM, it is torsion-free over R and embeds in $W^{-1}M$. It suffices to show that $f \cdot (m/1) \in M$. Since any element in the S_2-ification of R is multiplied back into R by an ideal in R of height at least 2 (see [HH, (2.3)]), the height of the ideal $R :_R f$ is at least two. Thus, there exist u and v in $R :_R f$ such that the sequence u, v is a part of a system of parameters for R. Since M is MCM, the sequence u, v is a regular sequence on M. Then $v \cdot ((uf) \cdot (m/1)) = u \cdot ((vf) \cdot (m/1)) \in vM$ implies that $(vf) \cdot (m/1) \in vM$. Since M is torsion-free over R, we have $f \cdot (m/1) \in M$. □

Theorem 4.2. *Let $(R, \mathfrak{m}, k)$ be a local domain. Suppose R has an S_2-ification S such that S is a regular local ring. Then every MCM module of R has the form $S^{\oplus h}$. Consequently, R has Ulrich modules if and only if S is an Ulrich module of R if and only if $IS = \mathfrak{m}S$ for any minimal reduction I of $\mathfrak{m}$.*

Proof. By Lemma 4.1, any MCM module M over R is MCM over S. But S is regular. Hence, $M \cong S^{\oplus h}$ by the Auslander–Buchsbaum formula. The second statement follows immediately because $S^{\oplus h}$ is an Ulrich module of R if and only if S is an Ulrich module of R. □

Theorem 4.3. *Let $S = k[[\underline{x}]] = k[[x_1, \ldots, x_d]]$, where $d \geq 2$. Let $\underline{u} = u_1, \ldots, u_d$ be a system for parameters of S such that $I = (\underline{u})S$ is not integrally closed. Let $\overline{I}$ be the integral closure of I in S. Let $\{g_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary collection of elements in $\overline{I}$ and $f \in \overline{I} - I$. For $1 \leq j \leq d$, let v_j, w_j be elements of the maximal ideal of $k[[\underline{u}]]$ that generate a height 2 ideal in $k[[\underline{u}]]$ (e.g., one can take powers of distinct elements in $\{u_1, \ldots, u_d\}$). Define R to be the domain*

$$R := k[[\underline{u}]][f][v_j x_j, w_j x_j]_{1\leq j\leq d}[g_\lambda]_{\lambda\in\Lambda}.$$

Then R has no Ulrich modules.

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Proof. First, notice that $k[[\underline{u}]] \subset k[[\underline{x}]]$ is a module-finite extension. Then R is (Noetherian) local and $R \subset k[[\underline{x}]]$ is a module-finite extension.

Let $\mathfrak{m}_R$ be the maximal ideal of R. From the construction of R, it is clear that $\underline{u} = u_1, \ldots, u_d$ is a system for parameters for R and, in fact, a minimal reduction of $\mathfrak{m}_R$ because all the other adjoined elements are integral over $(\underline{u})S$ in S and thus integral over $(\underline{u})R$ in R. Then for all $1 \le j \le d$, the element x_j is multiplied into R by v_j and w_j which generate a height 2 ideal in R. Thus x_j is in the S_2-ification of R for all $1 \le j \le d$. But this means that $S = k[[\underline{x}]]$ is the S_2-ification of R.

By Theorem 4.2, it suffices to show that S is not an Ulrich module of R. But $(\underline{u})S \ne \mathfrak{m}_R S$ because $f \notin (\underline{u})S$. Thus, R has no Ulrich modules. □

Remark 4.4. Similar constructions can be used to generate counterexamples that are essentially of finite type over fields. For example, see Counterexample 4.6.

Remark 4.5. In [IMW], Iyengar, Ma and Walker consider rings of the form $T = k + J$, where $S = k[[x, y]]$ and $J \subseteq S$ is an ideal primary to $(x, y)S$. If J has a minimal reduction $I = (u, v)S$, then the rings T have the form in Theorem 4.3. Thus, $T = k + J$ has no Ulrich modules if $J \ne IS$.

In the case where J does not have a minimal reduction, we can reduce to the previous case by taking a finite algebraic field extension of k so that J has a minimal reduction and then apply Lemma 3.5.

Counterexample 4.6. This is the first counterexample we found, which led to the general construction in Theorem 4.3. The local domain

$$R = k[x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy]_{\mathfrak{m}},$$

where $\mathfrak{m}$ is the maximal ideal $(x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy)$, and its completion

$$\widehat{R} = k[[x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy]]$$

do not have Ulrich modules for $n \ge 2$.

We can show that R and $\widehat{R}$ do not have Ulrich modules directly. The argument is essentially the same for R and $\widehat{R}$. We will work with R. The S_2-ification of R is $S = k[x, y]_{(x,y)}$, and the ideal $(xy, x^n - y^n)R$ is a minimal reduction for $\mathfrak{m}R$. But

$$(xy, x^n - y^n)S \ne (x^n, x^{n+1}, x^n y, y^n, y^{n+1}, xy^n, xy)S$$

as ideals in S.

We can recover this counterexample from the general construction in Theorem 4.3 by setting

$$\begin{aligned}
&\underline{x} = x, y\\
&\underline{u} = xy, x^n - y^n\\
&f = x^n\\
&v_j = (xy)^n \quad \text{and} \quad w_j = x^n - y^n \qquad 1 \le j \le 2\\
&g_1 = x^{n+1} \quad \text{and} \quad g_2 = y^{n+1}.
\end{aligned}$$

Remark 4.7. We can use R in Counterexample 4.6 to give a new counterexample to the localization of Ulrich modules (i.e., a local ring $(T, \mathfrak{n}, \ell)$ that has an Ulrich module M and a prime ideal $\mathfrak{p} \subseteq T$ such that $M_{\mathfrak{p}}$ is not an Ulrich module over $T_{\mathfrak{p}}$). Although a counterexample to localization was first given by Hanes in [Ha99], the following counterexample is stronger in the sense that T localizes to a ring that has no Ulrich modules, whereas Hanes's counterexample localizes to a ring that does have an Ulrich module. Note that Ulrich modules are called linear MCM modules in [Ha99].

Counterexample 4.8 (Localization). Consider the ring

$$T = k[s^{n+1}, sx^n, x^{n+1}, x^n y, sy^n, y^{n+1}, xy^n, s^{n-1}xy]_{\mathfrak{n}},$$

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where $\mathfrak{n} = (s^{n+1}, sx^n, x^{n+1}, x^n y, sy^n, y^{n+1}, xy^n, s^{n-1}xy)$ and $n \geq 2$. Let

$$\varphi : T \hookrightarrow k[s, x, y]_{(s,x,y)}$$

be the inclusion map and $\mathfrak{p} = \varphi^{-1}((x, y))$. Then the localization $T_{\mathfrak{p}}$ is the ring

$$k(s^{n+1})\left[\left(\frac{x}{s}\right)^n, \left(\frac{x}{s}\right)^{n+1}, \left(\frac{x}{s}\right)^n\left(\frac{y}{s}\right), \left(\frac{y}{s}\right)^n, \left(\frac{y}{s}\right)^{n+1}, \left(\frac{x}{s}\right)\left(\frac{y}{s}\right)^n, \left(\frac{x}{s}\right)\left(\frac{y}{s}\right)\right]$$

localized at the obvious maximal ideal. But $T_{\mathfrak{p}}$ has no Ulrich modules by Counterexample 4.6.

It remains to show that T has an Ulrich module. Let S be the localization of the $(n+1)$th Veronese subring of $k[s, x, y]$ at the homogeneous maximal ideal. One can compute $e_T(T) = (n+1)^2 = e_S(S)$. The rings T and S have the same fraction field and so $\operatorname{rank}_T(S) = 1$. Now S has an Ulrich module M by Proposition 3.6 in [Ha04]. Then M is MCM over T and $\operatorname{rank}_T(M) = \operatorname{rank}_S(M)$. Then

$$(n+1) \cdot \operatorname{rank}_T(M) = e_T(M) \geq \nu_T(M) \geq \nu_S(M) = e_S(M) = (n+1) \cdot \operatorname{rank}_T(M).$$

Thus, $e_T(M) = \nu_T(M)$ and M is Ulrich over T.

5. (Weakly) lim Cohen–Macaulay and (weakly) lim Ulrich sequences over domains of dimension 2

Definition 5.1. Let $(R, \mathfrak{m}, k)$ be a local ring. Let $\mathcal{M} = \{M_n\}$ be a sequence of nonzero finitely generated R-modules. Let $\nu_R(M_n)$ be the minimal number of generators of M_n. Let $\{a_n\}$ and $\{b_n\}$ be a sequence of positive integers. We define $\sim_{\mathcal{M}}$ to be the equivalence relation $\sim_{\mathcal{M}}$, where $\{a_n\} \sim_{\mathcal{M}} \{b_n\}$ if

$$\lim_{n\to\infty} \frac{a_n - b_n}{\nu_R(M_n)} = 0.$$

For the sake of simplicity, we write $a_n \sim_{\mathcal{M}} b_n$ instead of $\{a_n\} \sim_{\mathcal{M}} \{b_n\}$.

Lemma 5.2. *Let $(R, \mathfrak{m}, k)$ be a local ring. Let $\mathcal{M} = \{M_n\}$ and $\mathcal{N} = \{N_n\}$ be two sequences of nonzero finitely generated R-modules. Let $\{a_n\}$ and $\{b_n\}$ be a sequence of non-negative integers. Suppose $\nu_R(N_n) \sim_{\mathcal{M}} \nu_R(M_n)$. If $a_n \sim_{\mathcal{M}} b_n$, then $a_n \sim_{\mathcal{N}} b_n$. In particular, $\nu_R(M_n) \sim_{\mathcal{N}} \nu_R(N_n)$.*

Proof. Since $\nu_R(N_n) \sim_{\mathcal{M}} \nu_R(M_n)$, we have $\lim_{n\to\infty} \frac{\nu_R(M_n)}{\nu_R(N_n)} = 1$. Then

$$\begin{aligned} 0 = \lim_{n\to\infty} \frac{a_n - b_n}{\nu_R(M_n)} = \left(\lim_{n\to\infty} \frac{a_n - b_n}{\nu_R(M_n)}\right)\left(\lim_{n\to\infty} \frac{\nu_R(M_n)}{\nu_R(N_n)}\right) &= \lim_{n\to\infty}\left(\frac{a_n - b_n}{\nu_R(M_n)} \cdot \frac{\nu_R(M_n)}{\nu_R(N_n)}\right) \\ &= \lim_{n\to\infty} \frac{a_n - b_n}{\nu_R(N_n)}. \end{aligned}$$

□

Theorem 5.3. *Let $(R, \mathfrak{m}, k)$ be a local domain of dimension 2. Let $\{M_n\}$ be a weakly lim Cohen–Macaulay (resp. weakly lim Ulrich) sequence over R. Let $C_n \subseteq M_n$ be a torsion submodule such that the quotient $\overline{M}_n := M_n/C_n$ has no finite length submodules. Then the sequence $\{\overline{M}_n\}$ is a lim Cohen–Macaulay (resp. lim Ulrich) sequence over R.*

Proof. Let $\mathcal{M} := \{M_n\}$ be a weakly lim Cohen–Macaulay sequence over R. Let $I = (\underline{x})$ be a system of parameters of the maximal ideal $\mathfrak{m}$. Recall that $\nu_R(M_n)$ denotes the minimal number of generators of M_n. First, we check that

$$\nu_R(M_n) \sim_{\mathcal{M}} \nu_R(\overline{M}_n).$$

Consider the short exact sequence

$$0 \to C_n \to M_n \to \overline{M}_n \to 0.$$

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We know that $\nu_R(\overline{M}_n) \leq \nu_R(M_n) \leq \nu_R(\overline{M}_n) + \nu_R(C_n)$. So it suffices to show that

$$\nu_R(C_n) \sim_{\mathcal{M}} 0.$$

From the short exact sequence, we get the long exact sequence of Koszul homology

$$\begin{aligned} 0 &\to H_2(\underline{x}; C_n) \to H_2(\underline{x}; M_n) \to H_2(\underline{x}; \overline{M}_n) \\ &\to H_1(\underline{x}; C_n) \to H_1(\underline{x}; M_n) \to H_1(\underline{x}; \overline{M}_n) \to H_0(\underline{x}; C_n) \to H_0(\underline{x}; M_n) \to H_0(\underline{x}; \overline{M}_n) \to 0. \end{aligned}$$

Now $\overline{M}_n$ has no finite length torsion submodules, so $H_2(\underline{x}; \overline{M}_n) = 0$. We observe the following:

(a) $H_2(\underline{x}; C_n) \cong H_2(\underline{x}; M_n)$
(b) $h_1(\underline{x}; C_n) \leq h_1(\underline{x}; M_n)$
(c) $\chi_1(\underline{x}; M) \geq 0$ for any finitely generated R-module M [S, Corollary on pg. 90]
(d) $\chi(\underline{x}; C_n) = e(\underline{x}; C_n) = 0$, where $e(\underline{x}; C_n)$ is the multiplicity of C_n with respect to $(\underline{x})$, and the first equality is by [S, Theorem 1 on pg. 57]
(e) $0 \leq h_0(\underline{x}; M_n) - h_0(\underline{x}; \overline{M}_n) \leq h_0(\underline{x}; C_n)$

From (a), (b) and (c), it follows that

$$0 \leq \chi_1(\underline{x}; C_n) \leq \chi_1(\underline{x}; M_n),$$

and because $\mathcal{M}$ is weakly lim Cohen–Macaulay, we have

$$\chi_1(\underline{x}; C_n) \sim_{\mathcal{M}} 0. \tag{1}$$

But $\chi(\underline{x}; C_n) = 0$ and so, $\chi_1(\underline{x}; C_n) = h_0(\underline{x}; C_n) = \ell(C_n/(\underline{x})C_n)$. Then the inequality

$$\nu_R(C_n) = \ell(C_n/\mathfrak{m}C_n) \leq \ell(C_n/(\underline{x})C_n)$$

yields

$$\nu_R(C_n) \sim_{\mathcal{M}} 0.$$

Next, we show that $\{\overline{M}_n\}$ is a lim Cohen-Macaulay sequence over R. We already know that $h_2(\underline{x}; \overline{M}_n) = 0$. It remains to show

$$\lim_{n\to\infty} \frac{h_1(\underline{x}; \overline{M}_n)}{\nu_R(\overline{M}_n)} = 0.$$

By Lemma 5.2, it is enough to show that

$$\lim_{n\to\infty} \frac{h_1(\underline{x}; \overline{M}_n)}{\nu_R(M_n)} = 0$$

because $\nu_R(\overline{M}_n) \sim_{\mathcal{M}} \nu_R(M_n)$. Take the alternating sum of the lengths of the Koszul homology in the exact sequence

$$0 \to H_1(\underline{x}; C_n) \to H_1(\underline{x}; M_n) \to H_1(\underline{x}; \overline{M}_n) \to H_0(\underline{x}; C_n) \to H_0(\underline{x}; M_n) \to H_0(\underline{x}; \overline{M}_n) \to 0.$$

This is the sum

$$h_1(\underline{x}; C_n) - h_1(\underline{x}; M_n) + h_1(\underline{x}; \overline{M}_n) - h_0(\underline{x}; C_n) + h_0(\underline{x}; M_n) - h_0(\underline{x}; \overline{M}_n) = 0.$$

Then

$$\begin{aligned}
h_1(\underline{x};\overline{M}_n) &= -h_1(\underline{x};C_n) + h_0(\underline{x};C_n) + h_1(\underline{x};M_n) - h_0(\underline{x};M_n) + h_0(\underline{x};\overline{M}_n) \\
&= -h_2(\underline{x};C_n) + h_1(\underline{x};M_n) - h_0(\underline{x};M_n) + h_0(\underline{x};\overline{M}_n) \\
&= -h_2(\underline{x};M_n) + h_1(\underline{x};M_n) - h_0(\underline{x};M_n) + h_0(\underline{x};\overline{M}_n) \\
&= \chi_1(\underline{x};M_n) - (h_0(\underline{x};M_n) - h_0(\underline{x};\overline{M}_n)).
\end{aligned}$$

Now we know that

$$\chi_1(\underline{x};M_n) \sim_{\mathcal{M}} 0,$$

and by (e) and (1) above, we have

$$0 \le h_0(\underline{x};M_n) - h_0(\underline{x};\overline{M}_n) \le h_0(\underline{x};C_n) = \chi_1(\underline{x};C_n) \sim_{\mathcal{M}} 0.$$

Thus,

$$\lim_{n\to\infty} \frac{h_1(\underline{x};\overline{M}_n)}{\nu_R(M_n)} = 0,$$

and the sequence $\{\overline{M}_n\}$ is lim Cohen–Macaulay.

In the case where $\{M_n\}$ is a weakly lim Ulrich sequence, it remains to check that

$$\lim_{n\to\infty} \frac{e_R(\overline{M}_n)}{\nu_R(\overline{M}_n)} = 1.$$

But $e_R(\overline{M}_n) = e_R(M_n)$ and $\nu_R(\overline{M}_n) \sim_{\mathcal{M}} \nu_R(M_n)$, so the condition immediately follows, and thus, $\{\overline{M}_n\}$ is a lim Ulrich sequence of R. □

Definition 5.4. Let $(R,\mathfrak{m},k)$ be a local domain and let M be finitely generated torsion-free R-module. Let $(S,\mathfrak{n},\ell)$ be a local module-finite extension domain of R. Suppose $\mathcal{K} = frac(R) = frac(S)$. Then we define MS to be the S-module generated by M in $M \otimes_R \mathcal{K}$.

Remark 5.5. In the case where R is a local domain with a S_2-ification S that is local, if M is an MCM module of R, then $MS = M$ by Lemma 4.1.

Lemma 5.6. *Let $(R,\mathfrak{m},k)$ be a local domain of dimension 2 and let M be a finitely generated torsion-free R-module. Let $(S,\mathfrak{n},\ell)$ be a local module-finite extension domain of R. Suppose $S \subseteq frac(R)$ and S/R has finite length. Choose a fixed constant t such that $\mathfrak{m}^t S \subseteq R$. Let x, y be a system of parameters for R. Then*

(a) $MS \subseteq M :_{\mathcal{K}\otimes_R M} (x^t, y^t)R$,
(b) $(M :_{M\otimes_R\mathcal{K}} (x^t, y^t))/M \cong H_1(x^t, y^t; M)$,
(c) $\ell_R(MS/M) \le h_1(x^t, y^t; M)$.

Proof. Part (a) is clear by the choice of t. Part (c) follows immediately from parts (a) and (b). It remains to prove part (b). Define

$$\varphi : H_1(x^t, y^t; M) \to (M :_{M\otimes_R\mathcal{K}} (x^t, y^t))/M$$

to be the map

$$[(u,v)] \mapsto \left[\frac{u}{y^t}\right] = \left[\frac{-v}{x^t}\right],$$

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where the equality follows from the relation $ux^t + vy^t = 0$. This map is well-defined. If $[(u, v)]$ is trivial, then there exists $w \in M$ such that $[(u, v)] = [y^t w, -x^t w]$. But $[y^t w, -x^t w]$ is mapped to $[(y^t w)/y^t] = [w/1] = 0$.

For the map going the other direction, define

$$\psi : (M :_{M \otimes_R \mathcal{K}} (x^t, y^t))/M \to H_1(x^t, y^t; M)$$

to be the map

$$[f] \mapsto [(y^t f, -x^t f)].$$

This is clearly well-defined. The maps φ and ψ are inverses, so we are done. □

Theorem 5.7. *Let $(R, \mathfrak{m}, k)$ be a local domain of dimension 2. Let $(S, \mathfrak{n}, \ell)$ be a local module–finite extension domain of R such that $S \subseteq frac(R)$ and S/R has finite length. Let $\mathcal{M} = \{M_n\}$ be a lim Cohen–Macaulay (resp. lim Ulrich) sequence of torsion-free R-modules. Then the sequence $\mathcal{N} = \{M_n S\}$ is a lim Cohen–Macaulay (resp. lim Ulrich) sequence of R-modules and also a lim Cohen–Macaulay sequence of S-modules.*

Proof. We first prove that

$$\nu_R(M_n) \sim_{\mathcal{M}} \nu_R(M_n S).$$

Let $Q_n = M_n S/M_n$. Note that Q_n has finite length because S/R has finite length. The short exact sequence

$$0 \to M_n \to M_n S \to Q_n \to 0$$

yields the long exact sequence

$$\ldots \to \operatorname{Tor}_1^R(Q_n, k) \to M_n \otimes_R k \to M_n S \otimes_R k \to Q_n \otimes_R k \to 0.$$

Then

$$\nu_R(M_n) \le \nu_R(M_n S) + \ell(\operatorname{Tor}_1^R(Q_n, k)) \le \nu_R(M_n) + \nu_R(Q_n) + \ell(\operatorname{Tor}_1^R(Q_n, k)),$$

and so it suffices to show that

$$\ell(\operatorname{Tor}_1^R(Q_n, k)) \sim_{\mathcal{M}} 0 \quad \text{and} \quad \nu_R(Q_n) \sim_{\mathcal{M}} 0.$$

Let $\underline{x} = x_1, x_2$ be a system of parameters for R. By Lemma 5.6, we know that $\ell(Q_n) \le h_1(x_1^t, x_2^t; M_n)$ for some fixed t. But $\mathcal{M}$ is a lim Cohen-Macaulay sequence, so $h_1(x_1^t, x_2^t; M_n) \sim_{\mathcal{M}} 0$ and

$$\ell(Q_n) \sim_{\mathcal{M}} 0.$$

Then

$$\nu_R(Q_n) = \ell(Q_n/\mathfrak{m}Q_n) \sim_{\mathcal{M}} 0.$$

Next, by taking a prime cyclic filtration of Q_n, one can observe that

$$\ell(\operatorname{Tor}_1^R(Q_n, k)) \le \ell(Q_n)\ell(\operatorname{Tor}_1^R(k, k)).$$

Then it immediately follows that

$$\ell(\operatorname{Tor}_1^R(Q_n, k)) \sim_{\mathcal{M}} 0.$$

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We now show that $\mathcal{N} = \{M_nS\}$ is a lim Cohen–Macaulay sequence of R-modules. It is enough to show that

$$h_1(\underline{x}; M_nS) \sim_{\mathcal{M}} 0.$$

Because M_n and M_nS are torsion-free over R, we have the long exact sequence

$$\begin{aligned} 0 \to H_2(\underline{x}; Q_n) &\to H_1(\underline{x}; M_n) \to H_1(\underline{x}; M_nS) \to H_1(\underline{x}; Q_n) \\ &\to H_0(\underline{x}; M_n) \to H_0(\underline{x}; M_nS) \to H_0(\underline{x}; Q_n) \to 0. \end{aligned} \tag{2}$$

Next we show that for all $i \geq 0$,

$$h_i(\underline{x}; Q_n) \sim_{\mathcal{M}} 0.$$

We see that

$$h_2(\underline{x}; Q_n) \sim_{\mathcal{M}} 0$$

because $H_2(\underline{x}; Q_n)$ injects into $H_1(\underline{x}; M_n)$. We already proved that $\ell(Q_n) \sim_{\mathcal{M}} 0$. It immediately follows that

$$h_0(\underline{x}; Q_n) = \ell(Q_n/\underline{x}Q_n) \sim_{\mathcal{M}} 0.$$

Then it follows from $\chi(\underline{x}; Q_n) = 0$ that

$$h_1(\underline{x}; Q_n) \sim_{\mathcal{M}} 0.$$

From the long exact sequence (2), we have

$$h_1(\underline{x}; M_nS) \leq h_1(\underline{x}; M_n) + h_1(\underline{x}; Q_n).$$

But $h_1(\underline{x}; M_n) \sim_{\mathcal{M}} 0$ and $h_1(\underline{x}; Q_n) \sim_{\mathcal{M}} 0$. Therefore, $\mathcal{N} = \{M_nS\}$ is a lim Cohen–Macaulay sequence over R.

If $\mathcal{M}$ is lim Ulrich, it immediately follows that $\mathcal{N}$ is lim Ulrich because $e_R(M_n) = e_R(M_nS)$ and $\nu_R(M_n) \sim_{\mathcal{M}} \nu_R(M_nS)$. It remains to check that $\mathcal{N} = \{M_nS\}$ is a lim Cohen–Macaulay sequence for S.

For any i, the Koszul homology $H_i(\underline{x}; M_nS)$ does not change whether we think of M_nS as an R-module or an S-module. We also have

$$\nu_R(M_nS) \leq \nu_R(S)\nu_S(M_nS),$$

which yields

$$\frac{\nu_R(M_nS)}{\nu_R(S)} \leq \nu_S(M_nS).$$

Then

$$\lim_{n\to\infty} \frac{h_1^S(\underline{x}; M_nS)}{\nu_S(M_nS)} \leq \lim_{n\to\infty} \frac{\nu_R(S)h_1^R(\underline{x}; M_nS)}{\nu_R(M_nS)} = \nu_R(S) \lim_{n\to\infty} \frac{h_1^R(\underline{x}; M_nS)}{\nu_R(M_nS)} = 0.$$

Thus, $\mathcal{N} = \{M_nS\}$ is a lim Cohen-Macaulay sequence over S. $\square$

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Theorem 5.8. *A sequence of finitely generated nonzero torsion–free modules* $\{N_n\}$ *over a regular local ring* S *of dimension* 2 *is lim Cohen–Macaulay if and only if for the minimal free resolution*

$$0 \to S^{b_n} \to S^{a_n} \to N_n \to 0$$

we have $\lim_{n\to\infty} b_n/a_n = 0$. *Such a sequence is always lim Ulrich.*

Proof. Let x, y be a regular system of parameters for S. We have

$$a_n = \nu_S(N_n) = h_0(x, y; N_n)$$

and

$$b_n = h_1(x, y; N_n).$$

Then $\{N_n\}$ is lim Cohen–Macaulay over S if and only if

$$\lim_{n\to\infty} \frac{h_1(x, y; N_n)}{\nu_S(N_n)} = \lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

Moreover, we have

$$\lim_{n\to\infty} \frac{e_S(N_n)}{\nu_S(N_n)} = \lim_{n\to\infty} \frac{a_n - b_n}{a_n} = 1.$$

Note that $e_S(N_n) = a_n - b_n$ by [S, Theorem 1 on pg. 57]. Thus, $\{N_n\}$ is a lim Ulrich sequence for S. □

6. Weakly lim Ulrich sequences do not always exist for local domains

Theorem 6.1. *Let* $(R, \mathfrak{m}, k)$ *be a local domain of dimension* 2. *Suppose* R *has an* S_2*-ification* S *that is a regular local ring. The following are equivalent:*

(a) *R has a weakly lim Ulrich sequence.*
(b) *R has an Ulrich module.*
(c) *S is an Ulrich module of R.*
(d) *For any minimal reduction I of $\mathfrak{m}$, we have $IS = \mathfrak{m}S$.*

Proof. First, (c) and (d) are equivalent by definition. Next (b) and (c) are equivalent by Theorem 4.2. It is clear that (b) implies (a). It remains to show that (a) implies (b).

Suppose R has a weakly lim Ulrich sequence. Then by Theorems 5.3 and 5.7, there exists a lim Ulrich sequence $\mathcal{M} = \{M_n\}$ of torsion-free R modules that are also S modules. Consider the minimal free resolution

$$0 \to S^{b_n} \to S^{a_n} \to M_n \to 0,$$

where $a_n = \nu_S(M_n)$. Now

$$\nu_R(S^{b_n}) = b_n \nu_R(S)$$

and

$$e_R(S^{b_n}) = b_n e_R(S).$$

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Then Theorem 5.8 yields

$$\lim_{n\to\infty} \frac{\nu_R(S^{b_n})}{\nu_R(M_n)} = \lim_{n\to\infty} \frac{\nu_R(S)b_n}{\nu_R(M_n)} \le \lim_{n\to\infty} \frac{\nu_R(S)b_n}{\nu_S(M_n)} = \lim_{n\to\infty} \frac{\nu_R(S)b_n}{a_n} = 0,$$

and

$$\lim_{n\to\infty} \frac{e_R(S^{b_n})}{\nu_R(M_n)} = \lim_{n\to\infty} \frac{e_R(S)b_n}{\nu_R(M_n)} \le \lim_{n\to\infty} \frac{e_R(S)b_n}{\nu_S(M_n)} = \lim_{n\to\infty} \frac{e_R(S)b_n}{a_n} = 0.$$

Consequently, by the minimal free resolution above, we have

$$\nu_R(M_n) \sim_{\mathcal{M}} \nu_R(S^{a_n}) = \nu_R(S)a_n, \tag{3}$$

and

$$e_R(M_n) \sim_{\mathcal{M}} e_R(S^{a_n}) = e_R(S)a_n. \tag{4}$$

Combining equivalences 3 and 4, we have

$$\frac{e_R(S)}{\nu_R(S)} = \lim_{n\to\infty} \frac{e_R(S)}{\nu_R(S)} = \lim_{n\to\infty} \frac{e_R(M_n)}{\nu_R(M_n)} = 1.$$

Thus, S is an Ulrich module of R. □

Theorem 6.2. *Weakly lim Ulrich sequences do not always exist for (complete) local domains.*

Proof. This is immediate by Theorem 4.3 by taking $d = 2$ and applying Theorem 6.1. □

Corollary 6.3 (Localization). *Weakly lim Ulrich sequences do not always localize for local domains. More precisely, there exist local domains $(R, \mathfrak{m}, k)$ that have a weakly lim Ulrich sequence $\{M_n\}$ and a prime ideal $\mathfrak{p}$ such that $\{(M_n)_\mathfrak{p}\}$ is not a weakly lim Ulrich sequence for $R_\mathfrak{p}$. Moreover, there exist local domains that have weakly lim Ulrich sequences and a prime ideal $\mathfrak{p}$ such that $R_\mathfrak{p}$ has no weakly lim Ulrich sequences.*

Proof. This is immediate by taking k to be perfect and $\operatorname{char}(k) > 0$ in Counterexample 4.8 and applying Theorem 6.1. □

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Competing interest. The authors have no competing interest to declare.

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