ELEMENTARY ALGEBRAIC TREATMENT OF THE QUANTUM MECHANICAL SYMMETRY PROBLEM

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1. Stating the problem

A function $\eta(i_1, \ldots, i_f)$ of f quantities i, varying over the finite range $i = 1,2, \ldots, n$, is usually called an n-dimensional tensor of rank f. Any permutation p: $1 \to 1', \ldots, f \to f'$ changes this tensor into a tensor $\mathbf{p}\eta$ according to the equation $\mathbf{p}\eta(i_1, \ldots, i_f) = \eta(i_{1'}, \ldots, i_{f'})$. Thus the permutation p appears as a linear operator $\mathbf{p}$ in the n-dimensional space $\Sigma = \Sigma_{n,f}$ of all n-dimensional tensors of rank f. η is symmetric if $\mathbf{p}\eta = \eta$ for all permutations p, it is antisymmetric if $\mathbf{p}\eta = \delta_p . \eta$ where $\delta_p = +1$ for the even and -1 for the odd permutations. Let a linear transformation A in Σ,

(1.1) $\quad \eta' = A\eta, \quad \eta'(i_1 \ldots i_f) = \sum_k a(i_1 \ldots i_f\, ; k_1 \ldots k_f) \cdot \eta(k_1 \ldots k_f),$

be called symmetric [1] if

$$a(i_{1'} \ldots i_{f'}\, ; k_{1'} \ldots k_{f'}) = a(i_1 \ldots i_f\, ; k_1 \ldots k_f)$$

for all permutations p. A is symmetric if and only if it commutes with all the permutation operators $\mathbf{p}$. The symmetric transformations A form an algebra $\mathfrak{A}$. The general symmetry problem posed by the quantum theory of an aggregate of f equal physical entities is this:

(I) *to decompose the tensor space Σ as far as possible into subspaces* Π *that are invariant with respect to all symmetric transformations A.*

An epistemological principle basic for all theoretical science, that of projecting the actual upon the background of the possible, is here followed by asking what happens under any *possible* Schrödinger law of dynamics $\frac{h}{i}\frac{d\eta}{dt} = A\eta$, before taking up the specific law involving the *actual* energy operator $A = H$. We have here ignored the further condition which physics imposes on all energy operators A, to wit their Hermitean nature,

$$a(k_1 \ldots k_f\, ; i_1 \ldots i_f) = \bar{a}(i_1 \ldots i_f\, ; k_1 \ldots k_f).$$

Essential for the theory of eigenvalues (terms) and eigenfunctions, this condition is irrelevant for our purposes. For what is invariant under all Hermitean symmetric transformations stays so even when the Hermitean restriction is lifted. As algebraists we are glad to get rid of it. For we propose to carry our investigation through in any number field in which the equation $f!a = 0$ for a number a implies $a = 0$ (field of characteristic 0 or of a prime characteristic dividing none of the natural numbers $1, 2, \ldots, f$).

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[1] We shall adhere to this terminology and not use the word symmetric in the sense presently to be mentioned under the name Hermitean.

It is no wonder that the complete solution of the above symmetry problem depends on the theory of representations of the symmetric group of all permutations and Young's symmetry operators.[2]

Let Σ^+, Σ^- denote the linear manifolds of all symmetric or antisymmetric tensors respectively. Nature has most wisely put a stop to the breaking-up of Σ into isolated compartments Π by letting but one of them, the invariant subspace Σ^-, come into existence. Such at least is the case if the f entities of which the aggregate is composed are electrons (Pauli's exclusion principle). Thereby the symmetry problem (I) loses its significance for physics. Part of it, however, is restored, if the existence of the spin of the electron is taken into account but its dynamical influence disregarded—a procedure which is at least approximately permissible. The situation is then as follows. The argument i is replaced by a pair $(i\rho)$ with the range $i = 1, \ldots, n$ for the "positional" variable i and the range $\rho = 1, \ldots, \nu$ for the "spin" variable ρ. (Actually $\nu = 2$ while the positional variable varies over the continuum of all possible positions in the physical three-dimensional space.) Set $N = n\nu$. The possible wave states of the aggregate of f electrons are described by the antisymmetric N-dimensional tensors $\psi(i_1\rho_1, \ldots, i_f\rho_f)$ of rank f, forming the space $\Sigma^-_{N,f} = \Omega$. Moreover we envisage the space $\Sigma = \Sigma_{n,f}$ of all n-dimensional tensors $\eta(i_1 \ldots i_f)$ of rank f, and the space $\mathrm{P} = \Sigma_{\nu,f}$ of all ν-dimensional tensors $\phi(\rho_1, \ldots, \rho_f)$ of rank f. Any symmetric transformation A in Σ,

$$\eta'(i_1 \ldots i_f) = \Sigma_k a(i_1 \ldots i_f\,;\, k_1 \ldots k_f) \cdot \eta(k_1 \ldots k_f)$$

induces a transformation A^* in Ω,

$$\psi'(i_1\rho_1, \ldots, i_f\rho_f) = \Sigma_k a(i_1 \ldots i_f\,;\, k_1 \ldots k_f) \cdot \psi(k_1\rho_1, \ldots, k_f\rho_f).$$

The central problem is

(II) *to decompose Ω as far as possible into subspaces that are invariant under the transformations A^* thus induced in Ω by all symmetric transformations A in Σ.*

These A^* form an algebra $\mathfrak{A}^*$. It is also true that any symmetric transformation B in P,

$$\phi'(\rho_1 \ldots \rho_f) = \Sigma_\sigma b(\rho_1 \ldots \rho_f\,;\, \sigma_1 \ldots \sigma_f) \cdot \phi(\sigma_1 \ldots \sigma_f), \tag{1.3}$$

induces a corresponding transformation B^* in Ω,

$$\psi'(i_1\rho_1, \ldots, i_f\rho_f) = \Sigma_\sigma b(\rho_1 \ldots \rho_f\,;\, \sigma_1 \ldots \sigma_f) \cdot \psi(i_1\sigma_1, \ldots, i_f\sigma_f). \tag{1.4}$$

The B^* form an algebra $\mathfrak{B}^*$. Every A^* of $\mathfrak{A}^*$ commutes with every B^* of $\mathfrak{B}^*$.

Not only the problem (I), but also this new symmetry problem (II) may be solved by means of Young's symmetry operators; cf. GQ, chap. v, § 12. However, as shall be discussed here in detail, a more elementary approach is available for the physically important case $\nu = 2$. Indeed the decomposition of the spin tensor space $\mathrm{P} = \Sigma_{2,f}$ into irreducible invariant subspaces under the algebra $\mathfrak{B}$ of all its symmetric transformations B is readily derived from

[2]Cf. H. Weyl, *Gruppentheorie und Quantenmechanik* (2nd ed. Leipzig, 1931) [quoted as GQ], chap. V, §§ 1-7 and 13-14.

the classical Clebsch-Gordan expansion. From the algebra $\mathfrak{B}$ in P we may pass to its representation $\mathfrak{B}^*$ in Ω. Because of the commutability of the elements A^* and B^* of $\mathfrak{A}^*$ and $\mathfrak{B}^*$, decomposition of the generic matrix of $\mathfrak{B}^*$ entails a "dual" decomposition for $\mathfrak{A}^*$. The deeper lying fact that vice versa any linear transformation in Ω that commutes with all $B^* \epsilon \mathfrak{B}^*$ lies in $\mathfrak{A}^*$ is needed in order to show that the latter decomposition is also one into irreducible parts.

All linear transformations (matrices) in a g-dimensional vector space Ξ form an algebra $\mathfrak{M}_g$ of order g^2, the complete matric algebra of degree g. Throughout our investigation irreducibility for matric algebras will be sharpened to completeness. Decomposition of a matrix C into two matrices $C_1|C_2$ is defined by the equation

$$C = \begin{Vmatrix} C_1 & 0 \\ 0 & C_2 \end{Vmatrix}$$

$1\circ C$, $2\circ C$, $3\circ C$, ... are the abbreviations for C, $C|C$, $C|C|C$, ..., and $\mathbf{S}$ is the summation sign for the addition $|$ of matrices. Let $\mathfrak{C}$ be a matric algebra of order m in a g-dimensional vector space Ξ. Suppose that, relative to a suitably chosen coordinate system for Ξ, the generic matrix C of $\mathfrak{C}$ decomposes into $m_1\circ C_1|m_2\circ C_2|$ ..., the matrix C_r of degree g_r occurring with the multiplicity $m_r > 0$, $g = m_1g_1 + m_2g_2 + \ldots$. The $g_1^2 + g_2^2 + \ldots$ coefficients of the matrices $C_1, C_2, \ldots, C_h$ are linear forms of the m parameters of $\mathfrak{C}$. We speak of *complete decomposition* if these coefficients are all linearly independent and thus $m = g_1^2 + g_2^2 + \ldots$. For $\nu = 2$ we shall prove the following

MAIN THEOREM. *Relative to a suitably chosen coordinate system for the space* Ω, *the generic matrix* A^* *of* $\mathfrak{A}^*$ *suffers complete decomposition*

$$A^* = \mathbf{S}\,(v+1)\circ A^*_u\,; \tag{1.5}$$

u and v are two non-negative integers related by the equation $2u + v = f$. The part A^*_u of "valence defect" u and the corresponding "valence" v occurs with the multiplicity $v + 1$. Set $d = n - f$, $\bar{u} = d + u$. The degree g^*_u of the matrix A^*_u is given by the formula

$$g^*_u = \binom{n}{u}\binom{n}{\bar{u}}\frac{(n+1)(n+1-u-\bar{u})}{(n+1-u)(n+1-\bar{u})}, \tag{1.6}$$

$\binom{n}{u}$ *denoting the binomial coefficient* $\dfrac{n!}{u!(n-u)!}$. *Only those u occur in the sum* (1.5) *for which* $u \geq 0$, $\bar{u} \geq 0$, $v = n - (u + \bar{u}) \geq 0$.

Spectroscopically this theorem establishes the existence of non-intercombining term systems corresponding to the various valences v. The terms of valence v are of multiplicity $v + 1$. Only when the actually existing weak interactions between the spins are taken into account, each term of valence v splits into a "multiplet" of $v + 1$ slightly different terms; whereas the weak interaction between position and spin accounts for weak intercombinations between the several term systems. The significance of the valence v for chemistry is sufficiently indicated by its name.

After some preliminaries in **2** the decomposition (1.5) is derived from the Clebsch-Gordan expansion in **3.** Its completeness will be proved in **4** and **5.**

2. Auxiliary propositions

Schur's lemma for complete instead of irreducible matric algebras is a triviality; nevertheless it may be stated as our

LEMMA 1. Complete decomposition of the generic matrix C of a matric algebra $\mathfrak{C}$, $C = m_1{\circ}C_1 | m_2{\circ}C_2 | \ldots | m_h{\circ}C_h$, implies the same for its commutator algebra $\mathfrak{D}$, $D = g_1{\circ}D_1 | g_2{\circ}D_2 | \ldots$. But degree and multiplicity are interchanged: the degree g_r of C_r is the multiplicity with which D_r occurs in the generic matrix D of $\mathfrak{D}$, and the multiplicity m_r of C_r is the degree of D_r.

As one knows, the commutator algebra of a given matric algebra $\mathfrak{C}$ consists of those matrices D that commute with all elements C of $\mathfrak{C}$. As an abstract algebra $\mathfrak{c}$ the completely decomposed matric algebra $\mathfrak{C}$ of Lemma **1** is the direct sum of a number of complete matric algebras; indeed $\mathfrak{c}$ consists of all h-uples $(C_1, \ldots, C_h)$ of arbitrary matrices $C_1, \ldots, C_h$ of the respective degrees $g_1, \ldots, g_h$. We need the following classical proposition, for the simple proof of which I refer the reader to GQ, p. 271, Satz (6.1).

LEMMA 2. Every representation of the direct sum $\mathfrak{c}$ of h complete matric algebras is of the form

$$(C_1, \ldots, C_h) \rightarrow m^*_1{\circ}C_1 | \ldots | m^*_h{\circ}C_h.$$

(Here some of the multiplicities m^*_r may be zero; this will happen if the representation is not faithful and hence the representing matric algebra $\mathfrak{C}^*$ is of lower order than $\mathfrak{C}$.)

Any *antisymmetric* n-dimensional tensor η of rank f is completely characterized by its components $\eta(\iota_1 \ldots \iota_f)$ with $\iota_1 < \iota_2 < \ldots < \iota_f$, and these are independent. We have

$$\eta(i_1 \ldots i_f) = \delta_i \cdot \eta(\iota_1 \ldots \iota_f)$$

for any permutation $i_1 \ldots i_f$ of $\iota_1 \ldots \iota_f$, $\delta_i = \pm 1$ distinguishing the even from the odd permutations $\begin{pmatrix} \iota_1 \ldots \iota_f \\ i_1 \ldots i_f \end{pmatrix}$, and $\eta(i_1 \ldots i_f) = 0$ if the numbers $i_1 \ldots i_f$ are not all distinct. Hence $\Sigma^- = \Sigma^-_{n,f}$ does not exist unless $n \geq f$, and its dimensionality is

$$M^-_n(f) = \frac{n!}{f!(n-f)!}.$$

LEMMA 3. Any linear transformation in Σ^- may be written in the form (1.1) where $a(i_1 \ldots i_f\,; k_1 \ldots k_f)$ is antisymmetric in the f arguments i, antisymmetric in the f arguments k [and hence symmetric in the f pairs (ik)].

Indeed a linear transformation in Σ^-,

$$\eta'(\iota_1 \ldots \iota_f) = \textstyle\sum_\kappa a(\iota_1 \ldots \iota_f\,; \kappa_1 \ldots \kappa_f) \cdot \eta(\kappa_1 \ldots \kappa_f)$$

(with the sum extending over the possible sequences $\kappa_1 < \ldots < \kappa_f$ chosen from the range $1, 2, \ldots, n$) may be written as (1.1) when one puts

$$a(i_1 \ldots i_f\,;\, k_1 \ldots k_f) = \frac{\delta_i \delta_k}{f!} \cdot a(\iota_1 \ldots \iota_f\,;\, \kappa_1 \ldots \kappa_f)$$

for any permutation $i_1 \ldots i_f$ of $\iota_1 \ldots \iota_f$ and any permutation $k_1 \ldots k_f$ of $\kappa_1 \ldots \kappa_f$, and puts $a(i_1 \ldots i_f\,;\, k_1 \ldots k_f) = 0$ in case the numbers $i_1 \ldots i_f$ or $k_1 \ldots k_f$ are not all distinct.

It follows from this lemma that the algebra $\mathfrak{A}$ of symmetric transformations is a complete matric algebra *in the invariant subspace* Σ^- of Σ.

Any *symmetric* tensor η may be completely characterized by its components $\eta(i_1 i_2 \ldots i_f)$ with $i_1 \leq i_2 \leq \ldots \leq i_f$, and these are independent. On changing the labels $i_1 \ldots i_f$ into $i_1 + 0,\ i_2 + 1,\ i_3 + 2, \ldots,\ i_f + (f-1)$ one sees at once that the dimensionality of the space $\Sigma^+ = \Sigma^+_{n,f}$ of symmetric tensors equals

$$M^+_n(f) = \frac{(n + f - 1)!}{f!\ (n - 1)!}.$$

Set $\eta(i_1 \ldots i_f) = \eta_{f_1 \ldots f_n}$ if f_1 of the f arguments $i_1 \ldots i_f$ equal 1, f_2 of them equal 2, $\ldots$, f_n of them equal n. These numbers $\eta_{f_1 \ldots f_n}$ corresponding to the various partitions $f_1 + f_2 + \ldots f_n$ of f can also be used as the independent components of η. A typical symmetric tensor arises from a vector $(x_1, \ldots, x_n)$ by the formula

(2.1) $$\eta(i_1 \ldots i_f) = x_{i_1} \ldots x_{i_f} \quad \text{or} \quad \eta_{f_1 \ldots f_n} = x_1^{f_1} \cdots x_n^{f_n}.$$

A linear form $l(\eta)$ depending on a variable symmetric tensor η is to be written as

$$l(\eta) = \Sigma l_{f_1 \ldots f_n} \cdot \eta_{f_1 \ldots f_n}$$

with a constant coefficient $l_{f_1 \ldots f_n}$ for each partition $f_1 + \ldots + f_n$ of f. We make the altogether trivial remark that $l(\eta)$ vanishes identically in η provided it vanishes identically in x by dint of the substitution (2.1).

The symmetric transformation $B = \|b(\rho_1 \ldots \rho_f\,;\, \sigma_1 \ldots \sigma_f)\|$ of the algebra $\mathfrak{B}$ may be looked upon as a symmetric ν^2-dimensional tensor $b(\omega_1, \ldots, \omega_f)$ of rank f, if each pair $(\rho\sigma)$ is taken as a single argument ω capable of ν^2 values. Hence the order of the matric algebra $\mathfrak{B}$ in P is $M^+_{\nu^2}(f)$ [and the order of $\mathfrak{A}$ is $M^+_{n^2}(f)$]. The linear transformation $t = \| t_{\rho\sigma} \|$ in the ν-dimensional vector space induces the symmetric transformation $B(t)$,

(2.2) $$b(\rho_1 \ldots \rho_f\,;\, \sigma_1 \ldots \sigma_f) = t_{\rho_1\sigma_1} \cdots t_{\rho_f\sigma_f}$$

in the tensor space P. Considering the ν^2 coefficients $t_{\rho\sigma}$ as indeterminates, we speak of t as the generic element of the linear group ζ and of $t \to B(t)$ as the representation ζ^f of ζ. Equation (2.2) is in complete analogy to (2.1), and the "altogether trivial remark" made above amounts to the following

LEMMA 4. A linear form $l(B)$ depending on an arbitrary element B of $\mathfrak{B}$ vanishes identically if it vanishes identically in the parameters $t_{\rho\sigma}$ for $B = B(t)$.

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As a final lemma we write down a simple formula for the case $\nu = 2$, $\nu^2 = 4$:
LEMMA 5.

$$M^+_4(f) = \frac{(f+1)(f+2)(f+3)}{1 \cdot 2 \cdot 3} = \Sigma(v+1)^2 \tag{2.3}$$

where the sum extends over the non-negative members v of the sequence $f, f-2, f-4, \ldots$.

Proof. Verify (2.3) for $f = 0, 1$ and the relation

$$M^+_4(f) - M^+_4(f-2) = (f+1)^2$$

for all $f \geq 2$.

3. The Clebsch-Gordan expansion and the decomposition of $\mathfrak{A}^*$

In this section we assume $\nu = 2$.

The symmetric 2-dimensional tensors $\phi(\rho_1 \ldots \rho_v)$ $(\rho = 1,2)$ of rank v $(\leq f)$ form a linear manifold $\mathrm{P}^+_v = \Sigma^+_{2,v}$ of $v+1$ dimensions. In agreement with a usage established above denote by ϕ_h the component $\phi(\rho_1 \ldots \rho_v)$ in which h of the v arguments ρ have the value 1 and $h - v$ have the value 2 $(h = 0,1, \ldots, v)$. The indeterminate transformation $t = \| t_{\rho\sigma} \|$ $(\rho,\sigma = 1,2)$ in the 2-dimensional vector space induces the transformation

$$\phi'(\rho_1 \ldots \rho_v) = \Sigma_\sigma t_{\rho_1\sigma_1} \ldots t_{\rho_v\sigma_v} \cdot \phi(\sigma_1 \ldots \sigma_v)$$

in P^+_v, and thus P^+_v appears as the representation space of a definite representation Z_v of ζ of degree $v+1$. By multiplying the transformed components ϕ'_h by a fixed power Δ^u $(u = 0, 1, 2, \ldots)$ of the determinant $\Delta = t_{11}\,t_{22} - t_{12}\,t_{21}$ one obtains a representation $\Delta^u \mathrm{Z}_v$ of ζ of the same degree $v+1$. Envisage the subgroup ζ_0 of ζ, the generic element of which is the substitution

$$\begin{Vmatrix} t_{11}, & t_{12} \\ t_{21}, & t_{22} \end{Vmatrix} = \begin{Vmatrix} \lambda, & 0 \\ 0, & 1 \end{Vmatrix} \tag{3.1}$$

with one indeterminate parameter λ. That substitution multiplies ϕ_h by λ^h according to the representation Z_v, by λ^{u+h} according to the representation $\Delta^u \mathrm{Z}_v$. Hence the coordinates in the representation space Π of $\Delta^u \mathrm{Z}_v$ are so chosen that they are distinguished by a signature ("magnetic quantum number") $w = u + h$. This signature is the exponent of the factor λ^w taken on by the coordinate with the label w under the influence of (3.1) and ranges over the values $w = u, u+1, \ldots, u+v$. [Decomposition of Π into one-dimensional parts invariant with respect to the subgroup ζ_0 of ζ.]

The 2-dimensional tensors $\phi(\rho_1 \ldots \rho_a, \rho_{a+1} \ldots \rho_{a+b})$ of rank $a+b$ $(\leq f)$ which are symmetric in the first a and symmetric in the last b arguments form the substratum of the representation $\mathrm{Z}_a \times \mathrm{Z}_b$ of ζ of degree $(a+1)(b+1)$. The latter breaks up into parts in accordance with the Clebsch-Gordan formula

$$\mathrm{Z}_a \times \mathrm{Z}_b = \mathfrak{S}\Delta^u \mathrm{Z}_v, \tag{3.2}$$

the sum extending over all non-negative integers u, v for which $2u + v = a + b$ and $u \leq \min(a, b)$. This follows by induction from the equation

$$\mathrm{Z}_a \times \mathrm{Z}_b = \mathrm{Z}_{a+b} \,|\, \Delta(\mathrm{Z}_{a-1} \times \mathrm{Z}_{b-1}).$$

A simple proof is to be found, for instance, on pp. 115-117 of GQ.

Repeated application of (3.2) leads to a formula of this type:

$$Z_1 \times Z_1 \times \ldots \times Z_1 \ (f \text{ factors}) = \mathsf{S} g_u \circ \Delta^u Z_v \qquad (2u + v = f).$$

$Z_1 \times \ldots \times Z_1$ is nothing but the representation ζ^f, $t \to B(t)$, of ζ in P, and our formula states that the matrix $B(t)$ breaks up in the manner described by

$$B(t) = \mathsf{S} g_u \circ B_u(t) \tag{3.3}$$

into partial matrices $B_u(t)$ of degree $v + 1$. Here u, v range over all non-negative integers satisfying the equation $2u + v = f$, and each component $B_u(t)$ occurs with a certain multiplicity $g_u \geq 0$.

If we now make use of Lemma 4, which also states that two linear forms $l(B)$ are identical if they become identical by the substitution $B = B(t)$, we see at once that the generic matrix B of $\mathfrak{B}$ itself breaks up in the same fashion

$$B = \mathsf{S} g_u \circ B_u. \tag{3.4}$$

Lemma 5 then shows that none of the valences $v = f, f - 2, f - 4, \ldots$ is left out, $g_u > 0$ for $0 \leq u \leq \frac{1}{2}f$, and that all the coefficients of the various matrices B_u are independent linear forms of the $M^+{}_4(f)$ parameters $b_{f_1 f_2 f_3 f_4}$ of B. Hence (3.4) is a *complete decomposition.*

$\mathfrak{B}^*$ is a representation of $\mathfrak{B}$, and thus Lemma 2 leads to a similar formula

$$B^* = \mathsf{S} g^*{}_u \circ B_u \qquad (g^*{}_u \geq 0) \tag{3.5}$$

for the generic matrix B^* of $\mathfrak{B}^*$.

It is not difficult to determine the multiplicities $g^*{}_u$ explicitly. Specialize the element t of ζ by (3.1) in $B = B(t)$ and the corresponding $B^*(t)$. The effect of this specialized $B^*(t)$ upon a tensor component $\psi(i_1\rho_1, \ldots, i_f\rho_f)$ is multiplication by λ^w if w of the f indices $\rho_1, \ldots, \rho_f$ are 1 (and $f - w$ of them equal 2). A complete set of independent components of ψ of that type is obtained by choosing

$$\left.\begin{array}{l} \rho_1 = \ldots = \rho_w = 1 \\ i_1 < \ldots < i_w \end{array}\right\} \text{ and } \left\{\begin{array}{l} \rho_{w+1} = \ldots = \rho_f = 2 \\ i_{w+1} < \ldots < i_f \end{array}\right.$$

Hence their number N_w equals

$$N_w = \binom{n}{w} \cdot \binom{n}{f - w} = \binom{n}{w} \cdot \binom{n}{\overline{w}} \tag{3.6}$$

where $d = n - f$ and $\overline{w} = w + d$. According to (3.5) the space Ω breaks up into subspaces $\Pi^*{}_u$ of dimensionality $v + 1$ in each of which $B^*(t)$ induces the transformation $B_u(t)$. Every one of these $g^*{}_u$ subspaces $\Pi^*{}_u$, therefore, contributes exactly *one* coordinate of signature w to Ω provided $u \leq w \leq u + v = f - u$. This simple argument yields the recursive formula

$$N_w = \Sigma g^*{}_u$$

where u ranges over all integers satisfying the inequalities $u \geq 0$ and $u \leq w$, $u \leq f - w$. Consequently

$$g^*{}_u = N_u - N_{u-1} \qquad (0 \leq u \leq \tfrac{1}{2}f). \tag{3.7}$$

Put $\overline{u} = d + u$ so that $v = n - u - \overline{u}$. Now (1.6) readily follows from (3.6) and (3.7), and one sees from this explicit expression that $g^*{}_u$ is positive provided $u \geq 0$, $\bar{u} \geq 0$ and $u + \bar{u} \leq n$. The range of the valences v actually occurring in the decomposition of $\mathfrak{B}^*$ is thus circumscribed by the relations

$$v \geq 0, \quad v \leq n \pm d, \quad v \equiv n \pm d \,(\text{mod } 2).^3$$

$\mathfrak{B}^*$ serves merely as a jumping board for $\mathfrak{A}^*$. But since every A^* commutes with all the transformations B^* of $\mathfrak{B}^*$ the decomposition (1.5) of the generic matrix A^* of $\mathfrak{A}^*$ is now inferred from Lemma 1. A definite decomposition according to valences is thus obtained, and for physics this is the most essential result. However, as long as we have not yet convinced ourselves that $\mathfrak{A}^*$ is not only contained in, but identical with, the commutator algebra of $\mathfrak{B}^*$, completeness for the decomposition (1.5) is not ensured. In order to settle this point (**5**) one first has to prove that the only operators in P that commute with the symmetric transformations B are the symmetry operators (**4**).

4. Symmetric transformations and permutations

Our present object is the space $\Sigma = \Sigma_{n,f}$ of the n-dimensional tensors $\eta(i_1 \ldots i_f)$ of rank f. The permutations $\mathbf{p}$ and any linear combinations of them, $\mathbf{a} = \sum_p a(p)\mathbf{p}$, are linear operators in Σ, $\eta' = \mathbf{a}\eta$, which commute with all the symmetric linear transformations $\dot{\eta} = A\eta$,

$$\dot{\eta}(i_1 \ldots i_f) = \textstyle\sum_k a(i_1 \ldots i_f\,; k_1 \ldots k_f)\cdot\eta(k_1 \ldots k_f).$$

We introduce the symmetry quantities $\underset{.}{a} = \sum_p a(p)p$ (with arbitrary numbers $a(p)$ as coefficients)[4] quite independently from their usage as operators in Σ. They form an abstract algebra of order $f!$, the "group ring of the symmetric group."

Let η be a tensor and $i_1, \ldots, i_f$ a given sequence of integers from the interval $1 \leq i \leq n$. We consider the $f!$ numbers $\mathbf{p}\eta(i_1 \ldots i_f) = x(p)$ as the coefficients of a symmetry quantity $\underset{.}{x} = \sim\eta(i_1 \ldots i_f)$. The tensor equation $\eta' = \mathbf{a}\eta$ is equivalent with $\sim\eta' = (\sim\eta)\cdot\hat{a}$ where $\hat{a}$ is the symmetry quantity with the coefficients $\hat{a}(p) = a(p^{-1})$. Here $\sim\eta$ may be interpreted as the symmetry quantity with the tensorial coefficients $\sim\eta(p) = \mathbf{p}\eta$, or one may replace $\sim\eta$ and $\sim\eta'$ in our equation by the ordinary symmetry quantities $\sim\eta(i_1 \ldots i_f)$ and $\sim\eta'(i_1 \ldots i_f)$ corresponding to any argument combination $i_1, \ldots, i_f$.

The group ring is an $f!$-dimensional vector space. In it we envisage those symmetry quantities $\sim\eta(i_1 \ldots i_f)$ that arise from arbitrary tensors η and arbitrary argument combinations $(i_1, \ldots, i_f)$, and we determine their linear closure $\kappa = \kappa_n$, i.e. the smallest linear subspace that comprises them all.

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[3]In passing we notice that the order of the algebra $\mathfrak{B}^*$ may now be evaluated as $\Sigma(v + 1)^2$, the sum extending over the non-negative v of the sequence f', $f' - 2, \ldots$ where $f' = \min(n - d,\ n + d) = \min(f, 2n - f)$, and hence equals $(f' + 1)(f' + 2)(f' + 3)/1\cdot2\cdot3$. It should be easily possible to confirm this directly.

[4]The dot under a letter merely serves to indicate that it stands for a symmetry quantity.

Let γ_s ($s = 1, 2, \ldots, n^f$) be a basis for the space Σ. Then the elements $\underset{\cdot}{x}$ of κ are given by the equation

$$\underset{\cdot}{x} = \sum_{s;i} \xi_s(i_1 \ldots i_f) \cdot \sim\gamma_s(i_1 \ldots i_f) \tag{4.1}$$

where the $\xi_s(i_1 \ldots i_f)$ are arbitrary coefficients. Write more explicitly

$$x(p) = \sum_{s;i} \xi_s(i_1 \ldots i_f) \cdot \mathbf{p}\gamma_s(i_1 \ldots i_f) = \sum_{s;i} \mathbf{p}^{-1}\xi_s(i_1 \ldots i_f) \cdot \gamma_s(i_1 \ldots i_f),$$

hence

$$\hat{\underset{\cdot}{x}} = \sum_{s;i} \gamma_s(i_1 \ldots i_f) \cdot \sim\xi_s(i_1 \ldots i_f). \tag{4.2}$$

Since $\gamma'_s = \mathbf{a}\gamma_s$ implies $\sim\gamma'_s = (\sim\gamma_s)\cdot\hat{a}$ one sees that $\underset{\cdot}{x}\hat{\underset{\cdot}{a}}$ lies in κ if $\underset{\cdot}{x}$ does; κ is therefore not only an algebra, but even a right-ideal. But in (4.2) one may consider ξ_s as a tensor and the $\gamma_s(i_1 \ldots i_f)$ as coefficients ; consequently $\hat{\underset{\cdot}{x}}$ lies in κ if $\underset{\cdot}{x}$ does, and thus κ is also a left-ideal. Introduce $\xi'_s = \mathbf{a}\xi_s$; then (4.2) yields

$$\hat{\underset{\cdot}{x}}\cdot\hat{\underset{\cdot}{a}} = \sum_{s;i} \gamma_s(i_1 \ldots i_f) \cdot \sim\xi'_s(i_1 \ldots i_f),$$

$$\underset{\cdot}{a}\cdot\underset{\cdot}{x} = \sum_{s;i} \xi'_s(i_1 \ldots i_f) \cdot \sim\gamma_s(i_1 \ldots i_f). \tag{4.3}$$

As a left-ideal κ has a generating idempotent $\underset{\cdot}{e}$. This means that $\underset{\cdot}{z}\underset{\cdot}{e}$ is in κ whatever the symmetry quantity $\underset{\cdot}{z}$, and if $\underset{\cdot}{z}$ lies in κ then $\underset{\cdot}{z} = \underset{\cdot}{z}\underset{\cdot}{e}$. Similar statements hold for multiplication by $\hat{e}$ on the left. The ensuing equations $\hat{e} = \hat{e}\cdot\underset{\cdot}{e}$ and $\underset{\cdot}{e} = \hat{e}\cdot\underset{\cdot}{e}$ show that $\underset{\cdot}{e} = \hat{e}$. Every tensor η satisfies the equation $\mathbf{e}\eta = \eta$.

One more fact about κ is of importance. Introduce as the trace $\mathrm{tr}(\underset{\cdot}{a})$ of a symmetry quantity $\underset{\cdot}{a}$ the coefficient $a(\mathbf{1})$ corresponding to the identical permutation $\mathbf{1}$. The scalar product $\mathrm{tr}(\underset{\cdot}{a}\underset{\cdot}{b}) = \sum_p a(p^{-1})\cdot b(p)$ is clearly a symmetric and non-degenerate bilinear form of the two arbitrary symmetry quantities $\underset{\cdot}{a}$ and $\underset{\cdot}{b}$. This non-degeneracy is preserved under restriction to κ; i.e. an $\underset{\cdot}{a} \epsilon \kappa$ such that $\mathrm{tr}(\underset{\cdot}{a}\underset{\cdot}{b}) = 0$ for every $\underset{\cdot}{b} \epsilon \kappa$ is necessarily zero. Indeed let $\underset{\cdot}{z}$ be an arbitrary symmetry quantity; then $\underset{\cdot}{b} = \underset{\cdot}{z}\underset{\cdot}{e}$ is in κ, hence $\mathrm{tr}(\underset{\cdot}{a}\underset{\cdot}{z}\cdot\underset{\cdot}{e}) = 0$. But with $\underset{\cdot}{a}$ also $\underset{\cdot}{a}\underset{\cdot}{z}$ lies in κ, therefore $\underset{\cdot}{a}\underset{\cdot}{z}\cdot\underset{\cdot}{e} = \underset{\cdot}{a}\underset{\cdot}{z}$. Thus our equation turns into $\mathrm{tr}(\underset{\cdot}{a}\underset{\cdot}{z}) = 0$ for every $\underset{\cdot}{z}$, and that implies $\underset{\cdot}{a} = 0$.

THEOREM I. *The symmetry quantities* $\mathbf{a}$ *if interpreted as operators in* Σ *are the only ones that commute with all symmetric transformations* A. *The symmetry quantity* $\underset{\cdot}{a}$ *expressing such an operator can be uniquely normalized by requiring* $\underset{\cdot}{a}$ *to lie in* κ.

Proof (cf. GQ, pp. 266-267).[5] Let L be a linear operator in Σ, $\eta \to L\eta$,

[5] By using deeper algebraic resources than we care to employ in this elementary approach, Theorem I could be obtained as an immediate consequence of the following two facts: (α) Every representation $\underset{\cdot}{a} \to \mathbf{a}$ of the group ring of the symmetric group breaks up into irreducible parts (is "fully reducible"); (β) A fully reducible matric algebra coincides with the commutator algebra of its commutator algebra (R. Brauer).—Another variant: Explicit construction by means of Young's symmetry operators shows that the inequivalent irreducible parts of the representation $a \to \mathbf{a}$ are *absolutely* irreducible and inequivalent, and consequently (α) yields a *complete* decomposition. With this additional knowledge (β) can be replaced by the trivial fact that complete decomposition of a matric algebra implies its identity with the commutator algebra of its commutator algebra.

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commuting with all symmetric A. Let $L\gamma_s = \beta_s$, and with the same coefficients $\xi_s(i_1 \dots i_f)$ as in (4.1) form

$$\underset{\cdot}{y} = \sum_{s;i} \xi_s(i_1 \dots i_f) \cdot \sim\beta_s(i_1 \dots i_f).$$

I am going to show that the equation $\underset{\cdot}{x} = 0$ for the arbitrary coefficients $\xi_s(i_1 \dots i_f)$ implies $\underset{\cdot}{y} = 0$. Let η be any tensor and set

$$\theta = \Sigma_p x(p^{-1}) \cdot \mathbf{p}\eta, \qquad \tilde{\theta} = \Sigma_p y(p^{-1}) \cdot \mathbf{p}\eta.$$

Then

$$\theta(i_1 \dots i_f) = \Sigma_s \Sigma_k a_s(i_1 \dots i_f\,; k_1 \dots k_f) \cdot \gamma_s(k_1 \dots k_f),$$

$$\tilde{\theta}(i_1 \dots i_f) = \Sigma_s \Sigma_k a_s(i_1 \dots i_f\,; k_1 \dots k_f) \cdot \beta_s(k_1 \dots k_f)$$

where

$$a_s(i_1 \dots i_f\,; k_1 \dots k_f) = \Sigma_p \mathbf{p}\eta(i_1 \dots i_f) \cdot \mathbf{p}\xi_s(k_1 \dots k_f)$$

is clearly the matrix of a symmetric operator A_s in Σ. As A_s commutes with L we conclude that $\tilde{\theta} = L\theta$. Consequently $\theta = 0$ implies $\tilde{\theta} = 0$, and $\underset{\cdot}{x} = 0$ implies $\Sigma_p y(p^{-1}) \cdot \mathbf{p}\eta(i_1 \dots i_f) = 0$, or tr $(\underset{\cdot}{y}\underset{\cdot}{y}^*) = 0$ for every $\underset{\cdot}{y}^* \in \kappa$. The quantity $\underset{\cdot}{y}$ itself is in κ, and hence the last equation forces $\underset{\cdot}{y}$ to vanish.

This settled, one concludes that the correspondence $\underset{\cdot}{x} \to \underset{\cdot}{y} = R\underset{\cdot}{x}$ defines a linear mapping R of κ into itself. Formula (4.3) and its parallel

$$\underset{\cdot}{a} \cdot \underset{\cdot}{y} = \sum_{s;i} \xi'_s(i_1 \dots i_f) \cdot \sim\beta_s(i_1 \dots i_f)$$

prove the mapping R to be a similarity; i.e. it carries $\underset{\cdot}{a}\underset{\cdot}{x}$ into $\underset{\cdot}{a}\underset{\cdot}{y}$ whatever $\underset{\cdot}{a}$. Replace $\underset{\cdot}{x}$ and $\underset{\cdot}{a}$ by $\underset{\cdot}{e}$ and $\underset{\cdot}{x}$. Setting $R\underset{\cdot}{e} = \hat{a}$ one finds that $\underset{\cdot}{x} = \underset{\cdot}{x}\underset{\cdot}{e}$ goes into $\underset{\cdot}{x} \cdot R\underset{\cdot}{e} = \underset{\cdot}{x}\hat{a}$. This statement is equivalent with the n^f equations $\beta_s = \mathbf{a}\gamma_s$, or $L\eta = \mathbf{a}\eta$ for every tensor η. The symmetry quantities $\hat{a}$ and $\underset{\cdot}{a}$ lie in κ.

5. The reciprocity of $\mathfrak{A}^*$ and $\mathfrak{B}^*$

In this section ν is not assumed to have the special value 2.

THEOREM II. *$\mathfrak{A}^*$ is the commutator algebra of $\mathfrak{B}^*$.*

Proof. Let

$$C = \| c(i_1\rho_1, \dots, i_f\rho_f\,; k_1\sigma_1, \dots, k_f\sigma_f) \|$$

be the matrix of any linear transformation in Ω in the unique normalization established by Lemma 3. Hence C is antisymmetric in the f pairs $(i\rho)$, antisymmetric in the f pairs $(k\sigma)$, and thereby symmetric in the f quadruples $(i\rho,\ k\sigma)$. Let $X = \| x(\rho_1 \dots \rho_f\,; \sigma_1 \dots \sigma_f) \|$ be symmetric in the f pairs $(\rho\sigma)$. Then CX with the components

$$\Sigma_\tau c(i_1\rho_1, \dots, i_f\rho_f\,; k_1\tau_1, \dots, k_f\tau_f) \cdot x(\tau_1 \dots \tau_f\,; \sigma_1 \dots \sigma_f)$$

is certainly antisymmetric in the pairs $(i\rho)$, and since it is symmetric in the quadruples $(i\rho,\ k\sigma)$ it is also antisymmetric in the pairs $(k\sigma)$. The same is true for XC. Our hypothesis demands that CX and XC coincide as operators in Ω. Hence their matrices in normalized form must be identical. For fixed $i_1 \ldots i_f\,; k_1 \ldots k_f$ the coefficients

$$c(\rho_1 \ldots \rho_f\,; \sigma_1 \ldots \sigma_f) = c(i_1\rho_1, \ldots, i_f\rho_f\,; k_1\sigma_1, \ldots, k_f\sigma_f)$$

form a matrix $\| c(\rho_1 \ldots \rho_f\,; \sigma_1 \ldots \sigma_f) \|$ in P which may be denoted by $C(i_1 \ldots i_f\,; k_1 \ldots k_f)$. Theorem I when applied to P rather than Σ shows that this transformation is of the form $\sum_p t_p \mathbf{p}$ where

$$t(p) = t_p = t_p(i_1 \ldots i_f\,; k_1 \ldots k_f)$$

are the coefficients of a symmetry quantity

(5.1) $$\underset{\cdot}{t} = \underset{\cdot}{t}(i_1 \ldots i_f\,; k_1 \ldots k_f)$$

that lies in $\kappa = \kappa_\nu$. Introduce the transformation

(5.2) $$T_p = \| t_p(i_1 \ldots i_f\,; k_1 \ldots k_f) \|$$

in Σ. Our result may then be written in the form

$$C = \textstyle\sum_p (T_p \times \mathbf{p}),$$

the cross indicating the Kronecker product of a matrix in Σ (first factor) and a matrix in P (second factor). If we are not afraid of making use of a symmetry quantity $\underset{\cdot}{T}$ whose coefficients are the matrices T_p in Σ we can express the fact that each $\underset{\cdot}{t}$ lies in κ_ν by the equations

(5.3) $$\underset{\cdot}{T}\underset{\cdot}{e} = \underset{\cdot}{e}\underset{\cdot}{T} = \underset{\cdot}{T},$$

$\underset{\cdot}{e} = \underset{\cdot}{e}_\nu$ being the generating idempotent of $\kappa = \kappa_\nu$.

C is antisymmetric in the pairs $(k\sigma)$. Hence

(5.4) $$C(\mathbf{q} \times \mathbf{q}) = \delta_q \cdot C$$

for any permutation q. It is antisymmetric in the pairs $(i\rho)$; hence also

(5.5) $$(\mathbf{q} \times \mathbf{q})C = \delta_q \cdot C.$$

In more explicit form (5.4) reads

$$\textstyle\sum_p \{ T_p \mathbf{q} \times \mathbf{pq} \} = \delta_q \cdot \sum_p \{ T_p \times \mathbf{p} \}$$

or

(5.6) $$\textstyle\sum_p \{ T_{pq^{-1}} \mathbf{q} \times \mathbf{p} \} = \delta_q \cdot \sum_p \{ T_p \times \mathbf{p} \}.$$

In order to avoid confusion use for the moment

$$Q = \| q(i_1 \ldots i_f\,; k_1 \ldots k_f) \|$$

as a notation for the linear transformation $\mathbf{q}$ in Σ and its matrix. Set $T'_p = T_p Q$,

$$t'_p(i_1 \ldots i_f\,; k_1 \ldots k_f)$$
$$= \textstyle\sum_l t_p(i_1 \ldots i_f\,; l_1 \ldots l_f) \cdot q(l_1 \ldots l_f\,; k_1 \ldots k_f).$$

Given a combination $(i_1 \ldots i_f\,; k_1 \ldots k_f)$, the symmetry quantity $\underset{\cdot}{t}'$ with the coefficients $t'(p) = t'_p(i_1 \ldots i_f\,; k_1 \ldots k_f)$ lies in κ_ν because all the quantities $\underset{\cdot}{t}(i_1 \ldots i_f\,; l_1 \ldots l_f)$ do. (This is true for any linear transformation Q in Σ. What holds for $T'_p = T_p Q$ holds likewise for $T''_p = QT_p$.) For a fixed permutation q the numbers $t^*(p) = t'(pq^{-1})$ are the coefficients of the symmetry quantity $\underset{\cdot}{t}^* = \underset{\cdot}{t}'q$. $\| t^*_p(i_1 \ldots i_f\,; k_1 \ldots k_f) \|$ is the matrix $T'_{pq^{-1}} = T_{pq^{-1}}Q$. Hence (5.6) states that $\underset{\cdot}{t}^*$ and $\delta_q \cdot \underset{\cdot}{t}$ coincide as symmetry operators in P.

But $\underset{\cdot}{t}^* = \underset{\cdot}{t}'\underset{\cdot}{q}$ lies in κ_ν because $\underset{\cdot}{t}'$ does; coincidence as operators in P, therefore, implies identity of the symmetry quantities themselves, $\underset{\cdot}{t}^* = \delta_q \cdot \underset{\cdot}{t}$ or

$$T_{p q^{-1}}\,\mathbf{q} = \delta_q \cdot T_p.$$

Setting $q = p$, $T_1 = A$, one finds

$$T_p = \delta_p \cdot A\mathbf{p}.$$

In the same manner (5.5) leads to

$$T_p = \delta_p \cdot \mathbf{p}A.$$

The transformation A in Σ thus commutes with the permutation operators $\mathbf{p}$ in the same space and is therefore symmetric. Because of the antisymmetry of $\psi(i_1\rho_1, \ldots, i_f\rho_f)$ in the pairs $(i\rho)$ the equation

$$\psi' = C\psi = \textstyle\sum_p (T_p \times \mathbf{p})\psi$$

may be written as

$$\psi' = \textstyle\sum_p \delta_p \cdot (T_p \mathbf{p}^{-1} \times I)\psi = f!(A \times I)\psi$$

where I stands for identity, and thus Theorem II is proved.

The normalizing condition (5.3) takes on the form

$$\breve{\mathbf{e}}A = A\breve{\mathbf{e}} = A, \tag{5.7}$$

$\breve{e}$ being the idempotent with the coefficients $\delta_p \cdot e(p) = \delta_p \cdot e(p^{-1})$. This, however, is no surprise. As a matter of fact, A induces the same transformation A^* in Ω as $\breve{\mathbf{e}}A\breve{\mathbf{e}}$, and hence, whether or not A satisfies (5.7), it can always be so modified as to fulfil that relation, without change in the corresponding A^*.

Application of Theorem II to $\nu = 2$ shows that the decomposition (1.5) by valences is complete.

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