Can. J. Math., Vol. XXII, No. 4, 1970, pp. 863–874

AN ABSTRACT CONCEPT OF THE SUM OF A NUMERICAL SERIES

WILLIAM H. RUCKLE

1. Introduction. Our aim in this paper, generally stated, is to formulate an abstract concept of the notion of the sum of a numerical series. More particularly, it is a study of the type of sequence space called "sum space". The idea of sum space arose in connection with two distinct problems.

1.1 The Köthe-Toeplitz dual of a sequence space T consists of all sequences t such that $st \in l^1$ (absolutely summable sequences) for each $s \in T$. It is known that if cs or bs is used in place of l^1, an analogous theory of duality for sequence spaces can be developed (cf. [**2**]). What other spaces of sequences can play a rôle analogous to l^1? This problem is treated in [**6**].

1.2. Let $\{x_n, f_n\}$ be a complete biorthogonal sequence in (X, X^*), where X is a locally convex linear topological space and X^* is its topological dual space. For $x \in X$ and f in X^* the formal series corresponding to $f(x)$ is

$$f(x) \sim \sum_{n=1}^{\infty} f_n(x) f(x_n).$$

Is there a topological sequence space S and a continuous linear functional E on S such that the sequence $(f_n(x)f(x_n))$ is in S for each $x \in X$ and $f \in X^*$ and $E((f_n(x)f(x_n))) = f(x)$? This problem will be treated for X a Banach space in [**7**].

The definitions of sum space and the related concept of abstract series method are given in § 2.

Most of the results in this paper are directed toward the construction of sum spaces or criteria under which a given type of sequence space is a sum space. For instance, the processes of permuting and mixing sum spaces to obtain new sum spaces is given at the end of § 2. In § 4, conditions are given under which the sum and intersection of sum spaces is a sum space.

Previous formulations of a generalized concept of sum have been given in terms of series-sequence and series-series summability matrices. An account of what is known about such matrices can be found in [**1**, Chapter 4]. In § 5, criteria are developed under which certain matrices determine a sum space, and the series-sequence method of the arithmetic mean is shown to be such a matrix.

Received September 16, 1969 and in revised form, December 15, 1969. This paper was presented at the Conference on FK-spaces in Albany, New York, in July, 1969. This research is partially supported by National Science Foundation Grant GP-12638.

One of the implicit results of this paper is that the concept of sum space is in a sense dual to the concept of multiplier algebra. Multiplier algebras of sequence spaces were studied in [**4**] and the definition is repeated preceding Definition 2.4. Concepts allied with that of multiplier algebra are those of B-invariance and solidity studied by Garling in [**2**].

A very simple illustration of the utility of the abstract notion of sum is the following. Let $\{x_n\}$ be a conditional Schauder basis of a Banach space and let $\{f_n\}$ be the biorthogonal coordinate functionals. Let π be a permutation on the integers such that $\{x_{\pi(n)}\}$ is no longer a basis. It is easy to verify that there is no regular matrix which takes the partial sums of each sequence of the form

$$\{f(x_{\pi(n)})f_{\pi(n)}(x)\}, \qquad f \in X^*, \quad x \in X,$$

into its "correct" sum, i.e., $f(x)$. However, each such sequence is in the sum space cs_π (see Example 3.2 and the end of § 2), and the sum E on cs_π takes each such sequence into its correct sum.

Acknowledgement. I am indebted to the referee for several constructive suggestions.

2. Sum spaces and abstract series methods. The letters s, t, u, and v with or without subscripts will denote sequences. If s is the sequence $\{a_1, a_2, \ldots\}$, then $s(j)$ denotes the jth coordinate of s, namely a_j. A K-space is a locally convex linear space of sequences upon which each functional E_j, given by $E_j(s) = s(j)$, is continuous. An FK-(BK-)space is a K-space which is a Fréchet (Banach) space.

2.1. *Definition.* A *sum* is a continuous linear functional E defined on a K-space S containing ϕ (finitely non-zero sequences) such that

$$E(s) = \sum_j s(j)$$

for s in ϕ.

The following facts about sums follow very quickly from the definition, and their proof is omitted.

2.2. PROPOSITION. (a) *Suppose that S and T are* K-*spaces containing ϕ such that $S \subset T$ and the inclusion is continuous.* (*This occurs, for instance, when S and T are* FK-*spaces.*) *If E is a sum on T, then E restricted to S is a sum.*

(b) *A sum defined on a* K-*space S is unique if and only if ϕ is dense in S.*

The definition of "sum" as given by 2.1 is very inclusive. In fact, if S is any BK-space, it is easy to find a sequence t such that $t(j) \neq 0$ for each j and $tS \equiv \{ts: s \in S\}$ is contained in l^1 (absolutely summable sequences). Here, ts is the sequence whose jth coordinate is $t(j)s(j)$. Then tS is a BK-space equivalent to S and the usual sum on l^1 is a sum when restricted to tS.

For a K-space S containing ϕ, S^0 denotes the closure of ϕ in S, and S^f denotes the space of all sequences $(x'(e_j))$ as x' ranges over S^*. Here e_j denotes the jth coordinate vector, the sequence with 1 in the jth place and 0s elsewhere.

Let S be a K-space containing ϕ on which the sum E is defined. Roughly speaking, the most natural set arising in connection with S on which a sum is required is the set SS^f. If $s \in S$ and $t = (x_t'(e_j))$ is in S^f, it would be desirable that the following conditions hold (see 1.2):

$$st \in S, \tag{2–1}$$

$$E(st) = x_t'(s). \tag{2–2}$$

The first condition requires that st be in the domain of the sum E. The second condition requires that E gives st its "correct" value. It cannot be expected that (2–2) will be valid unless $s \in S^0$ because of Proposition 2.2 (b). On the other hand, if $s \in \phi$, it is easy to see that (2–1) and (2–2) are valid for each $t \in S^f$.

Definition 2.1 is not sufficient to yield condition (2–1) even when $S = S^0$. For example, let $S = tc_0$, where $t(j) = 2^{-j}$ for each j and c_0 consists of all sequences which converge to zero. Then $S^f = sl^1$, where $s(j) = 2^j$ for each j. The sequence $(2^{-j}/j^2)$ is in S and $(2^j/j^2)$ is in S^f, but their coordinatewise product $(1/j^4)$ is not in S.

The *multiplier algebra* $M(S)$ of a sequence space S is the set of all sequences u such that $us \in S$ whenever $s \in S$. Multiplier algebras are studied in [**4**].

2.3. Proposition. (a) *For S a* K-*space containing ϕ, $SS^f \subset S$ if and only if $S^f \subset M(S)$.*

(b) *A sum is defined upon S, and* FK-*space, if and only if $M(S) \subset S^f$.*

Proof. (a) is obvious.

(*b*) If $M(S) \subset S^f$, then $e \equiv (1, 1, 1, \ldots) \in S^f$ since e is always contained in $M(S)$. Thus there is a continuous linear functional E defined on S such that $E(E_j) = 1$ for each j.

If a sum is defined on S, then $e \in S^f$, and so $M(S^f) \subset S^f$, and by [**4**, Proposition 3.5], $M(S) \subset M(S^f)$.

2.4. *Definition.* A K-space S is called a *sum space* if and only if S contains ϕ and $S^f = M(S)$.

If S is a sum space, then a sum E is defined upon S because of Proposition 2.3 (b) and (2–1) is satisfied by Proposition 2.3 (a). However, the sum is not unique nor is (2–2) satisfied for all s in S and $t \in S^f$ unless $S = S^0$ (i.e., S has AD).

2.5 Proposition. *If S is an* FK-*sum space, then so is S^0.*

Proof. By an easy application of the Hahn-Banach theorem, $S^{0f} = S^f$. By [**4**, Propositions 3.2 (a) and 3.4], $M(S) \subset M(S^0)$. Thus

$$S^{0f} = S^f = M(S) \subset M(S^0)$$

https://doi.org/10.4153/CJM-1970-098-5 Published online by Cambridge University Press

because S is a sum space. On the other hand, S^0 has a sum defined upon it, namely the restriction to it of any sum on S. Thus $M(S^0) \subset S^{0f}$ by Proposition 2.3 (b).

The condition that S have AD is not required in Definition 2.4 since certain spaces without AD, e.g. bs, are useful in constructing a Köthe-Toeplitz type duality theory such as discussed in 1.1.

The BK-spaces l^1 and cs (summable series) as well as the space ϕ with either its weak or Mackey topologies are examples of sum spaces. In [**4**, following Theorem 5.6] it is noted that for the space X of all real sequences s for which

$$p_k(s) = \sum_j |s(j)|k^j < \infty, \qquad k = 1, 2, \ldots,$$

it is true that $X^f = M(S)$. Thus X is also a sum space. Two processes will now be described by which a proliferation of sum spaces can be generated.

If S is a K-space and π is any permutation on the integers, denote by S_π the space of all sequences $s_\pi = \{s(\pi(1)), s(\pi(2)), \ldots\}$ as s ranges over S with the topology generated by the seminorms $p \circ \pi^{-1}$ as p ranges over the continuous seminorms on S. It is clear that $(S_\pi)^f = (S^f)_\pi$ and $M(S_\pi) = (M(S))_\pi$ so that if S is a sum space, then so is S_π.

Let S_1 and S_2 be two K-spaces, and let N_1 and N_2 be two disjoint complementary sets of indices, say $N_1 = \{k_1, k_2, \ldots\}$, $N_2 = \{n_1, n_2, \ldots\}$. Denote by $S_1 \oplus S_2(N_1, N_2)$ the space of all sequences s such that

$$(s(k_1), s(k_2), \ldots) \in S_1 \qquad \text{and} \qquad (s(n_1), s(n_2), \ldots) \in S_2.$$

Determine a topology on this space by means of the seminorms

$$p \oplus q(s) = p((s(k_1), s(k_2), \ldots)) + q((s(n_1), s(n_2), \ldots))$$

as p ranges over the continuous seminorms on S_1 and q ranges over the continuous seminorms on S_2. It is not hard to calculate that

$$(S_1 \oplus S_2(N_1, N_2))^f = S_1{}^f \oplus S_2{}^f(N_1, N_2)$$

and $M(S_1 \oplus S_2(N_1, N_2)) = M(S_1) \oplus M(S_2)(N_1, N_2)$. Hence, if S_1 and S_2 are sum spaces, so are $S_1 \oplus S_2(N_1, N_2)$. This process can be extended to any finite number of summands.

2.6. *Definition*. An *abstract series method* is a pair $\{S, E\}$, where S is a sum space and E is a sum on S. The *abstract sum* of a sequence $s \in S$ is $E(s)$.

If S is an FK-space, $\{S, E\}$ is called an FK-abstract series method.

2.7. *Definition*. An abstract series method is said to *apply* to the K-space T containing ϕ if $TT^f \subset S$. It is said to apply *correctly* to S if for each $t \in T$ and $u \in T^f$, $tu \in S$ and $E(tu) = f_u(t)$, where $f_u \in S^*$ if any function such that $f_u(e[j]) = u(j)$ for each j.

https://doi.org/10.4153/CJM-1970-098-5 Published online by Cambridge University Press

2.8. PROPOSITION. *Let $\{S, E\}$ be an abstract* FK-*series method. For T an* FK-*space with* AD, *the following statements are equivalent:*

(a) *$\{S, E\}$ applies to T;*

(b) *$\{S, E\}$ applies correctly to T;*

(c) *$T^f = \{u\colon us \in S$ for each $t \in T\}$.*

Proof. (a) $\Rightarrow$ (b). Since $\{S, E\}$ applies to T, $TT^f \subset S$. For each $u \in T^f$ the mapping defined by $F_u t = ut$ is linear and closed, hence continuous from T into S. For each $t \in \phi$, $F_u t \in \phi$ so that for $t \in T^0$,

$$E(ut) = \sum_j u(j)t(j) = \sum_j f_u(e_j)t_j = f_u(t)$$

since the sum is finite. But ϕ is dense in S so that $E(ut) = f_u(t)$ for each $t \in S$.

(b) $\Rightarrow$ (c). Denote the set on the right-hand side of (c) by T^S. Since $\{S, E\}$ applies to T, $T^f \subset T^S$. On the other hand, if $u \in T^S$, the mapping F_u is continuous from T into S so that $f_u(t) = E(ut)$ is continuous on S. Since $f_u(e_j) = u(j)$ for each j, $u \in T^f$.

(c) $\Rightarrow$ (a). If $T^f = T^S$, then $TT^f = TT^S \subset S$ by definition of T^S.

3. Lattice properties of S^f. If S and T are K-spaces, $S \cap T$ is a K-space with the topology generated by all continuous seminorms on S or T. With the topology generated by all seminorms of the form

$$(3\text{–}1) \qquad r(u) = \inf\{p(s) + q(t)\colon s \in S, t \in T, s + t = u\}$$

as p ranges over all continuous seminorms on S and q ranges over all continuous seminorms on T, $S + T$ ($\equiv \{s + t\colon s \in S, t \in T\}$) is also a K-space. Some properties of such intersections and sums were derived in [**5**].

In this section, additional properties concerning the sum and intersection of K-spaces will be developed. These properties have some interest in themselves, but they are given here primarily so that they can be applied in the following section.

3.1. THEOREM. *For* K-*spaces S and T:*

(a) *$(S + T)^f \subset S^f \cap T^f$;*

(b) *If ϕ is dense in $S \cap T$, then $(S + T)^f = S^f \cap T^f$;*

(c) *$(S \cap T)^f = S^f + T^f$.*

Proof. (a) This follows immediately since the inclusions of S and T into $S + T$ are continuous.

(b) Suppose that $S \cap T$ has AD, i.e., ϕ is dense in $S \cap T$. If $u \in S^f \cap T^f$, there is f_1 in S^* and f_2 in T^* such that $f_1(e_i) = f_2(e_i) = u(i)$ for each i. Define f on $S + T$ by

$$f(s + t) = f_1(s) + f_2(t), \qquad s \in S, \quad t \in T.$$

To see that f is well-defined, let $s + t = s' + t'$, where $s, s' \in S$ and $t, t' \in T$.

https://doi.org/10.4153/CJM-1970-098-5 Published online by Cambridge University Press

Then $s - s' = t' - t$, and so this vector is in $S \cap T$ Both f_1 and f_2 are continuous on $S \cap T$ and $f_1(v) = f_2(v)$ for $v \in \phi$. Since $S \cap T$ has AD, $f_1(s - s') = f_2(t' - t)$.

To see that f is continuous, let p and q be continuous seminorms on S and T, respectively, such that $|f_1(s)| \leqq p(s)$, $s \in S$, and $|f_2(t)| \leqq q(t)$, $t \in T$. Then if r is given by (3–1), r is a continuous seminorm on $S + T$ such that $|f(v)| \leqq r(v)$ for $v \in S + T$. Since $f(e_i) = u(i)$ for each i, $u \in (S + T)^f$.

(c) The fact that $S^f + T^f \subset (S \cap T)^f$ follows from the fact that the inclusions from $S \cap T$ into S and T are continuous.

If $u \in (S \cap T)^f$, there is $f \in (S \cap T)^*$ such that $f(e_i) = u(i)$ for each i. There are continuous seminorms p on S and q on T such that for each $u \in S \cap T$,

$$|f(u)| \leqq \max\{p(u), q(u)\}.$$

By [3, § 19, p. 229, 6 (3)], there are linear functionals f_1 and f_2 on $S \cap T$ such that $f_1 + f_2 = f$, and

$$|f_1(u)| \leqq p(u), \qquad |f_2(u)| \leqq q(u),$$

for each $u \in S \cap T$. By the Hahn-Banach theorem, f_1 has an extension F_1 to all of S such that $|F_1(s)| \leqq p(s)$ for each s, and there is an analogous extension F_2 of f_2 to all of T. If $u_1 = (F_1(e_i))$ and $u_2 = (F_2(e_i))$, then $u_1 \in S^f$, $u_2 \in T^f$, and $u = u_1 + u_2$. Thus $(S \cap T)^f \subset S^f + T^f$.

3.2. *Example.* If $s \in cs - l^1$, there is a permutation on the positive integers such that $s_\pi \in cs$ but

$$\sum_{j=1}^{\infty} s_\pi(j) \neq \sum_{j=1}^{\infty} s(j).$$

It was noted in § 2 that cs_π is a sum space so that $e \in cs^f \cap cs_\pi{}^f$. But e is not in $(cs + cs_\pi)^f$. If there were a continuous linear functional E on $cs + cs_\pi$ such that $E(e_j) = 1$ for each j, it would be continuous when restricted to both cs and cs_π. Since cs and cs_π have AD, $E(t)$ would be uniquely defined on cs and on cs_π. Thus $E(t) = \sum_{j=1}^{\infty} t(j)$ for t in cs and $E(t) = \sum_{j=1}^{\infty} t(\pi^{-1}(j))$ for t in cs_π. But s_π is in $cs \cap cs_\pi$ and

$$\sum_{j=1}^{\infty} s_\pi(j) \neq \sum_{j=1}^{\infty} s_\pi(\pi^{-1}(j)).$$

Thus no such E can exist.

Therefore $(cs + cs_\pi)^f$ is strictly contained in $cs^f + cs_\pi{}^f$, and $cs \cap cs_\pi$ does not have AD even though both cs and cs_π do. An alternative proof that $cs \cap cs_\pi$ does not have AD would note that the continuous linear functional

$$f(s) = \sum_{j=1}^{\infty} s(j) - \sum_{j=1}^{\infty} s(\pi^{-1}(j))$$

is 0 on each e_j but non-zero on $cs \cap cs_\pi$.

Problem. Is the converse of Theorem 3.1 (b) true if both S and T have AD?

4. Lattice properties of sum spaces. Our aim in this section is to determine conditions under which the sum and intersection of FK-sum spaces are sum spaces. Theorem 4.2 provides a rather complete solution in the case of the sum. In the case of the intersection we have Proposition 4.3 in which one of the spaces need not be a sum space. Although this proposition is somewhat contrived, it yields many examples of sum spaces as evidenced by its Corollary 4.4 which is widely applicable. See also the statement preceding Proposition 5.4.

The following lemma has a trivial proof which is omitted.

4.1. Lemma. *For sequence spaces S and T,*

$$M(S) \cap M(T) \subset M(S+T) \cap M(S \cap T).$$

4.2. Theorem. *Let S and T be* FK-*sum spaces. The space $S + T$ is a sum space if and only if there is a sum defined on it.*

Proof. The necessity that a sum be defined on $S + T$ in order that it be a sum space is clear.

If a sum is defined on $S + T$, then $e \in (S+T)^f$ so that

$$M((S+T)^f) \subset (S+T)^f.$$

But by [**4**, Proposition 3.5], $M(S+T) \subset M((S+T)^f)$ so that

$$M(S+T) \subset (S+T)^f.$$

On the other hand,

$$M(S+T) \supset M(S) \cap M(T) = S^f \cap T^f$$

by Lemma 4.1 and the fact that S and T are sum spaces. By Theorem 3.1 (a), $S^f \cap T^f \supset (S+T)^f$ so that $M(S+T) \supset (S+T)^f$.

4.3. Proposition. *If S is an* FK-*sum space and T is an* FK-*space such that $M(S) \subset M(T)$ and $TT^f \subset (S \cap T)$, then $S \cap T$ is a sum space.*

Proof. By Theorem 3.1 (c), $(S \cap T)^f = S^f + T^f$ and since $e \in S^f$, $M(S^f + T^f) \subset S^f + T^f$. By [**4**, Proposition 3.5], $M(S \cap T) \subset M((S \cap T)^f)$. Since $M(S) \subset M(T)$,

$$S^f = M(S) = M(S) \cap M(T) \subset M(S \cap T)$$

because of Lemma 4.1. And $TT^f \subset (S \cap T)$ implies $T^f \subset M(S \cap T)$. In view of these conclusions, it follows that

$$M(S \cap T) \subset M((S \cap T)^f) \subset S^f + T^f \subset M(S \cap T)$$

so that equality holds throughout.

4.4. Corollary. *If S is an* FK-*sum space with $l^1 \subset S$ and T is an* FK-*space in which $\{e_1, e_2, \ldots\}$ is an unconditional basis, then $S \cap T$ is a sum space.*

https://doi.org/10.4153/CJM-1970-098-5 Published online by Cambridge University Press

Proof. Since $e_1, e_2, \ldots$ is an unconditional basis for T, it easily follows that $TT^f \subset l^1 \subset S$. Since $l^1 \subset S$, $M(S) = S^f \subset m$ and since $\{e_1, e_2, \ldots\}$ is an unconditional basis for T, $m \subset M(T)$. Thus Proposition 4.3 is applicable.

4.5. *Example.* For $1 < p < \infty$, $cs \cap l^p$ is a sum space because of Corollary 4.4. This furnishes a positive answer to the question in [**4**, the end of § 6]. Every γ-perfect BK-algebra in the smallest class containing m, bv, and closed under finite application of [**4**, Theorem 6.2] is either m or contains a coordinate subspace equal to bv. In fact, any finite number of reapplications of [**4**, Theorem 6.2] is equivalent to a single application. The BK-space $cs \cap l^p$ for $1 < p < \infty$ has $\mathscr{E}$ as a basis by [**4**, Theorem 5.4]. By Theorem 3.1 (c), $(cs \cap l^p)^f = bv + l^q$, where $1/p + 1/q = 1$ and bv is the set of all sequences s such that

$$\sum_{j=1}^{\infty} |s(j) - s(j+1)| < \infty.$$

Thus $l^q + bv$ for $1 < q < \infty$ is a multiplier algebra of a Schauder basis and it clearly does not contain a coordinate subspace equal to bv. Thus the collection of multiplier algebras of bases in Banach spaces is not exhausted by the smallest class of BK-algebras containing m, bv, and being closed under the mixing operations described in [**4**, Theorem 6.2].

5. Criteria for a matrix to determine a sum space. Let $\{t_n\}$ be a sequence in ϕ such that

$$\lim_n t_n(j) = 1 \tag{5–1}$$

for each j. Let $\mathscr{T}$ be the infinite matrix whose *rows* are $t_1, t_2, \ldots$, i.e., $\{t_n(j)\}$, $1 \leqq n, j \leqq \infty$. Let $S_{\mathscr{T}}$ consist of all s in ω such that

$$p(s) = \sup_n |(t_n, s)| < \infty. \tag{5–2}$$

Here $(t, s) = \sum_j t(j)s(j)$ for t in ϕ and s in ω. In the terminology of Wilansky [**9**, p. 227], $S_{\mathscr{T}} = m_{\mathscr{T}}$.

Suppose that $\mathscr{T}$ has an inverse $\mathscr{S}$ whose *columns* are $\{s_1, s_2, \ldots\} \subset \phi$, i.e., $\mathscr{S} = \{s_n(k)\}$, $1 \leqq k, n \leqq \infty$. Also assume that the correspondence of s in S to

$$\mathscr{T}s = \{(t_1, s), (t_2, s), \ldots\}$$

in m is an isomorphism of $S_{\mathscr{T}}$ onto m. It then follows that

$$p(s) = ||\mathscr{T}s||_m.$$

In this case, $S_{\mathscr{T}}$ is a BK-space with norm p. Because of (5–1),

$$\left|\sum_j s(j)\right| = \lim_n (s, t_n) \leqq p(s)$$

https://doi.org/10.4153/CJM-1970-098-5 Published online by Cambridge University Press

for $s \in \phi$. Thus by the Hahn-Banach theorem there are sums defined on $S_{\mathscr{I}}$. Whenever the space $S_{\mathscr{I}}$ is mentioned in this section it will be assumed to have the properties mentioned above.

5.1. PROPOSITION. *The closure of* ϕ *in* $S_{\mathscr{I}}$, *i.e.,* $S_{\mathscr{I}}{}^0$, *consists of all* $s \in \omega$ *such that* $\lim_n (s, t_n)$ *exists.*

Proof. The space of all s such that $\lim_n (s, t_n)$ exists is called $c_{\mathscr{I}}$ in [**9**, Chapter 12, § 4]. It is a closed subspace of $S_{\mathscr{I}}$ by [**9**, p. 229, Lemma 3], and by the same lemma, every continuous linear functional f of $c_{\mathscr{I}}$ has the form

$$f(s) = a \lim_n (s, t_n) + \sum_j a_j (s, t_j), \tag{5–3}$$

where $\sum_j |a_j| < \infty$. Since $(s_n, t_j) = \delta_{nj}$, it follows that $a_j = f(s_j)$ for each j. Suppose that $f(e_j) = 0$ for each j. Then $f(s_j) = 0$ for each j since each $s_j \in \phi$ by our permanent hypothesis. Also, $\lim_n (s, e_j) = 1$ because of (5–1) so that $a = 0$. Hence f is the zero functional so that $c_{\mathscr{I}} = \mathscr{S}_{\mathscr{I}}{}^0$.

The space $S_{\mathscr{I}}{}^f = (S_{\mathscr{I}}{}^0)^f$ is a BK-space with the norm

$$||t||' = ||f_t||^*,$$

where $||\ ||^*$ is the norm on $(S_{\mathscr{I}}{}^0)^*$ and $t(j) = f_t(e_j)$ for each j. If $t \in \phi$,

$$||f_t||^* = \sum_{j=1}^{\infty} |f_t(s_j)| = \sum_{j=1}^{\infty} |(t, s_j)|.$$

For if $\mathscr{I}^{-1}$ is the inverse of $\mathscr{I}$ considered as an operator, the equation

$$F(u) = f_t(\mathscr{I}^{-1}u), \qquad u \in c_0,$$

determines a continuous linear functional on c_0 whose norm is

$$\sum_{j=1}^{\infty} |F(e_j)| = \sum_{j=1}^{\infty} |f_t(s_j)|.$$

5.2. THEOREM. *When* $\mathscr{I}$ *has an inverse* $\mathscr{S}$ *as above, the following conditions are equivalent:*

(a) $S_{\mathscr{I}}$ *is a sum space;*

(b) $S_{\mathscr{I}}{}^0$ *is a sum space;*

(c) $\{t_j s: j = 1, 2, \ldots\}$ *is bounded in* $S_{\mathscr{I}}{}^0$ *for each* s *in* $S_{\mathscr{I}}{}^0$;

(d) $\lim_n t_j s = s$ *for each* s *in* $S_{\mathscr{I}}{}^0$;

(e) $\{t_j t: j = 1, 2, \ldots\}$ *is bounded in* $S_{\mathscr{I}}{}^f$ *for each* $t \in S_{\mathscr{I}}{}^f$;

(f) $\sup_{k,n} ||t_k t_n||' = \sup_{k,n} \sum_j |(s_j, t_k t_n)| < \infty$, *where* $\{s_j\}$ *and* $||\ ||'$ *are as above.*

Proof. (a) $\Rightarrow$ (b). This results from Proposition 2.5.

(b) $\Rightarrow$ (c). If $S_{\mathscr{I}}{}^0$ is a sum space, then the topologies generated by the norms

$$||t||' = \sup\{|(t, s)|: s \in \phi, p(s) \leqq 1\}$$

and

$$||t||'' = \sup\{p(ts): s \in S_{\mathscr{I}}^0, p(s) \leqq 1\}$$

are equivalent on $S_{\mathscr{I}}^f$ $(=S_{\mathscr{I}}^{0f})$ since they both make it a BK-space. Here p is the norm on $S_{\mathscr{I}}$ given by (5–2). Since $||t_j||' \leqq 1$ for each j, $\sup_j ||t_j||'' < \infty$. Thus

$$p(t_j s) \leqq ||t_j||'' p(s) \leqq \sup_j ||t_j||'' p(s)$$

for each s in S.

(c) $\Rightarrow$ (d). Since $\lim_n t_n(j) = 1$ for each j, it follows that

$$\lim_n t_n e_j = e_j$$

for each j. If (c) is valid, (d) holds by the Banach-Steinhaus theorem since the span of $\{e_j\}$ is dense in $S_{\mathscr{I}}^0$.

(d) $\Rightarrow$ (c). This is a result of the uniform boundedness principle.

(c) $\Rightarrow$ (e). This follows from the fact that the conjugate of the operator determined by $s \to t_j s$ on $S_{\mathscr{I}}^0$ is the operator on $S_{\mathscr{I}}^f$ determined by $t \to t_j t$.

(e) $\Rightarrow$ (f). Obvious.

(f) $\Rightarrow$ (c). If $s \in S_{\mathscr{I}}^0$, then for each j,

$$\begin{aligned} p(t_j s) &= \sup_k |(t_k, t_j s)| \\ &= \sup_k |(t_k t_j, s)| \\ &\leqq \sup_k ||t_k t_j||' p(s). \end{aligned}$$

Thus if (f) is valid,

$$\sup_j p(t_j s) < \infty.$$

(e) $\Rightarrow$ (a). It has already been noted that a sum is defined on $S_{\mathscr{I}}$. Hence $S_{\mathscr{I}}^f$ contains $M(S_{\mathscr{I}}^f)$ and thus $M(S_{\mathscr{I}})$ by [**4**, Proposition 3.5].

If $t \in S_{\mathscr{I}}^f$ and $s \in S_{\mathscr{I}}$,

$$\sup_j |(t_j, ts)| = \sup |(t_j t, s)| < \infty$$

since $\{t_j t\}$ is bounded in $S_{\mathscr{I}}^f$. Thus $ts \in S_{\mathscr{I}}$ so that $t \in M(S_{\mathscr{I}})$.

5.3. PROPOSITION. *For $\mathscr{I}$, the series-sequence matrix of the arithmetic mean whose rows are*

$$t_n(j) = \begin{cases} \dfrac{n-j+1}{n}, & j \leqq n, \\ 0, & j > n, \end{cases}$$

$S_{\mathscr{I}}$, and hence $S_{\mathscr{I}}^0$, is a sum space.

Proof. A direct calculation shows that the inverse of $\mathscr{I}$ has the columns

$$\begin{aligned} s_n(n) &= n, \\ s_n(n+1) &= -2n, \\ s_n(n+2) &= n, \\ s_n(j) &= 0, \qquad \text{otherwise.} \end{aligned}$$

Now for $k \leqq n$,

$$t_k t_n(j) = \begin{cases} \left(\dfrac{k-j+1}{k}\right)\left(\dfrac{n-j+1}{n}\right), & j \leqq k, \\ 0, & \text{otherwise,} \end{cases}$$

so that

$$(t_k t_n, s_j) = j t_k t_n(j) - 2j t_k t_n(j+1) + j t_k t_n(j+2)$$

$$= \begin{cases} \dfrac{j}{kn}\{(k-j+1)(n-j+1) - 2(k-j)(n-j) + (k-j-1) \\ \qquad\qquad \times (n-j-1)\} = \dfrac{2j}{kn} & \text{if } j+2 \leqq k, \\ j\left(\dfrac{2}{k}\right)\left(\dfrac{n-j+1}{n}\right) - 2j\left(\dfrac{1}{k}\right)\left(\dfrac{n-j}{n}\right) = \dfrac{2(k-1)}{kn} & \text{if } j+1 = k, \\ j\left(\dfrac{1}{k}\right)\left(\dfrac{n-j+1}{n}\right) = \dfrac{n-k+1}{n} & \text{if } j = k, \\ 0 & \text{if } j > k. \end{cases}$$

Thus

$$\sum_{j=0}^{\infty} |(t_k t_n, s_j)| = \sum_{j=1}^{k-1} \frac{2j}{kn} + \frac{n-k+1}{n} = \frac{(k-1)k}{kn} + \frac{n-k+1}{n} = 1.$$

Therefore $S_{\mathscr{I}}$ is a sum space by Theorem 5.2 (f).

If a matrix is known to determine a sum space, various summability criteria can be stated using Proposition 4.3 or Corollary 4.4. For example, by combining Corollary 4.4 and Proposition 5.3 one obtains the following.

5.4. PROPOSITION. *Let $\mathscr{I}$ be the series-sequence matrix of the arithmetic mean and let p and q be numbers greater than one for which $1/p + 1/q = 1$. If s is a sequence in l^p which is mean series summable, then st is a mean series summable sequence in l^p if and only if $t \in l^q + S^f$. In other words, t is of the form $t_1 + t_2$, where $t_1 \in l^q$ and t_2 has the property that $t_2 s \in S_{\mathscr{I}}{}^0$ whenever $s \in S_{\mathscr{I}}{}^0$.*

REFERENCES

1. R. G. Cooke, *Infinite matrices and sequence spaces* (Macmillan, London, 1950).
2. D. J. H. Garling, *On topological sequence spaces*, Proc. Cambridge Philos. Soc. *63* (1967), 997–1019.
3. G. Köthe, *Topologische lineare Räume*. I (Springer, Berlin, 1960).
4. R. J. McGivney and W. Ruckle, *Multiplier algebras of biorthogonal systems*, Pacific J. Math. *29* (1969), 375–388.

5. W. Ruckle, *Lattices of sequence spaces,* Duke Math. J. *35* (1968), 491–504.
6. ——— *Topologies on sequence spaces* (in preparation).
7. ——— *Representation and series summability of complete biorthogonal sequences* (to appear in Pacific J. Math.).
8. H. H. Schaeffer, *Topological vector spaces* (Macmillan, New York, 1966).
9. A. Wilansky, *Functional analysis* (Blaisdell, New York, 1964).

Clemson University,
Clemson, South Carolina