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LOCAL TOPOLOGICAL PROPERTIES AND ONE POINT EXTENSIONS

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Introduction. In 1957, Mrowka [**12**] showed that a locally paracompact space admits a one point paracompactification (see also [**2**, Chapter 9, § 4, Exercise 27]). Similarly, in [**9**] Isiwata obtained a one point realcompactification for locally realcompact spaces. Recently a number of authors (see [**11**, **16**; **17**; **18**; **21**]) have constructed one point P-extensions of local P-spaces for a variety of topological properties P. It is the purpose of this paper to draw together the various techniques used by the above mentioned authors and to study the set (lattice) of all one point P-extensions of a particular space.

1. Axioms and basic definitions. Throughout this paper, space will mean completely regular Hausdorff space. A *topological property* is a class P of spaces such that if $X \in P$ and Y is homeomorphic to X, then $Y \in P$. An element of the class P will be called a *P-space* and a subspace of a P-space will be called *P-regular*. (Note that if P is the class of compact spaces then P-regularity is equivalent complete regularity.) In this note we shall restrict our attention to topological properties P which satisfy the following Axioms.

A_1 (closed-hereditary): Each closed subspace of a P-space belongs to P.

A_2 (compact-productive): Any product of compact P-spaces belongs to P.

A_3 (compactification): Each P-space is a subspace of a compact P-space.

A_4: The product of a P-space and a compact P-space belongs to P.

A_5 (Mrowka's condition (W); see [**12**]): If X is a space in which there is a point x_0 and a neighborhood base $\mathscr{B}$ at x_0 such that $X \backslash B \in P$ for all B in $\mathscr{B}$, then X belongs to P.

A_6: If Y_1, Y_2 are closed G_δ-subspaces of a compact P-space T and X is any subspace of T for which $X \cup Y_i \in P$, $i = 1, 2$, then $X \cup [Y_1 \cap Y_2] \in P$.

Remark. Axioms A_1, A_2, and A_3 insure the existence of a maximal compactification in the class P for each P-regular space (see 2.1 below) while A_1, A_5, and A_6 imply the existence of one point P-extensions (see 3.1 below) and A_4 enables us to characterize the one point P-extensions in terms of subspaces of the maximal compact P-extension.

1.1 LEMMA. *If P is a topological property which satisfies the above axioms, then P is productive if and only if P is closed with respect to arbitrary intersection.*

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Proof. The necessity follows from [**7**, Lemma 7.3, p. 242]. We shall prove that the condition is also sufficient. If $\{X_\alpha\}$ is a family of P-spaces indexed by a set Γ, then by axiom A_3 there is a family $\{K_\alpha\}$ of compact P-spaces indexed by the same set Γ such that $X_\alpha \subset K_\alpha$. Let Y_α be the product of X_α with all K_β, $\beta \neq \alpha$. Then A_2 and A_4 insure that each $Y_\alpha \in P$. Since the intersection of the Y_α is homeomorphic with the product of the spaces X_α, the proof is complete.

1.2 LEMMA. *If P_1 and P_2 are two topological properties satisfying axioms* A_1 *through* A_6, *then $P_1 \cap P_2$ satisfies, with the possible exception of* A_3, *these same* axioms.

2. Extensions of P-regular spaces. An extension of a space X is a space T which contains X as a dense subspace. An extension T of X is a *P-extension* of X if $T \in P$. In this section we shall show that there is a canonical P-extension of each P-regular space X. Moreover, this extension is closely related to the P-reflection of X whenever the latter exists.

2.1 THEOREM. *If X is a P-regular space, then there is a compactification $\beta_P X$ of X belonging to P for which any continuous map f from X to a compact P-space Y has an extension to a continuous map $\bar{f}$ from $\beta_P X$ to Y. Moreover, $\beta_P X$ is uniquely determined by this property.*

Proof. See [**7**, § 3, p. 234 and Satz 4.3, p. 237].

If all compact spaces have property P, then the space $\beta_P X$ in Theorem 2.1 is the Stone-Čech compactification of X. In any case $\beta_P X$ is the $P \cap K$-reflection of X [**8**] (here K denotes the compactness property).

A *perfect map* is a continuous closed map for which the pre-images of singletons are compact.

2.2 THEOREM. *If f is a perfect map from a P-regular space X to any space Y and T is a P-subspace of Y, then $f^{-1}(T)$ is a P-space.*

Proof. Let $S = f^{-1}(T)$. By 2.1, there is a continuous map g from $\beta_P S$ to $\beta_P T$ which extends the restriction of f to S. Since f is perfect, its restriction to S is perfect; whence it follows that $g^{-1}(T) = S$ (see [**5**, Lemma 1.5, p. 87]). By [**7**, Lemma 7.8, p. 243], $g^{-1}(T)$ is homeomorphic to a closed subspace of $\beta_P S \times T$. It now follows from axioms A_1 and A_4 that $S = f^{-1}(T)$ is a P-space.

Recall that a set is *regular closed* (*regular open*) if it is the closure of an open set (respectively, it is the interior of a closed set).

2.3 THEOREM. (a) *If X is a P-space and K is a compact subspace of X, then the quotient space of X obtained by identifying the points of K is a P-space.*

(b) *For P-regular spaces X, axiom* A_5 *remains valid when the point x_0 is replaced by a compact subspace K of X.*

(c) *If $X \cup K$ is a P-regular space for which $X \in P$ and K is compact, then $X \cup K \in P$.*

Proof. (a) Let Y be the quotient space, $f: X \to Y$ be the projection map, and y_0 be the image of K under f. If $\mathscr{B}$ is the set of all regular open sets in Y containing y_0, then $f^{-1}[Y \backslash B]$ is a regular closed subset of X; whence it belongs to P. Since f restricted to $X \backslash K$ is a homeomorphism, $Y \backslash B \in P$. Thus axiom A_5 insures that $Y \in P$.

(b) Let X be a P-regular space with a compact subspace K and base $\mathscr{B}$ for the neighborhoods of K such that $X \backslash B \in P$ for all $B \in \mathscr{B}$. Let Y be quotient space of X obtained by identifying the points of K. As in (a) it follows that $Y \in P$. Finally, since the quotient map is perfect, Theorem 2.2 implies that $X \in P$.

(c) This statement is an immediate corollary of (b).

Observe that up to this point we have not used axiom A_6. Thus we may use any of the previously derived results in proving that a particular topological property satisfies A_6. In the following lemma we shall give a condition which in the presence of the other axioms implies A_6.

2.4 LEMMA. *If P is a topological property which satisfies axioms A_1 through A_5 and is such that the union of two regular closed P-subspaces of a P-regular space belongs to P, then P satisfies axiom A_6.*

Proof. Let T be a compact P-space, Y_1 and Y_2 be closed G_δ-subspaces of T, and X any subspace of T such that $X \cup Y_i \in P$ for $i = 1, 2$. According to 2.3 (b) above, it is enough to show that there is a base $\mathscr{B}$ for the neighborhoods of $Y_1 \cap Y_2$ such that complement of B belongs to P for each $B \in \mathscr{B}$. Since Y_1, Y_2 are closed G_δ-sets in T, there are continuous functions f_i from T to $[0, 1]$ such that $Y_i = Z(f_i)(= f_i^{-1}(0)$ by definition) for $i = 1, 2$. Then $Z(f_1 + f_2) = Y_1 \cap Y_2$ and $\{x: (f_1 + f_2)(x) < 1/n\}$ is a neighborhood for $Y_1 \cap Y_2$. Now let $\mathscr{B}$ be the set of all regular open subsets B of $X \cup (Y_1 \cap Y_2)$ such that $B \supset Y_1 \cap Y_2$ and $X \backslash B \in P$. Now set $A_i = \mathrm{cl}_{X \cup Y_i} \{x: x \in X \cup Y_i, f_i(x) > 1/3n\}$. Then A_i is a regular closed subspace of the P-space $Y \cup Y_i$ which is contained in X. The hypotheses thus imply that $A_1 \cup A_2 \in P$. The complement of $A_1 \cup A_2$ is a neighborhood of $Y_1 \cap Y_2$ which is contained in $\{x: (f_1 + f_2)(x) < 1/n\}$. Thus $\mathscr{B}$ is a base with the required properties.

We shall define two canonical extensions of P-regular spaces. One of them will always be a P-space; the other is the largest space to which every continuous map to a P-space can be extended.

2.5 *Definition.* For a P-regular space X, define

$$\gamma X = \bigcap \{T: X \subset T \subset \beta_P X,\ T \in P\}$$

and define

$$\gamma_K X = \bigcap \{T: X \subset T \subset \beta_P X,\ T \in P,\ \mathrm{cl}_T(T \backslash X) \text{ is compact}\}.$$

Remark. Clearly, $X \subset \gamma X \subset \gamma_K X \subset \beta_P X$ and $X = \gamma X = \gamma_K X$ if $X \in P$. Also, if $\gamma X \in P$ (in particular, if P is productive), then γX is the P-reflection

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of X [**8**]. For this reason we shall call γX the *approximate P-reflection of X*. This terminology is further justified by the following theorem.

2.6 THEOREM. *Every continuous function from the P-regular space X to a P-space Y is continuously extendable to γX. Moreover, γX is the largest P-regular space with this mapping extension property. Specifically, if S is a P-regular space containing X as a dense subspace such that every continuous map from X to a P-space Y has an extension to a continuous map from S to Y, then there is a continuous homeomorphism h from S into γX leaving X pointwise fixed.*

Proof. If $f:X \rightarrow Y$ is continuous where Y is a P-space, then by 2.1, there is a continuous extension $\bar{f}: \beta_P X \rightarrow \beta_P Y$. By 2.2, $T = \bar{f}^{-1}(Y)$ is a P-space. Since $X \subset T \subset \beta_P X$, it follows that $T \supset \gamma X$. Thus the restriction of $\bar{f}$ to γX is the desired extension. Next, let S be as in the statement of the theorem. Let $h: S \rightarrow \beta_P X$ be the continuous extension of the inclusion of X in $\beta_P X$ and let $g: \beta_P X \rightarrow \beta_P S$ be the continuous extension of the inclusion of X in $\beta_P S$. Now $g \circ h$ is the identity since its restriction to X is the identity. Similarly, $h \circ g_1$ is the identity on $h(X)$ where g_1 is the restriction of g to $h(X)$. Thus h is a homeomorphism from S onto $h(S)$. Now if $T \in P$ and $X \subset T \subset \beta_P X$, then h may be thought of as a map from S into T; whence $h(S) \subset \gamma X$.

Remark. In general, γX fails to be a P-space. If P is the class of paracompact spaces, then γX is the Dieudonné completion of X which, in general, is not normal, let alone paracompact. To make up for this deficiency of γX, we have introduced the space $\gamma_K X$. Fortuitously, it is true that $\gamma_K X \in P$ as the next theorem shows.

2.7 THEOREM. *The space $\gamma_K X$ belongs to P for every P-regular space X.*

Proof. Let K be the intersection of all compact G_δ-sets H in $\beta_P X$ such that $X \cup H \in P$. First we show that $\gamma_K X = X \cup K$. It is immediate that $\gamma_K X \subset X \cup H$ for any of the above sets H. Thus $\gamma_K X \subset X \cup K$. Now let $p \in \beta_P X, p \notin \gamma_K X$. Then there exists $T \in P$ with $X \subset T \subset \beta_P X$ and $\mathrm{cl}_T(T \backslash X)$ compact such that $p \notin T$. By the complete regularity of $\beta_P X$, there is a compact G_δ-set $H \supset T \backslash X$ such that $p \notin X \cup H$. Since $X \cup H = T \cup H$, this space is in P by (c) of 2.3 above. Thus $p \notin X \cup K$; whence $\gamma_K X = X \cup K$. We shall now use 2.3 (b) to show that $\gamma_K X \in P$. Let G be a regular open neighborhood of K in $\beta_P X$. Then by the definition of K and the compactness of the sets H, there are compact G_δ-sets $H_1, H_2 \ldots H_n$ such that $X \cup H_i \in P$ for $i = 1, 2, \ldots, n$ and $\bigcap_{i=1}^n H_i \subset G$. By axiom A_6, $X \cup (\bigcap_{i=1}^n H_i) \in P$. Thus $X \backslash G \in P$ by A_1. Thus if $\mathscr{B}$ is taken to be the set of all regular open neighborhoods of K in $\gamma_K X$, then $\mathscr{B}$ has all the required properties for (b) of 2.3. Thus $\gamma_K X \in P$.

2.8 THEOREM. *If γX belongs to P, then $\gamma_K X = X \cup \mathrm{cl}_{\beta_P X}(\gamma X \backslash X)$. In particular, this will be the case when P is a productive property.*

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Proof. By 2.7, there is a compact set K such that $X \cup K = \gamma_K X$. Whence $\gamma X \backslash X \subset K$ and $X \cup \mathrm{cl}(\gamma X \backslash X) \subset \gamma_K X$. If $\gamma X \in P$, then by 2.3 (c), $X \cup \mathrm{cl}(\gamma X \backslash X)$ does also. Finally, 2.5 implies that $X \cup \mathrm{cl}(\gamma X \backslash X) \supset \gamma_K X$.

Remark. In the proof of 2.7, note that for finitely productive properties P, axiom A_6 is redundant. Thus Theorem 2.7 is valid for finitely productive properties that satisfy axioms A_1 through A_5.

Also, we observe that except for Theorem 2.2, axiom A_1 is used only to show that compact subspaces and regular closed subspaces of P-spaces belong to P. Thus if P is a compact hereditary and regular closed hereditary property for which 2.3 (b) and axiom A_2 through A_6 hold, then the conclusion of the theorems in this section, with the exception of 2.2, remain valid. In particular this is true for the class of pseudocompact spaces even though closed subspaces of pseudocompact spaces may fail to be pseudocompact (see [**4**, 8.20]).

3. Specific topological properties. In this section we give some examples of topological properties which satisfy the axioms A_1 through A_6. The list of topological properties given in Theorem 3.1 is not exhaustive but it is representative of the types of properties which satisfy our axioms.

3.1 THEOREM. *If P is the class of all completely regular Hausdorff spaces which are* (a) *compact,* (b) *Lindelöf,* (c) *paracompact,* (d) *countably compact,* (e) *countably paracompact,* (f) *realcompact (more generally, k-compact in the sense of Herrlich* [**6**]), (g) *topologically complete in the sense of Dieudonné* [**20**], (h) *N-compact* [**15**], (i) *m-bounded* [**22**], *or* (j) *zero-dimensional, then P satisfies axioms A_i, $1 \leqq i \leqq 6$. Moreover, the intersection of any number of the above classes satisfies these same axioms.*

Proof. Since this theorem represents little more than an amalgamation of known facts, we shall give proofs or references for only the less well known results. For the sake of brevity we shall use x_i to denote the statement that property (x) satisfies axiom A_i (e.g., j_2 denotes the statement: The class of compact zero-dimensional spaces is productive). Statements b_5 and c_5 are proved in [**12**]; d_5, e_5, h_5, and g_5 are in [**16**]; while f_5, g_5, f_6, and g_6 are proved in Lemmas 3.2, 3.4 and 3.5 below. The statements f_i, $1 \leqq i \leqq 4$ are either in Chapter 8 of [**4**] or in § 3 of [**6**]. The productive and closed hereditary axioms for topological completeness appear on p. 232 of [**7**] (see Remark following 3.4 below). For facts concerning m-bounded spaces see [**22**]. Finally, it is a simple matter to check that A_3 holds for the intersection of any of the above classes. Thus the final statement of the theorem follows immediately from Lemma 1.2.

Remark. There are a number of topological properties which fail to satisfy some of the axioms A_1 through A_6 but which none-the-less lend themselves to one point extensions for spaces which satisfy the property locally. Examples of such properties are normality [**12**], σ-compactness, H-closedness [**17**], and pseudo $\aleph$-compactness [**21**; **16**]. Note that according to the remark at the

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end of § 2, pseudo-$\aleph$-compactness can be brought into the purview of this study if axiom A_1 is weakened slightly.

3.2 LEMMA. *If X is a completely regular space and K is a compact subspace for which any closed subspace of X disjoint from K is k-compact (in the sense of Herrlich [**6**]), then X is k-compact.*

Proof. Let $\mathscr{U}$ be a free z-ultrafilter on X. Then there exists $Z_1 \in \mathscr{U}$ which is disjoint from K. Next, let F be a closed set disjoint from K such that $X \backslash F$ is completely separated from Z_1. It is easy to verify for any zero set W in F that $W \cap Z_1$ is a zero-set in X (cf. [**4**, 3C]). Define $\mathscr{V} = \{V: V \in Z(F), V \supset Z$ for some $Z \in \mathscr{U}\}$. Since $\mathscr{U}$ is free, $\mathscr{V}$ is a free z-filter on F. Now if $W \in Z(F)$ is not in $\mathscr{V}$, then $W \cap Z_1$ is a zero-set in X which is not in $\mathscr{U}$. Thus there is a $Z_2 \in \mathscr{U}$ for which $W \cap Z_1 \cap Z_2$ is empty. Since $Z_1 \cap Z_2 \in \mathscr{V}$, it follows that $\mathscr{V}$ is a z-ultrafilter on F. By hypothesis F is a k-compact space; hence there exists a family $\{V_\alpha\} \subset \mathscr{V}$ with cardinal $\leqq k$ such that $\bigcap V_\alpha$ is empty. Since $\{V_\alpha \cap Z_1\} \subset \mathscr{U}$, the latter z-ultrafilter fails to have the k-intersection property. Thus X is k-compact.

In order to show that topological completeness satisfies A_5, we will need the following characterization of completeness [**20**, p. 172].

3.3 (Tamano). *A completely regular space X is topologically complete if and only if for each $p \in \beta X \backslash X$, there is a locally finite partition of unity $\{f_\alpha\}$ on X for which the extensions f_α^β vanish at p.*

3.4 LEMMA. *If X is completely regular and K is a compact subspace for which any closed subspace disjoint from K is topologically completely, then X is topologically complete.*

Proof. We show that X satisfies the condition in 3.3. Let $p \in \beta X \backslash X$ and $h \in C(X)$, $h \geqq 0$, be such that $h(p) = 2$ while $h(K) = \{0\}$. Define $F = \{x: x \in X, h(x) \geqq 1\}$. By the hypothesis, F is topologically complete. Let $\phi: \beta F \rightarrow \mathrm{cl}_{\beta X} F$ be the continuous map that leaves F point-wise fixed. Then select $t \in \phi^{\leftarrow}(p)$. By 3.3 there is a locally finite partition of unity $\{h_\alpha\}$ for F such that h_α^β vanishes at t. Define $g = (h - 1)^+ \wedge 1$, and $f_\alpha(x) = g(x) h_\alpha(x)$ for $x \in F$ and $f_\alpha(x) = 0$ for x in $X \backslash F$. Clearly, f_α is continuous. Since $f_\alpha | F \leqq h_\alpha$, we have $f_\alpha^\beta \circ \phi \leqq h_\alpha^\beta$. Thus f_α^β vanishes at p. Finally, since $g(p) = 1$, it follows that $\{1 - g|X\} \cup \{f_\alpha\}$ is the desired partition of unity.

Remark. In the notation of Herrlich [**7**], Engelking and Mrowka [**3**], the topologically complete spaces are precisely the $\mathscr{E}$-compact spaces where $\mathscr{E}$ is the class of metric spaces. Thus by using the concept of P-embedding due to Arens [**1**] and Shapiro [**19**] (which is equivalent to Herrlich's $\mathscr{E}$-embedding), one can generalize 3.4 to an exact analogue of [**15**, 3.1].

3.5 LEMMA. *For a completely regular space X, let vX denote the Hewitt realcompactification and δX the Dieudonné completion of X. Then any realcompact* *cozero-subspace of X is a cozero-set in υX and any topologically complete cozero-subspace of X is a cozero-set in δX.*

Proof. Let V be a realcompact, cozero-subspace of X. Then $V \cup \mathrm{cl}_{\beta X}(X\backslash V)$ is realcompact (3.2 above, or [**4**, Chapter 8]). Thus $\upsilon X \subset V \cup \mathrm{cl}_{\beta X}(X\backslash V)$ since υX is the smallest realcompact subspace of βX which contains X. Whence $\upsilon X = V \cup \mathrm{cl}_{\upsilon X}(X\backslash V)$. By [**4**, 8.8(b)], $\mathrm{cl}_{\upsilon X}(X\backslash V)$ is a zero-set in υX. Thus $V = \upsilon X\backslash \mathrm{cl}_{\upsilon X}(X\backslash V)$ is a cozero-set in $_{\upsilon}X$. In view of 3.4, the fact $\delta X \subset \upsilon X$ implies that the above argument also applies to topologically complete cozero-sets.

3.6 THEOREM. *In any completely regular space X, a countable union Y of realcompact (topologically complete) cozero-subspaces is realcompact (topologically complete, respectively).*

Proof. Since countable unions of cozero-subspaces are cozero, 3.5 implies Y is a cozero-set in $\upsilon X(\delta X$, respectively). Since the topological properties considered are productive and enjoyed by every subspace of R, Y is an inverse image of a realcompact (topologically complete) space. The theorem now follows from [**7**, Lemma 7.8]. (Here, R stands for the real numbers.)

Remark. Theorem 3.6 should be contrasted with the example in [**15**] of a non-realcompact space X which is the union of two closed realcompact subspaces.

4. The lattice of one point extensions. In this section we shall characterize those P-regular spaces which admit a one point P-extension and then study the lattice of such extensions.

An extension T is a *one point extension* if $T\backslash X$ consists of at most one point. A *local P-space* is a space such that every point has a neighborhood which is a P-space with respect to the relative topology.

4.1 THEOREM. *For any P-regular space X, the following are equivalent:*

(a) *X is locally P.*

(b) *X is open in $\gamma_K X$.*

(c) *X has a one point P-extension.*

(d) *X is a dense subspace of a P-space T for which $T\backslash X$ is compact.*

(e) *X is an open subspace of some P-space.*

Moreover, if γX belongs to P, then each of the above statements is equivalent to X being open in γX.

Proof. If X is a P-space, then these condition are equivalent in a trivial way. Assume in the remainder of this proof that X is not a P-space.

(a) implies (b). For x in X, let Y be a P-space neighborhood of x. Since $\beta_p X$ is compact and Hausdorff, there is a compact set H in $\beta_P X$ such that $x \notin H$ while $X\backslash Y \subset H$. By 2.3 (c) and 2.5, $X \cup H = Y \cup H$ is a P-space that contains $\gamma_K X$. Thus $X\backslash H$ is an open neighborhood of x in the space $\gamma_K X$.

(b) implies (c). By 2.7, we have $\gamma_K X = X \cup K$ for some compact set K. The hypothesis (b) allows us to assume that K is disjoint from X. Let $^*X = X \cup \{w\}$ where w is a point not in X. Define $f: \gamma_K X \to {}^*X$ to be the identity on X and $f(K) = \{w\}$. When $^*X$ is assigned the quotient topology with respect to this map, it becomes a completely regular Hausdorff space. It is a simple matter to check that the set of all open neighborhoods of w in $^*X$ satisfies the requirements for $\mathscr{B}$ in axiom A_5. Thus $^*X$ is a P-space.

It is clear that (c) implies (d) and (d) implies (e). Finally it is easy to see, using regularity and axiom A_1, that (e) implies (a). With regard to the final statement, note that (b) implies X is open in γX while if X is open in γX and $\gamma X \in P$ then (e) holds.

In the remaining part of this section we shall assume that X is a P-regular, local P-space that is not in P and let $^*X$ denote the one point P-extension of X constructed in the proof of Theorem 4.1.

For a pair T_1, T_2 of one point P-extensions of X, define $T_1 \leqq T_2$ to mean there is a continuous map h from T_2 to T_1 which leaves X point-wise fixed. Note that this order is analogous to the usual order on the set of compactifications of a space [**10**]. We shall say that two one point P-extensions are identical if they are homeomorphic with respect a map that leaves X point-wise fixed. With this agreement, the relation defined above becomes a partial order on the set of all one point P-extensions of X.

An *anti-lattice isomorphism* between two partially ordered sets is an order reversing bijection ϕ such that

$$\phi(a \wedge b) = (\phi\, a) \vee (\phi\, b)$$

and

$$\phi(a \vee b) = (\phi\, a) \wedge (\phi\, b)$$

whenever $a \vee b$ and $a \wedge b$ exist.

4.2 THEOREM. *The partially ordered set of all one point P-extensions of X is anti-lattice isomorphic to the inclusion ordered collection of all compact subsets of $\beta_P X \backslash X$ which contain $\gamma_K X \backslash X$.*

Proof. For a point P-extension T of X, define $\phi T = f^{-1}(T \backslash X)$ where f is a continuous map from $\beta_P X$ to a P-space compactification αT of T which leaves X point-wise fixed. Since the restriction of f to $f^{-1}(T)$ is a perfect map, $f^{-1}(T \backslash X)$ is independent of the compactification αT (see [**5**, p. 87]). Thus ϕ is a well defined map. This map is injective since the topology on T is the quotient topology determined by the restriction of f to $f^{-1}(T)$. Since f is a perfect map and $T \backslash X$ is a single point, ϕT is a compact subset of $\beta_P X$ which is disjoint from X. It follows from Theorem 2.2 that $f^{-1}(T)$ is a P-space; whence $\gamma_K X \subset f^{-1}(T)$ and finally $\phi T \supset \gamma_K X \backslash X$. To see that ϕ is surjective let K be a compact set in $\beta_P X \backslash X$ which contains $\gamma_K X \backslash X$. Then $X \cup K = \gamma_K X \cup K$ is a P-space by Theorem 2.3 (c). Let $T = X \cup \{w\}$ be the quotient space of $X \cup K$ obtained by collapsing K to a single point and leaving X unchanged.

Then by 2.3 (a), T is a one point P-extension of X. Since the quotient map from $X \cup K$ to T is a perfect map it follows, as noted above, from [**5**] that $\phi T = K$. It remains to show that ϕ and ϕ^{-1} are order reversing maps. Let T_1 and T_2 be one point P-extensions of X with $T_1 \leqq T_2$ and let $h: \beta_P T_2 \to \beta_P T_1$ be the continuous map (the existence of which is insured by 2.1) which extends the continuous map from T_2 to T_1 leaving X pointwise fixed. Next, let $f_i: \beta_P X \to \beta_P T_i$ be the extension of the inclusion map of X in T_i for $i = 1, 2$. Then $f_1 = h \circ f_2$. Now $T_1 \leqq T_2$ implies that $h^{-1}(T_1 \backslash X) \supset T_2 \backslash X$. Thus

$$\phi T_2 = f_2^{-1}(T_2 \backslash X) \subset f_2^{-1}(h^{-1}(T_1 \backslash X)) = \phi T_1.$$

Conversely, if $\phi T_1 \supset \phi T_2$, then it is clear that $h: T_2 \to T_1$, defined by $h(x) = x$ for $x \in X$and $h(T_2 \backslash X) = T_1 \backslash X$, is continuous.

Remark. If $\gamma X \in P$, then any compact set which contains $\gamma X \backslash X$ also contains $\gamma_K X \backslash X$. Thus, in this case, $\gamma_K X$ may be replaced by γX in the statement of Theorem 4.2.

4.3 COROLLARY. *The partially ordered set of all one point P-extensions of a P-regular, local P-space X is a $\vee$-complete lattice for which $^*X$ is the maximal element. Moreover, this lattice has a smallest element (and thus is complete) if and only if X is locally compact; this smallest element (if it exists) is the one point compactification X^* of X.*

Proof. The first sentence follows immediately from 4.2. If T is the smallest one point P-extension, then ϕT must be a compact set in $\beta_P X \backslash X$ which contains every compact subset of $\beta_P X \backslash X$. This is possible only when $\beta_P X \backslash X$ is compact; that is when X is locally compact. Conversely, if X is locally compact, then $\beta_P X \backslash X$ is compact; whence by 2.3 (a), X^* is a one point P-extension.

4.4 *Remark.* The lattice of all one point P-extensions of X can be thought of as a family of topologies on a single set $X_w = X \cup \{w\}$. For one point P-extensions $T_i = \{X_w, \mathscr{T}_i\}$ where $\mathscr{T}_i$ is the topology for T_i, then $T_1 \leqq T_2$ if and only if $\mathscr{T}_1 \subset \mathscr{T}_2$.

We now study more closely the topology on the maximal P-extension $^*X$. For this purpose we define the concept of P-separated [**7**]. Two sets A, B in X are P-separated if there exists a P-space Y and a continuous map f from X to Y such that $f(A)$ and $f(B)$ have disjoint closures in Y.

4.5 THEOREM. *Let A be a subset of X. The following are equivalent*:

(a) *A is closed in $\gamma_K X$.*

(b) *A is closed in $^*X$.*

(c) *A is closed in X and is $P \cap K$-separated from $\gamma_K X \backslash X$ in $\gamma_K X$ (here, $P \cap K$ denotes the class of compact P-spaces).*

Proof. (a) and (b) are equivalent since the quotient map defining the topology on $^*X$ is a closed map.

(a) implies (c). By hypothesis, A and $\gamma_K X \backslash X$ have disjoint closures in the compact P-space $\beta_P X$. Thus the inclusion of $\gamma_K X$ into $\beta_P X$ is the map that shows A and B are $P \cap K$-separated.

(c) implies (a). Let Y be a compact P-space and $f: \gamma_K X \rightarrow Y$ be such that A and $f(\gamma_K X \backslash X)$ have disjoint closures. Then $f^{-1}\ \mathrm{cl}_Y f(A)$ is closed in $\gamma_K X$ and contained in X. Thus A is closed in $\gamma_K X$.

5. Examples. In this section we illustrate some of the possible relationships that exist among the one point compactification X^*, the maximal one point P-extension $*X$, and other one point P-extensions. In order to be specific, we shall let P be the class of Lindelöf spaces. Then $\beta_P X$ is the Stone-Čech compactification of X and γX is the Hewitt realcompactification vX.

5.1 *Example.* Let n and m be positive integers and let Y be a non-compact, Lindelöf space. As usual, W will denote the space of all ordinals less than the first uncountable ordinal ω_1. Define X to be the free union of n copies of W and m copies of Y. Now $\beta W = W^* = W \cup \{\omega_1\}$ (see [**4**, 5.13]); thus γX is the free union of n copies of W^* and m copies of Y, and $|\gamma X \backslash X| = n$; whence $\gamma_K X = \gamma X$. Obviously X is open in $\gamma_K X$, and so $*X$ exists; if $n > 1$ then $\gamma X \neq *X$. If Y is not locally compact then neither is X and so X^* will not exist (as a Hausdorff space). Thus the existence of $*X$ does not imply the existence of X^* (although the existence of X^*, for non-Lindelöf X, implies the existence of $*X$). As Y is not compact, neither is $*X$. Hence if Y is locally compact, then X^* exists but is distinct from $*X$. In this case, if $m \geqq 2$ there are one-point Lindelöf extensions of X that lie strictly between $*X$ and X^*; for, choose k copies of Y (where $1 \leqq k < m$), take the one-point compactification T of the free union of the n copies of W and k copies of Y, and then form the free union S of T and the remaining $m-k$ copies of Y. It is evident that S is a one-point Lindelöf extension of X distinct from both $*X$ and X^*. Variations on this general theme lead to examples of spaces X for which $|\gamma X \backslash X| = \aleph_0$, $*X$ exists, and X^* may or may not exist.

5.2 *Example.* We now construct a locally compact space X for which γX is Lindelöf but $\gamma X \neq \gamma_K X$. Let W be as in 5.1, and let N denote the positive integers with the discrete topology. Define an equivalence relation on $Y = W^* \times W^* \times N$ as follows: $(\alpha, \omega_1, n) \sim (\alpha, \omega_1, n+1)$ if $\alpha \in W^*$ and n is even, and $(\omega_1, \beta, n) \sim (\omega_1, \beta, n+1)$ if $\beta \in W^*$ and n is odd. All other points are equivalent only to themselves. Let S be the space of equivalence classes endowed with the quotient topology. The quotient map f from Y onto S is closed, and Y is a normal, T_1-space; hence S is a normal T_1-space and is thus a completely regular Hausdorff space. Note that in Y, $(\omega_1, \omega_1, m) \sim (\omega_1, \omega_1, n)$ for all $m, n \in N$; let s^* be the equivalence class containing these points, and set $T = S \backslash f[s^*]$. Then T is open in S, so $f|Y \backslash s^*$ is a perfect map from the locally compact space $Y \backslash s^*$ onto T (s^* is the only non-compact equivalence class). Hence T is locally compact and C-embedded in S; now S is σ-compact and hence Lindelöf; so $S = \gamma T = {}^*T$. Now put $X = T \times N$; then X is locally compact, and $\gamma X = S \times N$ is σ-compact, and hence Lindelöf. Evidently $\gamma X \backslash X$ is a countably infinite, closed, C-embedded subspace of γX, and so $\mathrm{cl}_{\beta X}\,(\gamma X \backslash X) = \gamma_K X \backslash X$ is homeomorphic to βN. Also, γX, $\gamma_K X$ and ${}^*X$ are all σ-compact but not locally compact.

References

1. R. Arens, *Extensions of coverings, of pseudometrics, and of linear-space-valued mappings*, Can. J. Math. *5* (1953), 211–215.
2. N. Bourbaki, *Topologie generale*, Chapitre IX (Herman, Paris, 1958).
3. R. Engelking and S. Mrowka, *On E-compact spaces*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. *6* (1958), 429–435.
4. L. Gillman and M. Jerison, *Rings of continuous functions* (Van Nostrand, Princeton, 1960).
5. M. Henriksen and J. Isbell, *Some properties of compactifications*, Duke Math. J. *25* (1958), 83–105.
6. H. Herrlich, *Forsetzbarheit stetiger Abbildungen und Kompaktheitsgrad topologischer Räume*, Math. Z. *96* (1967), 64–72.
7. ——— *$\mathscr{E}$-kompakte Räume*, Math. Z. *96* (1967), 228–255.
8. H. Herrlich and J. van der Slot, *Properties which are related to compactness*, Nederl. Akad. Wetensch. Proc. Ser. A *70* (1967), 524–529.
9. T. Isiwata, *On locally Q-complete spaces.* I, II, *and* III, Proc. Japan Acad. *35* (1959), 232–236, 263–267, 431–434.
10. K. D. Magill, *The lattice of compactifications of a locally compact space*, Proc. London Math. Soc. *18* (1968), 321–244.
11. F. Marin, *Convergence in E-completely regular spaces*, Doctorial Dissertation, Penn. State Univ., University Park, 1968.
12. S. Mrowka, *On local topological properties*, Bull. Acad. Polon. Sci. Cl. III *5* (1957), 951–956.
13. ——— *An outline of the theory of Q-spaces* (Warazawa, 1959) (Polish).
14. ——— *On the union of Q-spaces*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. *6* (1958), 161–163.
15. ——— *Some comments on the author's example of a non-R-compact space*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. *18* (1970), 443–448.
16. S. Mrowka and J. H. Tsai, *On local properties of topological spaces* (to appear).
17. J. Porter, *On locally H-closed spaces*, Proc. London Math Soc. *20* (1970), 193–204.
18. S. Salbany, *Completeness of $C_s(X)$*, Nederl. Akad. Wetensch. Proc. Ser. A*73* (= Indag. Math. *32*) (1970), 182–186.
19. H. Shapiro, *Extensions of pseudometrics*, Can. J. Math. *18* (1966), 991–998.
20. H. Tamano, *On compactifications*, J. Math. Kyoto Univ. *1* (1962), 162–193.
21. G. Vidossich, *Topological pseudo-$\aleph$-compact spaces*, Proc. Amer. Math. Soc. *27* (1971), 195–198.
22. R. Grant Woods, *Some $\aleph_0$-bounded subsets of Stone-Čech compactifications*, Israel J. Math. *9* (1971), 250–256.
23. P. Zenor, *A note on Z-mappings and WZ-mappings*, Proc. Amer. Math. Soc. *23* (1969), 273–275.

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