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# Differential Structure of Orbit Spaces

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*Abstract.* We present a new approach to singular reduction of Hamiltonian systems with symmetries. The tools we use are the category of differential spaces of Sikorski and the Stefan-Sussmann theorem. The former is applied to analyze the differential structure of the spaces involved and the latter is used to prove that some of these spaces are smooth manifolds.

Our main result is the identification of accessible sets of the generalized distribution spanned by the Hamiltonian vector fields of invariant functions with singular reduced spaces. We are also able to describe the differential structure of a singular reduced space corresponding to a coadjoint orbit which need not be locally closed.

## 1 Introduction

We consider a proper Hamiltonian action

(1) $$\Phi\colon G \times P \to P\colon (g,p) \mapsto \Phi(g,p) = \Phi_g(p) = g \cdot p$$

of a Lie group $G$ on a connected finite dimensional paracompact smooth symplectic manifold $(P,\omega)$ with a coadjoint equivariant momentum map $J\colon P \to \mathfrak{g}^*$. Here $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of $G$. The usual approach to reduction is to choose $\alpha \in \mathfrak{g}^*$, and then to study the space $J^{-1}(\alpha)/G_\alpha$ of orbits of the isotropy group $G_\alpha = \{g \in G \mid \mathrm{Ad}^t_{g^{-1}}\alpha = \alpha\}$ on $J^{-1}(\alpha)$. For a free action, $J^{-1}(\alpha)/G_\alpha$ is a quotient manifold of $J^{-1}(\alpha)$ endowed with a symplectic form which pulls back to the restriction of $\omega$ to $J^{-1}(\alpha)$ [14], [13]. For proper actions, the space $J^{-1}(0)/G$ is a stratified space with symplectic strata [2], [6], [5], [24]. Sjamaar and Lerman [24] have shown that the strata of $J^{-1}(0)/G$ are projections of the sets in $J^{-1}(0)$ consisting of points which can be joined by piecewise integral curves of Hamiltonian vector fields of $G$-invariant functions. The stratification of $J^{-1}(\alpha)/G_\alpha$ for $\alpha \neq 0$ has been studied in [3].

In this paper we study the differential structure of the space $\overline{P} = P/G$ of $G$-orbits on $P$. We begin with aspects of the structure which do not depend on the symplectic form $\omega$ on $P$. Let $\pi\colon P \to \overline{P}$ be the $G$-orbit map. If the action of $G$ on $P$ is free and proper, then $\overline{P}$ is a manifold, and $\pi\colon P \to \overline{P}$ is a (left) principal fibre bundle with structure group $G$. If the action of $G$ is proper but not free, then $\overline{P}$ need not be a manifold. In this case $\overline{P}$ is a stratified space. Smooth strata of $\overline{P}$ are connected components of the projections of the sets

(2) $$P_K = \{p \in P \mid G_p = K\},$$

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where $G_p$ denotes the isotropy group of $p$ under the $G$-action $\Phi$ [8]. Stratified spaces are defined within the category of topological spaces. Hence, the description how the smooth strata fit together is given by links, which are defined up to local homeomorphisms [9].

The space $\overline{P}$ of the $G$-orbits in $P$ is also a differential space in the sense of Sikorski [23]. Its differential structure is given by the space $C^\infty(\overline{P})$ of functions on $P$ which pull back under the $G$-orbit map $\pi\colon P \to \overline{P}$ to smooth $G$-invariant functions on $P$. In the category of differential spaces we obtain a finer description of the local differential geometry of $\overline{P}$.

Since the action of $G$ on $P$ is proper, we can introduce a $G$-invariant Riemannian metric g on $P$ [19]. Let ver $TP$ be the set of vectors in $TP$ tangent to $G$-orbits on $P$ and let hor $TP$ be its g-orthogonal complement. If the $G$-action on $P$ is free then ver $TP$ and hor $TP$ are distributions on $P$, and hor $TP$ defines a connection on the principal bundle $\pi\colon P \to \overline{P}$. Hence the tangent bundle $TP$ of $P$ is isomorphic to the fibre product of the vector bundles ver $TP$ and hor $TP$ over $P$, that is,

$$TP = \text{ver } TP \times_P \text{hor } TP. \tag{3}$$

If the $G$-action is not free, then neither ver $TP$ nor hor $TP$ are distributions, because their dimensions may vary from point to point. However, both ver $TP$ and hor $TP$ are differential spaces and the fibre product decomposition (3) holds at every point $p \in P$. Clearly, ver $TP$ is $G$-invariant. Because the metric g is $G$-invariant, it follows that hor $TP$ is also.

Let $\Psi$ be the prolongation of the $G$-action $\Phi$ to $TP$. In other words,

$$\Psi\colon G \times TP \to TP\colon (g,u) \mapsto T\Phi_g(u). \tag{4}$$

If the $G$-action on $P$ is free, the space $(TP)/G$ of $G$-orbits of $\Psi$ is the fibre product of smooth bundles (ver $TP)/G$ and (hor $TP)/G$ over $\overline{P}$ whose total space is the space $G$-orbits on ver $TP$ and hor $TP$, respectively, and whose base space is $\overline{P}$. In symbols

$$(TQ)/G = (\text{ver } TP)/G \times_{\overline{P}} (\text{hor } TP)/G. \tag{5}$$

In addition, (ver $TP)/G$ is naturally isomorphic to the adjoint bundle $P[\mathfrak{g}]$ and (hor $TP)/G$ is naturally isomorphic to the tangent bundle $T\overline{P}$ of $\overline{P}$. Thus (5) reads

$$(TP)/G = P[\mathfrak{g}] \times_{\overline{P}} T\overline{P}, \tag{6}$$

see [4] and [7], where the dual decomposition $(T^*P)/G = P[\mathfrak{g}^*] \times_{\overline{P}} T^*\overline{P}$ is investigated.

In this paper we analyze the structure of each factor on the right hand side of (5) when the action of $G$ on $P$ is proper but *not* free. We show that (ver $TP)/G$ and (hor $TP)/G$ are differential spaces with smooth projections $\pi_{\text{ver}}\colon (\text{ver } TP)/G \to \overline{P}$ and $\pi_{\text{hor}}\colon (\text{hor } TP)/G \to \overline{P}$ and smooth inclusions $\iota_{\text{ver}}\colon (\text{ver } TP)/G \hookrightarrow TP/G$ and $\iota_{\text{hor}}\colon (\text{hor } TP)/G \hookrightarrow TP/G$. We show that the fibre product decomposition on the right hand side of equation (5), is valid at every point of $\overline{P}$. A similar interpretation can be given to equation (6). Smooth sections of the fibration $\pi_{\text{ver}}$ correspond to infinitesimal automorphisms of the action of $G$ on $P$, which induce the identity transformation on $\overline{P}$. In order to emphasize the fibration $\pi_{\text{hor}}\colon (\text{hor}\, TP)/G \to \overline{P}$, we introduce the notation $T^{\text{w}}\overline{P} = (\text{hor}\, TP)/G$. We show that for each $\bar{p} \in \overline{P}$, the fibre $T^{\text{w}}_{\bar{p}}\overline{P} = \pi^{-1}_{\text{hor}}(\bar{p})$ is a direct sum of the (Zariski) tangent space $T_{\bar{p}}\overline{P}$ of $\overline{P}$ and a cone $T^{\text{c}}_{\bar{p}}\overline{P}$. For this reason, we refer to $T^{\text{w}}\overline{P}$ as the tangent wedge of $\overline{P}$ at $\bar{p}$. The space $T^{\text{w}}_{\bar{p}}\overline{P}$ is locally *diffeomorphic* to $\overline{P}$. In particular, the tangent cone $T^{\text{c}}_{\bar{p}}\overline{P}$ carries information describing the links at $\bar{p}$ of the stratification of $\overline{P}$.

Next we investigate the structure of the orbit space $\overline{P}$ induced by the coadjoint equivariant momentum map $J\colon P \to \mathfrak{g}^*$. Motivated by the results of Sjamaar and Lerman [24], we consider the generalized distribution $E$ on $P$ locally spanned by Hamiltonian vector fields of $G$-invariant functions on $P$. A subset $L$ of $P$ is called an accessible set of $E$ if every pair of points in $L$ can be joined by a piecewise integral curve of vector fields locally spanning $E$. A theorem of Stefan and Sussmann [26], [27] ensures that accessible sets of $E$ are immersed submanifolds of $P$. Moreover, the partition of $P$ by accessible sets of $E$ is a smooth foliation with singularities. We show that each accessible set $L$ of $E$ is a connected component of $J^{-1}(\alpha) \cap P_K$ for some $\alpha \in \mathfrak{g}^*$ and some compact subgroup $K$ of $G$. It should be noted that the standard proof that $J^{-1}(\alpha) \cap P_K$ is locally a manifold is fairly involved. Here, all technical points of the proof are taken care of by the Stefan-Sussmann theorem [26], [27].

The smooth foliation with singularities on $P$ given by accessible sets of $E$ projects to a partition of $\overline{P}$. Each set of this partition of $\overline{P}$ is a smooth submanifold of $\overline{P}$ endowed with a symplectic form. For each $\bar{p} \in \overline{P}$, the information about how the smooth parts of $\overline{P}$ fit together in a neighbourhood of $\bar{p}$ is encoded in the tangent cone at $\bar{p}$.

The space $C^\infty(P)$ has the structure of a Poisson algebra induced by the symplectic form $\omega$ on $P$. Since $\omega$ is $G$-invariant, it follows that the space $C^\infty(P)^G$ of $G$-invariant smooth functions on $P$ is a Poisson subalgebra of $C^\infty(P)$. Hence, the differential structure $C^\infty(\overline{P})$ inherits the structure of a Poisson algebra. This makes our approach analogous to Poisson reduction studied by several authors [1], [12], [17], [18]. The main difference between our approach and theirs is our systematic use of the category of differential spaces and the Stefan-Sussmann theorem. We obtain a description of geometry of the spaces under consideration up to a diffeomorphism, while stratifications are studied up only to a homeomorphism. Moreover, we resolve the problem of differential structures of $J^{-1}(\mathcal{O})/G$ for nonlocally closed coadjoint orbits $\mathcal{O} \subseteq \mathfrak{g}^*$.

## 2 Symmetry Type

In this section we describe the partition of $P$ by sets of points with the same symmetry type. For the action $\Phi$ of $G$ on $P$, we shall use the notation

$$\Phi(g, p) = \Phi_g(p) = \Phi_p(g) = g \cdot p.$$

For each $p \in P$, the isotropy group $G_p$ of $p$ is

$$G_p = \{g \in G \mid \Phi(g, p) = p\}.$$

Because the action $\Phi$ is proper, $G_p$ is a compact subgroup of $G$ for each $p \in P$. Let $K$ be a compact subgroup of $G$. The set of points of *symmetry type* $K$ is

$$P_K = \{p \in P \mid G_p = K\}.$$

***Theorem 2.1*** *Let $M$ be a connected component of $P_K$ and let $\iota_M \colon M \to P$ be the inclusion map. Then*

*i) $M$ is a submanifold of $P$ and $\omega_M = \iota_M^* \omega$ is a symplectic form on $M$.*
*ii) For each smooth $G$-invariant function $f$ on $P$, the flow $\varphi_t$ of the Hamiltonian vector field $X_f$ associated to $f$ preserves $M$.*
*iii) When $f$ is a smooth $G$-invariant function on $P$, the restriction to $(M, \omega)$ of the Hamiltonian vector field $X_f$ is a Hamiltonian vector field on $(M, \omega_M)$ associated to the restriction of $f$ to $M$.*

**Proof** i) The proof of i) can be found in [10], [5].

ii) Since $f$ is $G$-invariant, $g \cdot \varphi_t(p) = \varphi_t(g \cdot p)$ for all $g \in G$, and $p \in P$. Hence if $g \in G_p$, then $g \in G_{\varphi_t(p)}$. Since $\varphi_t$ is a local diffeomorphism, we find that, if $g \in G_{\varphi_t(p)}$, then $g \in G_{\varphi_t^{-1}(\varphi_t(p))} = G_p$. Hence $G_{\varphi_t(p)} = G_p$ and $\varphi_t(p) \in P_K$ for all $p \in M$. Since $\varphi_t(p)$ and $p$ are in the same connected component of $P_K$, it follows that $\varphi_t(p) \in M$ for all $p \in M$. This proves ii).

iii) Since $M$ is a symplectic submanifold of $P$ for each $p \in M$, the *symplectic annihilator* $T_p^\omega M$ of $T_p M$, defined by

$$T_p^\omega M = \{u \in T_p P \mid \omega(p)(u, v) = 0 \; \forall \, v \in T_p M\}, \tag{7}$$

is a symplectic subspace of $T_p P$ complementary to $T_p M$, that is,

$$T_p P = T_p M \oplus T_p^\omega M. \tag{8}$$

Let $f$ be a $G$-invariant function on $P$. Let $\varphi_t$ be the flow of the Hamiltonian vector field $X_f$, which satisfies the equation $X_f \mathbin{\lrcorner} \omega = df$. Since $\varphi_t$ preserves $M$, $X_f$ is tangent to $M$. Hence for every $u \in T_p^\omega M$,

$$\langle df(p) \mid u \rangle = \omega(p)\big(X_f(p), u\big) = 0.$$

Therefore for every $v \in T_p M$, $(X_f \mathbin{\lrcorner} \omega) v = \langle df \mid v \rangle$, which implies that $X_f \mathbin{\lrcorner} \omega_M = d(f|_M)$. This proves iii). ∎

The normaliser of $K$ in $G$ is

$$N^K = \{g \in G \mid gKg^{-1} = K\}.$$

For every $p \in P$, $G_{g \cdot p} = g G_p g^{-1}$. Hence $g \in G$ preserves $P_K$ if and only if $g \in N^K$. Let $N_M$ be the subgroup of $N^K$ preserving the component $M \subseteq P_K$, that is,

$$N_M = \{g \in N^K \mid g \cdot p \in M \; \forall \, p \in M\}.$$

Note that $K$ is a normal subgroup of $N_M$. The subgroup $N_M$ contains the connected component of the identity of $N^K$ and is a closed subgroup of $G$. Let $\mathfrak{n}$ be the Lie algebra of $N_M$. For each $\xi \in \mathfrak{n}$ and each $p \in M$, we have

$$\exp(t\xi) \cdot p = \Phi\big(\exp(t\xi), p\big) = \Phi_p\big(\exp(t\xi)\big) \in M.$$

Hence $X^\xi(p) = T_e\Phi_p(\xi) \in T_pM$. For each $k \in K$, there exists $k' \in K$ such that $k \cdot \exp(t\xi) = \exp(t\xi) \cdot k'$. Hence

$$\begin{aligned}\Phi_k\Big(\Phi_p\big(\exp(t\xi)\big)\Big) &= \Phi_k\big(\exp(t\xi)\cdot p\big) = \Phi\big(k, \exp(t\xi)\cdot p\big) = \Phi\big(k\exp(t\xi), p\big)\\ &= \Phi\big(\exp(t\xi)k', p\big) = \Phi\big(\exp(t\xi), k'\cdot p\big) = \Phi\big(\exp(t\xi), p\big)\\ &= \Phi_p\big(\exp(t\xi)\big).\end{aligned}$$

Therefore

$$T_p\Phi_k\big(X^\xi(p)\big) = X^\xi(p) \quad \forall\, k \in K, \xi \in \mathfrak{n}, \text{ and } p \in M. \tag{9}$$

The quotient group $G_M = N_M/K$ is a Lie group which acts on $M$ by

$$\Phi_M \colon G_M \times M \to M \colon ([g], p) \mapsto \Phi(g, p), \tag{10}$$

where $[g] \in G_M$ is the coset containing $g \in N_M$.

***Theorem 2.2*** *The action $\Phi_M$ of $G_M$ on $M$ is free and proper.*

**Proof** The action $\Phi_M$ is free by construction of $G_M$. To prove properness we argue as follows. Suppose that the sequence $\{p_n\}$ of points in $M$ converges to $p \in P_K$ and let $\{[g_n]\}$ be a sequence of elements of $G_M$ such that $\Phi_M([g_n], p_n) \to p' \in M$. Then $\Phi(g_n, p_n) = \Phi_M([g_n], p_n) \to p'$. By properness of the action of $G$ on $P$, there is a subsequence $\{g_{n_m}\}$ in $N_M$ converging to $g \in G$ such that $\Phi(g, p) = p'$. Since $N_M$ is closed, the limit $g$ lies in $N_M$ and $p \in M$. Hence, the subsequence $\{[g_{n_m}]\}$ converges to $[g] \in G_M$ and $\Phi_M([g], p) = p'$. Thus the action $\Phi_M$ is proper. ■

***Corollary 2.3*** *The space $\overline{M} = M/G_M$ of $G_M$-orbits on $M$ is a connected manifold. The space $\pi(M) \subseteq \overline{P} = P/G$ has the structure of a smooth manifold induced by the natural bijection $\tau_M \colon \pi(M) \to \overline{M}$.*

**Proof** Since the action of $G_M$ on $M$ is free and proper, $\overline{M} = M/G_M$ is a smooth manifold. Let $\pi_M \colon M \to \overline{M}$ be the $G_M$-orbit map. Since $M$ is connected and $\pi_M$ is continuous, it follows that $\overline{M}$ is connected.

For each $p \in M$, $\pi(p) = G \cdot p$ is the orbit of $G$ through $p$. The intersection of $G \cdot p$ with $M$ is the unique $G_M$-orbit $\pi_M(p) = G_M \cdot p$ through $p$. In other words,

$$\pi(p) \cap M = G \cdot p \cap M = G_M \cdot p = \pi_M(p).$$

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Consequently, the map

$$\tau_M \colon \overline{M} \to \pi(M) \colon G_M \cdot p \mapsto G \cdot p,$$

is bijective. Moreover, $\tau_M$ induces a manifold structure on $\pi(M)$. ■

It should be noted that, we can have $\pi(M) = \pi(M')$ with $M \neq M'$. This happens if $M' = g \cdot M$ for some $g \in G$. The manifold structures of $\pi(M)$ obtained from $M$ and $M'$ coincide. The manifold $\pi(M)$ is called a stratum of $\overline{P}$. In the following we shall identify $\pi(M)$ with $\overline{M}$, and shall refer to $\overline{M}$ as a *stratum* of $\overline{P}$.

For each $p \in P_K$, the action $\Phi \mid (K \times P)$ of $K$ on $P$ induces a $K$-action $\Psi_p^K$ on $T_pP$. In more detail, given $p \in P_K$ for each $k \in K$ we have $\Phi_k(p) = p$. Hence the tangent at $p$ of $\Phi_k$ defines an action $\Psi_p^K$ on $T_pP$. The tangent space $T_pP_K$ consists of vectors $v \in T_pP$ which are invariant under this induced action. In other words,

$$T_pP_K = \{v \in T_pP \mid \Psi_k(v) = \Psi_p^K(k, v) = T_p\Phi_k(v) = v \,\forall\, k \in K\}.$$

For every $u \in T_pP$, the average of $u$ over $K$ is

$$\overline{u} = \int_K \Psi_k(u)\, dk = \int_K T_p\Phi_k(u)\, dk, \tag{11}$$

where $dk$ denotes Haar measure of $K$ normalised so that $\operatorname{vol} K = 1$. Let

$$T_p^{\perp}P_K = \{u \in T_pP \mid \overline{u} = 0\}. \tag{12}$$

Note that the $G$-invariant metric $\mathrm{g}$ on $P$ is $K$-invariant. We have

***Lemma 2.4*** *For every $K$-invariant metric* $\mathrm{k}$ *on $P$, the space $T_p^{\perp}P_K$ is the* $\mathrm{k}$*-orthogonal complement of $T_pP_K$. Moreover, $T_p^{\perp}P_K \subseteq \ker df$ for every $K$-invariant $f \in C^\infty(P)$.*

**Proof** Let $\mathrm{k}$ be a $K$-invariant metric on $P$. For every $u, v \in T_pP$, and $k \in K$, we have $\mathrm{k}\big(\Psi_k(u), \Psi_k(v)\big) = \mathrm{k}(u, v)$. If $v \in T_pP_K$, then $\Psi_k(v) = v$ for all $k \in K$. Hence,

$$\mathrm{k}(\overline{u}, v) = \mathrm{k}\left(\int_K \Psi_k(u)\, dk, v\right) = \int_K \mathrm{k}\big(\Psi_k(u), v\big)\, dk = \mathrm{k}(u, v)$$

for all $v \in T_pP_K$.

Suppose $u$ is $\mathrm{k}$-orthogonal to $T_pP_K$. Then, $\mathrm{k}(u, v) = 0$ and, therefore, $\mathrm{k}(\overline{u}, v) = 0$ for all $v \in T_pP_K$. This implies that $\overline{u}$ is $\mathrm{k}$-orthogonal to $T_pP_K$. But, $\overline{u}$ is $K$-invariant, which implies that $\overline{u} \in T_pP_K$. Therefore, $\overline{u} = 0$ and $u \in T_p^{\perp}P_K$.

Conversely, suppose that $u \in T_p^{\perp}P_K$, which means that $\overline{u} = 0$. Hence, for every $v \in T_pP_K$, $\mathrm{k}(u, v) = \mathrm{k}(\overline{u}, v) = 0$, which implies that $u$ is $\mathrm{k}$-orthogonal to $T_pP_K$. This proves the first statement of the lemma.

If $f \in C^\infty(P)$ is $K$-invariant, and $u \in T_p^{\perp}P_K$, then

$$\langle df \mid u \rangle = \langle d\Phi_k^* f \mid u \rangle = \langle df \mid T\Phi_k(u) \rangle = \langle df \mid \Psi_k(u) \rangle$$

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for all $k \in K$. Averaging over $K$, we get $\langle df \mid u\rangle = \langle df \mid \overline{u}\rangle = 0$. This implies that $T_p^{\perp} P_K \subseteq \ker df$. ■

In the following we shall need the slice theorem for proper actions due to Palais [19]. We state it here for completeness. A *slice* through $p \in P$ for an action $\Phi\colon G \times P \to P\colon (g, p') \mapsto g \cdot p'$ is a submanifold $S_p$ of $P$ containing $p$ such that

1. $S_p$ is transverse and complementary to the orbit $G \cdot p$ through $p$ at the point $p$, that is
$$T_pP = T_pS \oplus T_p(G \cdot p).$$
2. For every $p' \in S_p$, $S_p$ is transverse to $G \cdot p'$, that is
$$T_{p'}P = T_{p'}S + T_{p'}(G \cdot p').$$
3. $S_p$ is $G_p$-invariant.
4. For $p' \in S_p$ and $g \in G$, if $g \cdot p' \in S$ then $g \in G_p$.

Consider the $G_p$-action $\Psi_p = T\Phi \mid (G_p \times T_pP)$ on $T_pP$ and the $G_p$-action $\Phi_p = \Phi \mid (G_p \times P)$ on $P$. Let $\exp_p\colon T_pP \to P$ be the exponential map determined by the $G$-invariant Riemannian metric g on $P$. This map is a local diffeomorphism from a neighbourhood of $0 \in T_pP$ onto a neighbourhood of $p \in P$ with the property that, for every $g \in G$ and every $v \in T_pP$,
$$\exp_{g\cdot p}(\Psi_g v) = \Phi_g(\exp_p v).$$
Thus $\exp_p$ intertwines the $G_p$-action $\Psi_p$ with the $G_p$-action $\Phi_p$. Here we have used the notation $\Psi_g$ instead of $(\Psi_p)_g$.

***Theorem 2.5*** *Since the $G$-action $\Phi$ on $P$ is proper, for each $p \in P$ there is a neighbourhood $V_p$ of zero in* hor $T_pP$ *such that $S_p = \exp_p(V_p)$ is a slice at $p$ for the $G$-action $\Phi$.*

**Proof** See [19] or [8].

It follows from Theorem 2.1, that we have a $G$-invariant partition of the manifold $P$ into smooth manifolds $M$, given by

$$P = \bigcup_{K \text{ c.s. } G} \; \bigcup_{M \text{ c.c. } P_K} M, \tag{13}$$

where $K$ runs over compact subgroups of $G$ and $M$ over connected components of $P_K$. Its projection by the orbit map $\pi\colon P \to \overline{P}$ gives rise to a corresponding partition of the orbit space

$$\overline{P} = \bigcup_{K \text{ c.s. } G} \; \bigcup_{M \text{ c.c. } P_K} \overline{M}, \tag{14}$$

where $\overline{M} = \pi(M)$. The orbit space $\overline{P}$ is a (topological) quotient space of $P$. Corollary 2.3 ensures that each set $\overline{M}$ is a manifold. Its manifold topology is the same as the topology induced by the inclusion map $\iota_{\overline{M}}\colon \overline{M} \to \overline{P}$. We want to describe how the manifolds $\overline{M}$ fit together in $\overline{P}$. In order to do so, we employ the notion of a differential space.

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## 3 Differential Spaces

In this section we review the notion of a differential space introduced by Sikorski [23] to describe the differential structure of the orbit space $\overline{P}$, and then prove that strata $\overline{M}$ are submanifolds of $\overline{P}$.

A *differential structure* on a topological space $Q$ is a set $C^\infty(Q)$ of continuous functions on $Q$ which has the following properties.

I. The topology of $Q$ is generated by functions in $C^\infty(Q)$, that is, the collection

$$\{f^{-1}(V) \mid f \in C^\infty(Q) \text{ where } V \text{ is an open subset of } \mathbb{R}\}$$

is a subbasis for the topology of $Q$.

II. For every $F \in C^\infty(\mathbb{R}^n)$ and every $f_1, \dots, f_n \in C^\infty(Q)$, $F(f_1, \dots, f_n) \in C^\infty(Q)$.

III. If $f\colon Q \to \mathbb{R}$ is a function such that, for every $p \in Q$ there is an open neighbourhood $U$ of $p$ in $Q$ and a function $f_U \in C^\infty(Q)$ satisfying $f|U = f_U|U$, then $f \in C^\infty(Q)$.

A topological space $Q$ endowed with a differential structure $C^\infty(Q)$ is called a *differential space* [23, Sec. 6]. An element of $C^\infty(Q)$ is called a *smooth* function. Thus $C^\infty(Q)$ is the set of smooth functions on $Q$. From property II it follows that $C^\infty(Q)$ is a commutative ring under addition and pointwise multiplication.

***Example 3.1*** If $Q$ is a smooth manifold, then the collection of smooth functions on $Q$, defined in terms of the manifold structure of $Q$, is a differential structure on $Q$ [23].

Let $N$ and $Q$ be differential spaces with differential structures $C^\infty(N)$ and $C^\infty(Q)$, respectively, and let $\mu\colon N \to Q$ be a continuous map. We say that $\mu$ is *smooth* if $f \circ \mu \in C^\infty(N)$ for every $f \in C^\infty(Q)$. Furthermore, a smooth map $\mu\colon N \to Q$ is a *diffeomorphism* if it is invertible and $\mu^{-1}\colon Q \to N$ is smooth.

***Theorem 3.2*** *For every subset $Q$ of a differential space $N$ the inclusion map $\iota_Q\colon Q \hookrightarrow N$ induces a differential structure on $Q$. A function $f\colon Q \to \mathbb{R}$ is in $C^\infty(Q)$ if and only if, for every $q \in Q$, there is an open neighbourhood $U$ of $q$ in $N$ and a function $f_U \in C^\infty(N)$ such that $f \mid (Q \cap U) = f_U \mid (Q \cap U)$. In this differential structure on $Q$, the inclusion map $\iota_Q\colon Q \hookrightarrow N$ is smooth.*

**Proof** See [23].

A differential space $\big(N, C^\infty(N)\big)$ is a *manifold of dimension $n$* if, for each $p \in N$, there exists a neighbourhood $U_p$ of $p$ in $N$ and functions $f_1, \dots, f_n$ in $C^\infty(N)$ such that $(f_1|_{U_p}, \dots, f_n|_{U_p})\colon U_p \to \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$. A subset $Q$ of a differential space $N$ is a *submanifold* of $N$ if it is a manifold in the differential structure on $Q$ induced by the inclusion map $Q \hookrightarrow N$.

Let $C^\infty(N)$ be a differential structure on $N$. For each $p \in N$, a *tangent vector* to $N$ at $p$ is a linear mapping $v\colon C^\infty(N) \to \mathbb{R}$ satisfying Leibniz' rule: $v(f_1 f_2) = v(f_1) f_2(p) + f_1(p) v(f_2)$ for all $f_1, f_2 \in C^\infty(N)$. In other words, tangent vectors at

$p \in N$ are derivations at $p$ of smooth functions on $N$. The space of vectors tangent at $p$ to $N$ is a vector space and will be denoted $T_pN$. If $N$ is not a manifold then dim $T_pN$ may depend on $p \in N$. The space of all tangent vectors to $N$ will be denoted by $TN$.

Let $\mu\colon N \to Q$ be a smooth map between differential spaces $N$ and $Q$. The *derived map* $T\mu\colon TN \to TQ$ associates to each vector $v \in T_pN$ a vector $T\mu(v) \in T_{\mu(p)}Q$ such that

$$\big(T_p\mu(v)\big)\, f = v(f \circ \mu) \quad \forall\, f \in C^\infty(Q).$$

For each $p \in N$, the restriction of $T\mu$ to $T_pN$ is a linear map $T_p\mu\colon T_pN \to T_{\mu(p)}Q$. A smooth map $\mu\colon N \to Q$ between differential spaces $N$ and $Q$ is an *immersion* if $T_p\mu\colon T_pN \to T_{\mu(p)}Q$ is injective for all $p \in N$. The map $\mu$ is a *submersion* if $T_p\mu\colon T_pN \to T_{\mu(p)}Q$ is surjective.

***Proposition 3.3*** *If $N$ is a closed subset of a smooth paracompact manifold $Q$ then smooth functions on $N$ extend to smooth functions on $Q$.*

**Proof** Let $f \in C^\infty(N)$ and $\{U_p \mid p \in N\}$ be a covering of $N$ by open sets in $Q$ such that for each $p \in N$, there exists an open set $U_p$ containing $p$ and a function $f_{U_p} \in C^\infty(Q)$ satisfying $f_{U_p}|U_p \cap N = f|U_p \cap N$. Since $N$ is closed in $Q$, its complement $N'$ is open in $Q$ and the family $\{U_p \mid p \in N\} \cup N'$ is an open covering of $Q$. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to this covering. Each $\varphi_\alpha \in C^\infty(Q)$ has support in some $U_{p_\alpha}$ or in $N'$. Moreover $\sum_\alpha \varphi_\alpha = 1$. Let $g = \sum_\alpha \varphi_\alpha f_{U_{p_\alpha}}$, where the sum is taken over $\alpha$ such that the support of $\varphi_\alpha$ has nonempty intersection with $N$. Clearly, $g \in C^\infty(Q)$. Since $N' \cap M = \varnothing$, it follows that $g|M = f$. ■

If $N$ is not closed in $Q$, and $\{p_n\}$ is a sequence of points in $N$ converging to $p \notin N$, then we can construct a smooth function $f$ on $N$ such that $f(p_n) \to \infty$ as $n \to \infty$. Hence $f$ cannot be the restriction to $N$ of a function on $C^\infty(Q)$.

***Theorem 3.4*** *Let $\overline{P} = P/G$ be the space of $G$-orbits of a smooth proper action $\Phi$ of a Lie group $G$ on a smooth manifold $P$ with orbit map $\pi\colon P \to \overline{P}$. Then $P$ is a differential space with differential structure $C^\infty(\overline{P})$ consisting of functions $\bar f\colon \overline{P} \to \mathbb{R}$ such that $\pi^*\bar f \in C^\infty(P)$.*

**Proof** Property I. It suffices to show that given $\bar p \in \overline{P}$ and an open neighbourhood $\overline{U}$ of $\bar p$ in $\overline{P}$, there is a smooth function $\bar f$ on $\overline{P}$ such that $\bar f^{-1}(0,1)$ is an open neighbourhood of $\bar p$ contained in $\overline{U}$. Let $p \in \pi^{-1}(\bar p)$ and let $S_p$ be a slice to the $G$-action on $P$ at $p$. Then $V = S_p \cap \rho^{-1}(\overline{U})$ is an open neighbourhood of $p$ in $S_p$. There is a smooth $G_p$-invariant nonnegative function $\widetilde{f}$ on $S_p$ whose support is a compact subset contained in $V$ which contains $p$ and whose range is contained in $[0, \frac{1}{2}]$. Define the function $f$ by $f\big(\Phi_g(v)\big) = \widetilde{f}(v)$ for every $g \in G$ and every $v \in V$. Then $f$ is a smooth $G$-invariant function on $P$ with support contained in $G \cdot V$ and whose range is contained in $[0, \frac{1}{2}]$. Thus $f$ induces a smooth function $\bar f$ on $\overline{P}$ such that $\bar f^{-1}(0,1)$ is an open subset of $\overline{U}$ containing $\bar p$.

Property II follows immediately from the fact that property II holds for the ring $C^\infty(P)^G$ of $G$-invariant smooth functions on $P$.

We now prove property III. Let $\bar{f}\colon \overline{P} \to \mathbb{R}$ be a function such that for each $\bar{p} \in \overline{P}$ there is an open neighbourhood $\overline{U}$ of $\bar{p}$ in $\overline{P}$ and a smooth function $\bar{f}_{\overline{U}}$ on $\overline{P}$ so that $\bar{f}|\overline{U} = \bar{f}_{\overline{U}}|\overline{U}$. Now $\pi^*\bar{f}\colon P \to \mathbb{R}$ is $G$-invariant and

$$\pi^*\bar{f}|\pi^{-1}(\overline{U}) = \pi^*\bar{f}_{\overline{U}}|\pi^{-1}(\overline{U}).$$

But $\pi^*\bar{f}_{\overline{U}} \in C^\infty(P)^G$. Hence $\pi^*\bar{f} \in C^\infty(P)^G$, which implies that $\bar{f} \in C^\infty(\overline{P})$. ■

***Lemma 3.5*** *Let $M$ be a connected component of $P_K$. For each $G_M$-invariant function $f_M \in C^\infty(M)$ and every $p \in M$, there exists a neighbourhood $U$ of $p$ in $P$ and a $G$-invariant function $f \in C^\infty(P)$ such that $f|M \cap U = f_M|M \cap U$.*

**Proof** Let $S$ be a slice through $p$ for the action of $G$ on $P$. Then $S \cap M$ is a slice through $p$ for the action of $G_M$ on $M$. Since $S \cap M$ is closed in $S$ we can extend $f_M|S \cap M$ to a smooth function $\widetilde{f}_S$ on $S$. The isotropy group $K$ of $p$ is compact and it acts on $S$. By averaging over $K$, we can construct a neighbourhood $V$ of $p$ in $S$ and a smooth $K$-invariant function $f_S$ on $S$ with compact support such that $f_S|V \cap M = f_M|V \cap M$.

The set $U = G \cdot V$ is $G$-invariant and open in $P$. We define a function $f$ on $P$ as follows. If $p' \notin U$ then $f(p') = 0$. If $p' \in U$, then $f(p') = f_S(g \cdot p')$ where $g \in G$ is such that $g \cdot p' \in S$. If $\overline{g}$ is another element of $G$ such that $\overline{g} \cdot p' \in S$, then $\overline{g}g^{-1}$ maps $g \cdot p' \in S$ to $\overline{g} \cdot p' \in S$, which implies that $\overline{g}g^{-1} \in K$. Hence $f_S(g \cdot p') = f_S\big((\overline{g}g^{-1})(g \cdot p')\big) = f_S(\overline{g} \cdot p')$ because $f_S$ is $K$-invariant. Therefore $f$ is well defined.

Next, we want to show that $f$ is $G$-invariant. If $p' \notin U$, then $g \cdot p' \notin U$ for all $g \in G$, and $f(g \cdot p') = f(p') = 0$. If $p' \in U$ and $g \cdot p' \in S$ then, for every $\overline{g} \in G$, $\overline{g} \cdot p' \in U$ and $(g\overline{g}^{-1})\overline{g} \cdot p' \in S$. Therefore, since $S$ is a slice $g\overline{g}^{-1} \in K$, which implies that $\overline{g}g^{-1} \in K$. Hence

$$f(\overline{g} \cdot p') = f_S\big((\overline{g}g^{-1})g \cdot p'\big) = f_S(g \cdot p') = f(p').$$

Therefore, $f$ is $G$-invariant.

Since $S \cap M$ is a slice at $p$ for the action of $G_M = N_M/K$ on $M$, if $p' \in M \cap U$ there exists $g \in G_M$ such that $g \cdot p' \in S \cap M$. Hence,

$$f(p') = f_S(g \cdot p') = f_M(g \cdot p') = f_M(p')$$

because $f_M$ is $G_M$-invariant. Therefore, $f$ is a $G$-invariant smooth function on $P$ such that $f|U \cap M = f_M|U \cap M$. ■

Let $\overline{M}$ be the space of $G_M$ orbits on $M$ and $\pi_M\colon M \to \overline{M}$ the orbit map. Since the action of $G_M$ on $M$ is free and proper, connected components of $\overline{M}$ are quotient manifolds of the corresponding connected components of $M$.

***Theorem 3.6*** *The map $\bar{\iota}_M\colon \overline{M} \to \overline{P}\colon G_M \cdot p \mapsto G \cdot p$ is smooth. It induces a diffeomorphism $\tau_M\colon \overline{M} \to \pi(M)$, where the differential structure on $\pi(M)$ is induced by the inclusion map $\pi(M) \hookrightarrow \overline{P}$. Hence, $\pi(M)$ is a submanifold of $\overline{P}$.*

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**Proof** Let $\iota_M\colon M \to P$ be the inclusion map. Then, $\bar{\iota}_M \circ \pi_M = \pi \circ \iota_M$. Moreover, for each $f \in C^\infty(P)$, $\iota_M^* f \in C^\infty(M)$ is the restriction of $f$ to $M$. If $f$ is $G$-invariant, then $\iota_M^* f$ is $G_M$-invariant.

Let $\bar{f} \in C^\infty(\overline{P})$, then $f = \pi^*\bar{f} \in C^\infty(P)$ is $G$-invariant. Therefore, $\iota_M^*\pi^*\bar{f} \in C^\infty(M)$ is $G_M$-invariant and it pushes forward to a function $\bar{f}_M \in C^\infty(\overline{M})$ such that $\iota_M^*\pi^*\bar{f} = \pi_M^*\bar{f}_M$. But $\iota_M^*\pi^*\bar{f} = \pi_M^*\bar{\iota}_M^*\bar{f}$. Hence, $\pi_M^*\bar{\iota}_M^*\bar{f} \in C^\infty(M)$ which implies that $\bar{\iota}_M^*\bar{f} \in C^\infty(\overline{M})$. Thus, $\bar{\iota}_M\colon \overline{M} \to \overline{P}$ is smooth. Hence, the induced map $\tau_M\colon \overline{M} \to \pi(M)$ is smooth with respect to the differential structure on $\pi(M)$ induced by its inclusion in $\overline{P}$.

Clearly, $\bar{\iota}_M\colon \overline{M} \to \overline{P}$ is a bijection of $\overline{M}$ onto $\pi(M)$. Hence, the induced map $\tau_M\colon \overline{M} \to \pi(M)$ is invertible. In order to show that $\tau_M$ is a diffeomorphism, it suffices to show that its inverse $\tau_M^{-1}\colon \pi(M) \to \overline{M}$ is smooth. In other words, it suffices to show that, for each function $\bar{f}_M \in C^\infty(\overline{M})$, $(\tau_M^{-1})^*\bar{f}_M$ is in $C^\infty\big(\pi(M)\big)$. Here $C^\infty\big(\pi(M)\big)$ is the differential structure of $\pi(M)$ induced by the inclusion map $\pi(M) \hookrightarrow \overline{P}$.

Since $\pi_M^*\bar{f}_M$ is a $G_M$-invariant function on $M$, Lemma 3.5 ensures that, for every $p \in M$, there exist an open $G$-invariant neighbourhood $U$ of $p$ in $P$ and a $G$-invariant function $f' \in C^\infty(P)$ such that $f'|U \cap M = \pi_M^*\bar{f}_M|U \cap M$. Let $\bar{f}' \in C^\infty(\overline{P})$ be the push forward of $f'$ by $\pi$. In other words, $\pi^*\bar{f}' = f'$.

The orbit map $\pi\colon P \to \overline{P}$ is open. Hence, $\overline{U} = \pi(U)$ is open in $\overline{P}$. Given $\bar{p}' \in \overline{U} \cap \pi(M)$ let $p' \in U \cap M$ be such that $\pi(p') = \bar{p}'$. Then, $\iota_M(p')$ is $p'$, considered as a point in $M$. So $\pi_M(p') = \pi_M\big(\iota_M(p')\big) = \bar{\iota}_M^{-1}(\bar{p}')$. We have

$$\begin{aligned}(\bar{\iota}_M^{-1})^*\bar{f}_M(\bar{p}') &= \bar{f}_M\big(\bar{\iota}_M^{-1}(\bar{p}')\big) = \bar{f}_M\big(\pi_M(p')\big)\\ &= \pi_M^*\bar{f}_M(p') = f'(p') = \pi^*\bar{f}'(\bar{p}').\end{aligned}$$

Hence, $(\tau_M^{-1})^*\bar{f}_M|\overline{U} \cap \pi(M) = \bar{f}'|\overline{U} \cap \pi(M)$, where $\bar{f}' \in C^\infty(\overline{P})$. This implies that $(\tau_M^{-1})^*\bar{f}_M \in C^\infty\big(\pi(M)\big)$. Hence, $\tau_M^{-1}$ is smooth.

Since $\tau_M\colon \overline{M} \to \pi(M)$ is a diffeomorphism in the differential structure on $\pi(M)$ induced by the inclusion map $\pi(M) \hookrightarrow \overline{P}$ and connected components of $\overline{M}$ are manifolds, it follows that connected components of $\pi(M)$ are submanifolds of $\overline{P}$. This completes the proof of Theorem 3.6. ■

Observe that Theorem 3.6 is almost a restatement of Corollary 2.3 in the category of differential spaces. The main difference is the statement that $\overline{M} = \pi(M)$ is a submanifold of $\overline{P}$. Here we used the identification of $\overline{M}$ with $\pi(M)$ given by the diffeomorphism $\tau_M\colon \overline{M} \to \pi(M)$. This ensures that the partition (14) is a partition of the differential space $\overline{P}$ into submanifolds.

In Theorem 3.4 we have shown that the $G$-orbit space $\overline{P}$ of a smooth and proper action $\Phi$ of a Lie group $G$ on a smooth manifold $P$ is a differential space. According to Theorem 3.6 it is partitioned into submanifolds $\overline{M}$. We now discuss how these submanifolds fit together. This requires some further preparation. In the next three sections we verify the decomposition given by (5), then discuss the notion of the tangent wedge, and finally describe the links of the stratification of $\overline{P}$.

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## 4 Lifted Action

We start by looking at the lifted action $\Psi\colon G \times TP \to TP$ (4). Since the $G$-action $\Phi$ on $P$ is proper, it follows from Proposition 3.3 that the space $(TP)/G$ of $G$-orbits on $TP$ is a differential space, and the orbit mapping $\rho\colon TP \to (TP)/G$ of the lifted action $\Psi$ is smooth. Proposition 3.4 implies that ver $TP$ and hor $TP$ are differential spaces. Let $\rho_{\mathrm{ver}}\colon \operatorname{ver} TP \to (\operatorname{ver} TP)/G$ and $\rho_{\mathrm{hor}}\colon \operatorname{hor} TP \to (\operatorname{hor} TP)/G$ be the restrictions of $\rho$ to ver $TP$ and hor $TP$, respectively. Denote by $C^\infty(\operatorname{ver} TP)$ and $C^\infty(\operatorname{hor} TP)$ the differential structures induced by the inclusions $j_{\mathrm{ver}}\colon \operatorname{ver} TP \hookrightarrow TP$ and $j_{\mathrm{hor}}\colon \operatorname{hor} TP \hookrightarrow TP$. Similarly, let $C^\infty\big((\operatorname{ver} TP)/G\big)$ and $C^\infty\big((\operatorname{hor} TP)/G\big)$ be the differential structures induced by the inclusions $\iota_{\mathrm{ver}}\colon (\operatorname{ver} TP)/G \hookrightarrow (TP)/G$ and $\iota_{\mathrm{hor}}\colon (\operatorname{hor} TP)/G \hookrightarrow (TP)/G$.

***Lemma 4.1*** *The mappings* $\rho_{\mathrm{ver}}\colon \operatorname{ver} TP \to (\operatorname{ver} TP)/G$ *and* $\rho_{\mathrm{hor}}\colon \operatorname{hor} TP \to (\operatorname{hor} TP)/G$ *are smooth.*

**Proof** By Proposition 3.4, for every function $\bar f \in C^\infty\big((\operatorname{ver} TP)/G\big)$ and every $\bar v \in (\operatorname{ver} TP)/G$, there exists a neighbourhood $\overline{U}$ of $\bar v \in (\operatorname{ver} TP)/G$ and $\bar f_{\overline{U}} \in C^\infty\big((TP)/G\big)$ such that

$$\bar f \mid \big(\overline{U} \cap (\operatorname{ver} TP)/G\big) = \bar f_{\overline{U}} \mid \big(\overline{U} \cap (\operatorname{ver} TP)/G\big).$$

Let $U = \rho_{\mathrm{ver}}^{-1}(\overline{U})$, $f_U = \rho_{\mathrm{ver}}^* \bar f_{\overline{U}}$, and $f = \rho_{\mathrm{ver}}^* \bar f$. Proposition 2.3 ensures that $f_U \in C^\infty(TP)$. Moreover,

$$f \mid (U \cap \operatorname{ver} TP) = f_U \mid (U \cap \operatorname{ver} TP). \tag{15}$$

Since $\{\overline{U}\}$ forms a covering of $(\operatorname{ver} TP)/G$, the collection $\{U = \rho_{\mathrm{ver}}^{-1}(\overline{U})\}$ forms a covering of ver $TP$. Thus from (15) and property III, it follows that $f = \rho_{\mathrm{ver}}^* \bar f \in C^\infty(\operatorname{ver} TP)$. Hence, the map $\rho_{\mathrm{ver}}\colon \operatorname{ver} TP \to (\operatorname{ver} TP)/G$ is smooth. A similar argument proves the smoothness of $\rho_{\mathrm{hor}}\colon \operatorname{hor} TP \to (\operatorname{hor} TP)/G$. ■

Let $\tau\colon TP \to P$ be the tangent bundle projection map. It intertwines the lifted $G$-action $\Psi$ on $TP$ with the $G$-action $\Phi$ on $P$, that is, $\tau\big(\Psi_g(u)\big) = \Phi_g\big(\tau(u)\big)$ for every $g \in G$ and every $u \in TP$. Hence, $\tau$ induces a smooth map $\overline{\tau}\colon (TP)/G \to P/G = \overline{P}$ between differential spaces. The maps $\pi_{\mathrm{ver}}\colon (\operatorname{ver} TP)/G \to \overline{P}$ and $\pi_{\mathrm{hor}}\colon (\operatorname{hor} TP)/G \to \overline{P}$, defined by $\pi_{\mathrm{ver}} = \pi \circ \overline{\tau} \circ \iota_{\mathrm{ver}}$ and $\pi_{\mathrm{hor}} = \pi \circ \overline{\tau} \circ \iota_{\mathrm{hor}}$, respectively, are smooth maps between differential spaces, because the maps $\pi, \overline{\tau}, \iota_{\mathrm{ver}}$ and $\iota_{\mathrm{hor}}$ are smooth.

Next we study the differential space $(\operatorname{ver} TP)/G$. Denote by Gauge$(P)$ the group of diffeomorphisms of $P$ which commute with the $G$-action $\Phi$ and induce the identity transformation on the $G$-orbit space $\overline{P}$. Let gauge$(P)$ be the set of infinitesimal gauge transformations. The elements of gauge$(P)$ are smooth $G$-invariant vector fields on $P$ with values in ver $TP$.

***Theorem 4.2*** *Let* $X \in \operatorname{gauge}(P)$. *Then the vector field* $X$ *on* $P$ *is complete.*

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**Proof** Let $q \in P$, where $X(q) \neq 0$, and suppose that $t \mapsto \varphi_t(q)$ is the integral curve $\gamma$ of $X$ starting at $q$. Then there is a positive time $t_0$ such that $\gamma$ is defined on $[-t_0, t_0]$. Since $X \in \mathrm{gauge}(P)$, the integral curve $\gamma$ lies on the $G$-orbit through $q$. Thus there is a $g_0 \in G$ such that

$$\varphi_{t_0}(q) = g_0 \cdot q = \Phi_{g_0}(q).$$

Now

$$\varphi_{2t_0}(q) = \varphi_{t_0}\big(\varphi_{t_0}(q)\big) = \varphi_{t_0}(g_0 \cdot q) = g_0 \cdot \varphi_{t_0}(q),$$

since $X$ is $G$-invariant. Therefore by induction, for every $n \in \mathbb{Z}$, we have

$$\varphi_{nt_0}(q) = g_0^{n-1} \cdot \varphi_{t_0}(q).$$

Hence the integral curve $\gamma$ is defined for all $t \in \mathbb{R}$, that is, the vector field $X$ is complete. ■

From Theorem 4.2 it follows that $\mathrm{gauge}(P)$ is the Lie algebra of the gauge group $\mathrm{Gauge}(P)$. Note that $\mathrm{gauge}(P)$ consists of smooth $G$-invariant sections of the bundle $\tau_{\mathrm{ver}} = \tau \circ j_{\mathrm{ver}} \colon \mathrm{ver}\, TP \to \overline{P}$. There is a natural bijection between smooth $G$-invariant sections of the bundle $\tau_{\mathrm{ver}}$ and smooth sections of $\pi_{\mathrm{ver}} \colon (\mathrm{ver}\, TP)/G \to \overline{P}$. Thus the first summand $(\mathrm{ver}\, TP)/G$ in (5) is closely related to the Lie algebra $\mathrm{gauge}(P)$.

## 5 Tangent Wedge

In this section we study $T^{\mathrm{w}}\overline{P} = (\mathrm{hor}\, TP)/G$, which is the second summand in (5). For each $\bar{p} \in \overline{P}$, $T^{\mathrm{w}}_{\bar{p}}\overline{P}$ is the *tangent wedge* of $\overline{P}$ at $\bar{p}$.

For each $\bar{p} \in \overline{P}$, $p \in \pi^{-1}(\bar{p})$ and each $g \in G$, we have $\mathrm{hor}\, T_{g\cdot p}P = \Psi_g(\mathrm{hor}\, T_pP)$. Hence,

$$T^{\mathrm{w}}_{\bar{p}}\overline{P} = \rho_{\mathrm{hor}}(\mathrm{hor}\, T_{\pi^{-1}(\bar{p})}P) = (\mathrm{hor}\, T_{\pi^{-1}(\bar{p})}P)/G = (\mathrm{hor}\, T_pP)/G_p.$$

For any $N \subseteq P$ we have used the notation $T_NP$ to denote $\{v \in T_pP \mid p \in N\}$. Theorem 2.6 ensures that there is a neighbourhood $V_p$ of zero in $\mathrm{hor}\, T_pP$ such that $S_p = \exp_p(V_p)$ is a slice at $p$ for the action of $G$ on $P$. By definition of a slice, $V_p$ is $\Psi_p$-invariant and $U = \pi(S_p)$ is a neighbourhood of $\bar{p} = \pi(p)$ in $\overline{P} = P/G$.

***Theorem 5.1*** *For each $\bar{p} \in \overline{P}$, there is a neighbourhood of* 0 *in $T^{\mathrm{w}}_{\bar{p}}\overline{P}$, which is* diffeomorphic *to a neighbourhood of $\bar{p}$ in $\overline{P}$.*

**Proof** Using the notation above, consider the map $\varphi = \pi \circ \exp_p \colon V_p \to U$. First we note that $\varphi$ is continuous. From the facts that $V_p$ is $G_p$-invariant and the map $\exp_p$ intertwines the $G_p$-actions $\Psi_p$ and $\Phi_p$, it follows that $\varphi$ is $G_p$-invariant. Consequently, $\varphi$ induces a map $\overline{\varphi} \colon V_p/G_p \to U$, which is a homeomorphism, since $\exp_p$

induces a homeomorphism between $V_p/G_p$ and $S_p/G_p$, and $S_p/G_p$ is homeomorphic to $U$, see [5].

Next we show that the map $\varphi$ is smooth. Suppose that $\bar{f} \in C^\infty(U)$. Then $f = \pi^* \bar{f} \in C^\infty\big(\pi^{-1}(U)\big)$. Hence for every $p \in \pi^{-1}(\bar{p})$, the function $f|S_p$ on the slice $S_p$ is smooth. Because $S_p = \exp_p V_p$ and the exponential map $\exp_q$ is smooth, the function $\exp_p^* f$ on $V_p$ is smooth. Consequently, the mapping $\varphi$ is smooth. Since $\varphi$ is $G_p$-invariant, it induces a smooth map $\overline{\varphi}\colon V_p/G_p \subseteq (\mathrm{hor}\, T_pP)/G_p \to U \subseteq \overline{P}$. The map $\overline{\varphi}$ is invertible and has a continuous inverse, which we denote by $\sigma$.

All we have to do is to show that $\sigma$ is smooth. Towards this goal, let $\bar{f} \in C^\infty(V_p/G_p)$, then $f = \rho_{\mathrm{hor}}^* \bar{f} \in C^\infty(V_p)$, which implies that $h = \big((\exp_p)^{-1}\big)^* f \in C^\infty(S_p)$. Since $h$ is $G_p$-invariant, it extends to a smooth $G$-invariant function $\widetilde{h} \in C^\infty(G \cdot S_p)^G$, which corresponds to a smooth function $\overline{h}$ on $C^\infty\big(\pi(S_p)\big) = C^\infty(U)$. From $\bar{f} \in C^\infty(V_p/G_p)$ we see that $\sigma^* \bar{f}$ is a *continuous* function on $U$ and hence that $\pi^*(\sigma^* \bar{f})$ is a continuous $G$-invariant function on $G \cdot S_p$. To finish the argument we need to show that $\pi^*(\sigma^* \bar{f})$ is a *smooth* function. This follows from:

***Lemma 5.2*** *We have*

$$\pi^*(\sigma^* \bar{f}) = \widetilde{h}. \tag{16}$$

**Proof** We use the notation of the preceding argument. Let $p' \in G \cdot S_p$, and $p'' = g \cdot p'$. Then

$$\begin{aligned}\widetilde{h}(p') = \widetilde{h}(g \cdot p') = h(p'') = \big((\exp_p)^{-1}\big)^* f(p'') = f\big((\exp_p)^{-1} p''\big)\\ = \rho_{\mathrm{hor}}^* \bar{f}\big((\exp_p)^{-1} p''\big) = \bar{f}\big(\rho_{\mathrm{hor}}(\exp_p)^{-1} p''\big) = \bar{f}\big(\psi(p'')\big),\end{aligned}$$

where $\psi\colon S_p \to V_p/G_p$ is the mapping $\rho_{\mathrm{hor}} \circ (\exp_p)^{-1}$.

The following computation shows that $\overline{\varphi}\big(\psi(p'')\big) = \pi(p'')$. For every $g, h \in G_p$ we have

$$\begin{aligned}\overline{\varphi}\big(\psi(p'')\big) &= \overline{\varphi}\Big(\rho_{\mathrm{hor}}\big(\exp_p^{-1}(p'')\big)\Big) = \overline{\varphi}\big(\Psi_{g\cdot p} \circ \exp_p^{-1}(p'')\big)\\ &= \overline{\varphi}\Big(\exp_{g\cdot p}^{-1}\big(\Phi_g(p'')\big)\Big) = \varphi\Big(\Psi_h \circ \exp_{g\cdot p}^{-1}\big(\Phi_g(p'')\big)\Big)\\ &= \varphi\Big(\exp_{(hg)\cdot p}^{-1}\big(\Phi_{hg}(p'')\big)\Big) = \pi\big(\Phi_{hg}(p'')\big) = \pi(p'').\end{aligned}$$

Consequently,

$$h(q'') = \bar{f}\big(\psi(q'')\big) = \sigma^* \bar{f}\Big(\overline{\varphi}\big(\psi(q'')\big)\Big) = \sigma^* \bar{f}\big(\pi(q'')\big) = \sigma^* \bar{f}\big(\pi(q'')\big),$$

which implies, $\widetilde{h} = \pi^*(\sigma^* \bar{f})$. This proves Lemma 5.2. ■

From Lemma 5.2 it follows that for every $\bar{f} \in C^\infty(V_p/G_p)$ the map $\sigma^* \bar{f} = \overline{h} \in C^\infty(U)$. Hence $\sigma$ is smooth. Since $\sigma = \overline{\varphi}^{-1}\colon U \to V_p/G_p$ and $\overline{\varphi}$ is smooth, we see that $V_p/G_p$ is diffeomorphic to $U$. This proves Theorem 5.1. ■

For every $p \in P_K$ the tangent space $T_pP$ has a decomposition

$$T_pP = T_pP_K \times T_p^{\perp}P_K, \tag{17}$$

where $T_p^{\perp}P_K$ is the g-orthogonal complement of $T_pP_K$ in $T_pP$. The space $T_pP_K$ consists of $\Psi_p^K$-invariant vectors in $T_pP$. Since the metric g is $K$-invariant, Lemma 2.4 shows that $T_p^{\perp}P_K$ consists of vectors $u \in T_pP$ whose $\Psi_p^K$-average over $K$ vanishes.

***Lemma 5.3*** *We have the following decomposition*

$$\text{hor } T_pP = (\text{hor } T_pP_K) \times (\text{hor } T_p^{\perp}P_K). \tag{18}$$

**Proof** Let $u \in \text{hor } T_pP$ and write $u = \overline{u} + (u - \overline{u})$ where $\overline{u}$ is the $\Psi_p^K$-average of $u$ over $K$, see (11). Since hor $T_pP$ is $G_p$-invariant, it is $\Psi_p^K$-invariant. Consequently, both $u$ and $\overline{u}$ lie in hor $T_pP$. But $(\Psi_p^K)_k\overline{u} = \overline{u}$ for every $k \in K$, which implies that $\overline{u} \in T_pP_K$. Hence $\overline{u} \in (\text{hor } T_pP) \cap T_pP_K = \text{hor } T_pP_K$. Also $\overline{(u - \overline{u})} = 0$, which implies that $u - \overline{u} \in \text{hor } T_p^{\perp}P_K = (\text{hor } T_pP) \cap T_p^{\perp}P_K$. Thus

$$\text{hor } T_pP = (\text{hor } T_pP_K) + (\text{hor } T_p^{\perp}P_K).$$

If $u \in (\text{hor } T_pP_K) \cap (\text{hor } T_p^{\perp}P_K)$, then $\overline{u} = 0$ and $(\Psi_p^K)_k u = u$, for every $k \in K$. Consequently,

$$u = \int_K u \, dk = \int_K (\Psi_p^K)_k u \, dk = \overline{u} = 0.$$

Thus (18) holds. ■

For $p \in P_K$ we have denoted by $M$ the connected component of $P_K$ containing $p$, $\overline{M} = \pi(M)$ and $\bar{p} = \pi(p) \in \overline{M}$. Then the tangent space $T_{\bar{p}}\overline{M}$, which is isomorphic to $(\text{hor } T_pP_K)/K$, is a subset of the tangent wedge $T_{\bar{p}}^{\mathrm{w}}\overline{P}$ since hor $T_pP_K$ is contained in hor $T_pP$ and $K = G_p$. Let

$$T_{\bar{p}}^{\mathrm{c}}\overline{P} = (\text{hor } T_p^{\perp}P_K)/K. \tag{19}$$

Clearly, $T_{\bar{p}}^{\mathrm{c}}\overline{P}$ is independent of the choice of $p \in \pi^{-1}(\bar{p})$. We refer to $T_{\bar{p}}^{\mathrm{c}}\overline{P}$ as the *tangent cone* to $\overline{P}$ at $\bar{p}$. The decomposition (18) gives

$$T_{\bar{p}}^{\mathrm{w}}\overline{P} = T_{\bar{p}}\overline{M} \times T_{\bar{p}}^{\mathrm{c}}\overline{P}.$$

In other words, the tangent wedge to $\overline{P}$ at $\bar{p}$ is the direct sum of the (Zariski) tangent space to the manifold $\overline{M}$ at $\bar{p}$ and the tangent cone to $\overline{P}$ at $\bar{p}$.

The following result characterizes the (Zariski) tangent space $T_{\bar{p}}\overline{M}$ to $\overline{M}$ at $\bar{p}$.

***Theorem 5.4*** *Let $\iota_{\overline{M}} \colon \overline{M} \hookrightarrow \overline{P}$ be the inclusion map. Then for each $\bar{p} \in \overline{M}$ the map $T_{\bar{p}}\iota_{\overline{M}} \colon T_{\bar{p}}\overline{M} \to T_{\bar{p}}\overline{P}$ is an isomorphism of vector spaces.*

**Proof** Clearly, the map $T_{\bar{p}}\iota_{\overline{M}}$ is a monomorphism of vector spaces. To show that it is onto, consider a derivation $u \in T_{\bar{p}}\overline{P}$. For $p \in \pi^{-1}(\bar{p})$, let $S_p$ be a slice at $p$ for the $G$-action $\Phi$ on $P$. Then $U = \pi(S_p)$ is a neighbourhood of $\bar{p}$ in $\overline{P}$. Let $\pi_{S_p}$ be the restriction of the $G$-orbit map $\pi$ to $S_p$ and let $S_K = S_p \cap P_K$. Then $T_pS_p$ has a decomposition

$$T_pS_p = T_pS_K \oplus T_p^{\perp}S_K.$$

Derivations at $p$ of the space of smooth $K$-invariant functions on $S_p$ form a subspace $(T_p^*S_p)^K$ of $T_p^*S_p$, which annihilates the space $T_p^{\perp}S_K$ (see Lemma 2.4). Let $T_p^{\circ}S_K \subseteq T_p^*S_p$ be the annihilator of $T_p^{\perp}S_K$. We have the decomposition

$$T_p^*S_p = T_p^{\circ}S_K \times (T_p^*S_p)^K.$$

The derivation $u \in T_{\bar{p}}\overline{P}$ at $\bar{p}$ lifts to a derivation $\widetilde{u}$ at $p$ on the space of $K$-invariant smooth functions on $S_p$. Since a derivation at $p$ is a linear function on the space of tangent covectors to $S_p$ at $p$, we can consider $\widetilde{u}$ to be the linear function $\widetilde{u}\colon (T_p^*S_p)^K \to \mathbb{R}$. Let $\widehat{u}\colon T_p^*S_p \to \mathbb{R}$ be an extension of $\widetilde{u}$ such that

$$\langle \widehat{u} \mid d_pf \rangle = 0, \quad \forall\, d_pf \in T_p^{\circ}S_K. \tag{20}$$

Since $(T_p^*S_p)^*$ and $T_pS_p$ are isomorphic, it follows that $\widehat{u} \in T_pS_p$. Moreover, equation (20) implies that $\widehat{u}$ is contained in the subspace $T_pS_K$ of $T_pS_p$. Hence there is a unique vector $v$ in $T_pS_K$ such that $T_p\iota_{S_K}(v) = \widehat{u}$, where $\iota_{S_K}\colon S_K \hookrightarrow S_p$ is the inclusion map. Clearly, $\widehat{u}$ is a derivation at $p$ of $C^\infty(S_p)$, which coincides with $\widetilde{u}$ when restricted to $C^\infty(S_p)^K$, that is, $T_p\pi_{S_p}(\widehat{u}) = u$. Hence $u = T_p\pi_{S_p}\big(T_p\iota_{S_K}(v)\big)$. But $\pi_{S_p}\circ\iota_{S_K} = \iota_{\overline{M}}\circ\pi_{S_K}$, where $\pi_{S_K}\colon S_K \to \overline{M}$ is the restriction of $\pi$ to $S_K$. This implies that $u = T_{\bar{p}}\iota_{\overline{M}}\big(T_p\pi_{S_K}(v)\big)$, where $T_p\pi_{S_K}(v) \in T_{\bar{p}}\overline{M}$. Hence the map $T_{\bar{p}}\iota_{\overline{M}}\colon T_{\bar{p}}\overline{M} \to T_{\bar{p}}\overline{P}$ is onto.

***Corollary 5.5*** *The tangent wedge $T^{\mathrm{w}}\overline{P}$ to $\overline{P}$ at $\bar{p}$ is the product of the (Zariski) tangent space to $T_p\overline{P}$ and the tangent cone $T^{\mathrm{c}}_{\bar{p}}\overline{P}$ to $\overline{P}$ at $\bar{p}$, that is,*

$$T^{\mathrm{w}}_{\bar{p}}\overline{P} = T_{\bar{p}}\overline{P} \times T^{\mathrm{c}}_{\bar{p}}\overline{P}.$$

## 6 Links

Information how the manifolds $\overline{M}$ in the partition of $\overline{P}$ fit together in a neighbourhood of a point $\bar{p}$ is encoded in the tangent wedge $T^{\mathrm{w}}_{\bar{p}}\overline{P}$ of $\overline{P}$, because it is locally diffeomorphic to $\overline{P}$. It is known that $\overline{P}$ is a stratified space (see [8] and [9]), that is, the manifolds $\overline{M}$, called *strata*, fit together in a special way forming a *stratification* of $\overline{P}$. In particular, each of point of the stratum $\overline{M}$ has a neighbourhood which is homeomorphic to the product of a smooth manifold and a neighbourhood of a vertex of a cone [9]. This conical neighbourhood is called a *link* of the stratum $\overline{M}$ in the stratification of $\overline{P}$. In this section we identify the links of the stratification of $\overline{P}$ with certain subsets of the tangent cone.

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Let $S_p$ be a slice at $p \in P_K$ for the $G$-action $\Phi$ on $P$, and $M$ be a connected component of $P_K$ containing $p$. Suppose that $\bar{p}' \in \mathrm{cl}(\overline{M}) \setminus \overline{M}$ (where $\mathrm{cl}(\overline{M})$ is the closure of $\overline{M}$) is contained in the open set $\pi(S_p)$ of $\overline{P}$. From the properties of a slice it follows that $\bar{p}' \in \pi(P_{K'})$, where $K'$ is conjugate in $G$ to a subgroup of $K$ not equal to $K$. Without loss of generality we may assume that $K'$ is a subgroup of $K$ not equal to $K$. Let $p' \in P_{K'} \cap \mathrm{cl}(P_K) \cap S_p$. Since $p' \in S_p$ it follows that $p' = \exp_p v$ for some $v \in \mathrm{hor}\, T_pP$. From the fact that the map $\exp_p\colon \mathrm{hor}\, T_pP \to P$ intertwines the $K$-action $\Psi_p^K$ on $\mathrm{hor}\, T_pQ$ with the $K$-action $\Phi^K = \Phi \mid (K \times P)$ on $P$ and $p' \in P_{K'}$, it follows that $K'$ is the $\Psi_p^K$-isotropy group of $v$. Let

$$(\mathrm{hor}\, T_pP)_{K'} = \{w \in \mathrm{hor}\, T_pP \mid \Psi_p(k, w) = w \text{ for every } k \in K'\},$$

and let

$$W_p^{K,K'} = \mathrm{hor}\, T_pP_K \cup (\mathrm{hor}\, T_pP)_{K'}.$$

***Lemma 6.1*** *For every $u \in \mathrm{hor}\, T_pP_K$, every $w \in W_p^{K,K'}$, and every $s \in \mathbb{R}$, we have $u + sw \in W_p^{K,K'}$. If $u \in \mathrm{hor}\, T_pP_K$, $w \in (\mathrm{hor}\, T_pP)_{K'}$, and $s \neq 0$, then $u + sw \in (\mathrm{hor}\, T_pP)_{K'}$.*

**Proof** Since $K'$ is a subgroup of $K$ not equal to $K$ and the $K$-action $\Psi_p^K$ on $T_pP$ is linear, for every $u \in \mathrm{hor}\, T_pP_K$, every $w \in (\mathrm{hor}\, T_pP)_{K'}$, every $s \in \mathbb{R}$, and every $k \in K'$, we have

$$\Psi_p^K(k, u + sw) = \Psi_p^K(k, u) + s\Psi_p^K(k, w) = u + sw.$$

Hence, the $\Psi_p^K$-isotropy group of $u + sw$ contains $K'$. Conversely, if $k \in K \setminus K'$ then

$$\Psi_p^K(k, u + sw) = \Psi_p^K(k, u) + s\Psi_p^K(k, w) = u + s\Psi_p^K(k, w) \neq u + sw,$$

if $s \neq 0$. Hence, $u+sw \in (\mathrm{hor}\, T_pP)_{K'}$ for every $u \in \mathrm{hor}\, T_pP_K$, every $w \in (\mathrm{hor}\, T_pP)_{K'}$ and every $s \neq 0$.

If $u, w \in \mathrm{hor}\, T_pP_K$, then $u+sw \in \mathrm{hor}\, T_pP_K$ for every $s \in \mathbb{R}$. Hence $u+sw \in W_p^{K,K'}$ for every $u \in \mathrm{hor}\, T_pQ_K$, every $w \in W_p^{K,K'}$, and every $s \neq 0$. ■

Let

$$V_p^{K,K'} = W_p^{K,K'} \cap \mathrm{hor}\, T_p^{\perp} P_K.$$

***Lemma 6.2*** *$V_p^{K,K'}$ is a cone with vertex at $0 \in \mathrm{hor}\, T_pP_K$. In addition,*

$$W_p^{K,K'} = \mathrm{hor}\, T_pP_K \times V_p^{K,K'}, \tag{21}$$

*where $\mathrm{hor}\, T_pP_K$ is identified with $\mathrm{hor}\, T_pP_K \times \{0\}$ and $(\mathrm{hor}\, T_pP)_{K'}$ is identified with $\mathrm{hor}\, T_pP_K \times (V_p^{K,K'} \setminus \{0\})$.*

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**Proof** Let $v \in V_p^{K,K'}$. Then either $v \in \mathrm{hor}\, T_pP_K \cap \mathrm{hor}\, T_p^{\perp}P_K$ or $v \in (\mathrm{hor}\, T_pP)_{K'} \cap \mathrm{hor}\, T_p^{\perp}P_K$. In either case $sv \in V_p^{K,K'}$ for every $s \in \mathbb{R}$ and $0v = 0 \in T_pP$. Hence $V_p^{K,K'}$ is a cone with vertex $0 \in T_pP$.

Equation (18) implies that every vector $w \in \mathrm{hor}\, T_pP_K \cup (\mathrm{hor}\, T_pP)_{K'}$ can be decomposed uniquely as $w = u + v$ with $u \in \mathrm{hor}\, T_pP_K$ and $v \in \mathrm{hor}\, T_pP_K^{\perp}$. Moreover, $v = -u + w \in \mathrm{hor}\, T_pP_K \cup (\mathrm{hor}\, T_pP)_{K'}$. Hence, $v \in V_p^{K,K'}$. Conversely, if $(u, v) \in \mathrm{hor}\, T_pQ_K \times V_p^{K,K'}$, then $u + v \in \mathrm{hor}\, T_pP_K \cup (\mathrm{hor}\, T_pP)_{K'}$. This shows that $\mathrm{hor}\, T_pP_K \cup (\mathrm{hor}\, T_pP)_{K'} = \mathrm{hor}\, T_pP_K \times V_p^{K,K'}$.

On the one hand, if $v = 0 \in V_p^{K,K'}$ and $u \in \mathrm{hor}\, T_pP_K$, then $u+0 = u \in \mathrm{hor}\, T_pP_K$. On the other hand, if $v$ is a nonzero vector in $V_p^{K,K'}$, then $u + v \in (\mathrm{hor}\, T_pP)_{K'}$ for all $u \in \mathrm{hor}\, T_pP_K$. Hence, $\mathrm{hor}\, T_qQ_K = \mathrm{hor}\, T_qQ_K \times \{0\}$ and $W_p^{K,K'} = \mathrm{hor}\, T_pP_K \times (V_p^{K,K'} \setminus \{0\})$. ■

The quotient $(T_{\bar{p}}^{c}\overline{P})_{K'} = (V_{\bar{p}}^{K,K'})/K$ is independent of the choice of $p \in \pi^{-1}(\bar{p})$. It is a cone contained in $T_{\bar{p}}^{c}\overline{P}$ with vertex at 0. It follows from Lemma 6.2 and Theorem 5.1 that the exponential map $\exp_p \colon T_pP \to P$ restricted to $\mathrm{hor}\, T_pP$ composed with the $G$-orbit map $\pi \colon P \to \overline{P}$ maps a neighbourhood of 0 in $T_{\bar{p}}\overline{M} \times (T_{\bar{p}}^{c}\overline{P})_{K'}$ homeomorphically onto a neighbourhood of $\bar{p}$ in $\overline{M} \cup \overline{M'}$. This describes precisely the link at $\bar{p}$ between the stratum $\overline{M}$ and the stratum $\overline{M'}$ of $\overline{P}$.

## 7 A Momentum Map

In this section we study the refinement of the partition of $P$ given in (13) by level sets of an equivariant momentum map.

First we discuss momentum maps. Recall that an action $\Phi$ of a Lie group $G$ on a connected symplectic manifold $(P, \omega)$ is *symplectic* if it preserves the form $\omega$. For a symplectic action $L_{X^{\xi}}\omega = 0$ for all $\xi \in \mathfrak{g}$, where $X^{\xi}(p) = T_e\Phi_p(\xi)$. Hence, $d(X^{\xi} \lrcorner\, \omega) = 0$ which implies that locally $X^{\xi} \lrcorner\, \omega = dJ_{\xi}$ for some function $J_{\xi}$ on $P$. A symplectic action is *Hamiltonian* if there exists a momentum map $J \colon P \to \mathfrak{g}^*$ such that $J_{\xi} = \langle J \mid \xi \rangle$ for each $\xi \in \mathfrak{g}$. $J$ is *coadjoint equivariant* if $J\big(\Phi_g(p)\big) = \mathrm{Ad}^t_{g^{-1}} J(p)$ for every $(g, p) \in G \times P$.

If the momentum map $J \colon P \to \mathfrak{g}^*$ is not coadjoint equivariant, then it is equivariant with respect to an action on $\mathfrak{g}^*$, which is defined as follows. For each $p \in P$ the map

$$\widetilde{\sigma}_p \colon G \to \mathfrak{g}^* \colon g \mapsto J\big(\Phi_g(p)\big) - \mathrm{Ad}^t_{g^{-1}} J(p)$$

does not depend on the choice of the point $p$. Thus $\widetilde{\sigma}_p$ defines a mapping $\sigma \colon G \to \mathfrak{g}^*$ which is a $\mathfrak{g}^*$-cocycle, that is, for every $g, h \in G$

$$\sigma(gh) = \sigma(g) + \mathrm{Ad}^t_{g^{-1}} \sigma(h).$$

Let

$$A \colon G \times \mathfrak{g}^* \to \mathfrak{g}^* \colon (g, h) \mapsto \mathrm{Ad}^t_{g^{-1}} \alpha + \sigma(g). \tag{22}$$

Then $A$ is an action of $G$ on $\mathfrak{g}^*$ called the *affine coadjoint action*. A momentum mapping $J$ is equivariant with respect to the action $A$, that is, for every $(g, p) \in G \times P$

$$J\big(\Phi_g(p)\big) = A_g\big(J(p)\big).$$

From the beginning we have assumed that the action $\Phi$ of $G$ on $(P, \omega)$ has a coadjoint equivariant momentum map $J$. However, analogous results to the ones we have used hold if $J$ were equivariant with respect to an affine coadjoint action. In particular, the regular reduction theorem holds when the momentum map is equivariant with respect to an affine coadjoint action [11].

***Theorem 7.1*** *The action of $G_M$ on $(M, \omega_M)$ has a momentum map $J_M\colon M \to \mathfrak{g}_M^*$, which is equivariant with respect to the affine coadjoint action*

$$A\colon G_M \times \mathfrak{g}_M^* \to \mathfrak{g}_M^*\colon ([g], \mu) \mapsto A_{[g]}\mu.$$

*For every G-coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ with $J^{-1}(\mathcal{O}) \cap M \neq \varnothing$, there exists an orbit $\mathcal{O}_M$ of the action $A$ such that*

$$J^{-1}(\mathcal{O}) \cap M = J_M^{-1}(\mathcal{O}_M).$$

**Proof** Let $\kappa\colon \mathfrak{k} \to \mathfrak{g}$, $\mu\colon \mathfrak{k} \to \mathfrak{n}$, and $\nu\colon \mathfrak{n} \to \mathfrak{g}$ be inclusion mappings and $\lambda\colon \mathfrak{n} \to \mathfrak{g}_M$ the natural projection map. Their transposes are the mappings $\kappa^*\colon \mathfrak{g}^* \to \mathfrak{k}^*$, $\mu^*\colon \mathfrak{n}^* \to \mathfrak{k}^*$, $\nu^*\colon \mathfrak{g}^* \to \mathfrak{n}^*$, and $\lambda^*\colon \mathfrak{g}_M^* \to \mathfrak{n}^*$, respectively. Let $J|_M\colon M \to \mathfrak{g}^*$ be the restriction of $J$ to $M$.

To complete the proof of Theorem 7.1 we need several lemmas.

***Lemma 7.2*** $\kappa^* \circ J|_M\colon M \to \mathfrak{k}^*$ *is constant.*

**Proof** For every $\xi \in \mathfrak{g}$, we have $X^\xi \lrcorner\, \omega = dJ_\xi$. Moreover, $\xi \in \mathfrak{k}$ implies that $X^\xi(p) = 0$ for all $p \in M$. Hence $d(\kappa^* \circ J|_M) = \kappa^* \circ dJ|_M = 0$, and $\kappa^* \circ J|_M$ is constant on $M$. ■

Since $\mu^*\colon \mathfrak{n}^* \to \mathfrak{k}^*$ is onto and $\kappa^* \circ J|_M\colon M \to \mathfrak{k}^*$ is constant, there exists a constant map $j_M\colon M \to \mathfrak{n}^*$ such that

$$\mu^* \circ j_M = \kappa^* \circ J|_M.$$

***Lemma 7.3*** *There exists a unique map $J_M\colon M \to \mathfrak{g}_M^*$ such that*

$$\lambda^* \circ J_M = \nu^* \circ J|_M - j_M. \tag{23}$$

**Proof** We have

$$\mu^* \circ (\nu^* \circ J|_M - j_M) = \kappa^* \circ J|_M - \kappa^* J|_M = 0.$$

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The existence of a unique lift $J_M\colon M \to \mathfrak{g}_M^*$ of $(\nu^* \circ J|_M - j_M)\colon M \to \mathfrak{n}^*$ follows from the exactness of the sequence

$$0 \longrightarrow \mathfrak{g}_M^* \xrightarrow{\lambda^*} \mathfrak{n}^* \xrightarrow{\mu^*} \mathfrak{k}^* \longrightarrow 0. \tag{24}$$

■

Continuing with the proof of the first assertion in Theorem 7.1, we now show that the map $J_M\colon M \to \mathfrak{g}_M^*$ is a momentum map for the action of $G_M$ on $M$. For each $\xi \in \mathfrak{n} \subseteq \mathfrak{g}$, the action of the one parameter subgroup $\exp t\lambda(\xi)$ of $G_M$ on $M$ coincides with the action of the subgroup $\exp t\xi$ of $G$. This latter action is generated by the Hamiltonian vector field $X^\xi$ of $J_\xi$ restricted to $M$. Hence

$$\begin{aligned} X^\xi \mathbin{\lrcorner} \omega_M &= d\langle J|_M \mid \nu(\xi)\rangle = d\langle \nu^* \circ J|_M \mid \xi\rangle \\ &= d\langle \lambda^* \circ J_M + j_M \mid \xi\rangle = \langle d(\lambda^* \circ J_M) \mid \xi\rangle + \langle d j_M \mid \xi\rangle \\ &= d\langle J_M \mid \lambda(\xi)\rangle. \end{aligned}$$

Thus $X^\xi$ is the Hamiltonian vector field of $\langle J_M \mid \lambda(\xi)\rangle$. Hence $J_M$ is a momentum map for the action $G_M$ on $M$. This completes the proof of the first assertion in Theorem 7.1.

***Remark 7.4*** We note that the momentum map $J_M\colon M \to \mathfrak{g}_M^*$ need *not* be coadjoint equivariant. However, there exists a $\mathfrak{g}_M^*$-cocycle $\sigma\colon G_M \to \mathfrak{g}_M^*$ such that the map

$$A\colon G_M \times \mathfrak{g}_M^* \to \mathfrak{g}_M^*\colon ([g],\mu) \mapsto A_M([g],\mu) = \mathrm{Ad}^t_{[g]^{-1}}\,\mu + \sigma([g]) \tag{25}$$

is an action of $G_M$ on $\mathfrak{g}_M^*$ and $J_M([g]\cdot p) = A_{[g]}\big(J_M(p)\big)$.

We now find an explicit expression for the cocycle $\lambda^*\sigma$, which will not be used in the remainder of the proof. Comparing equations (23) and (22) we see that for $\xi \in \mathfrak{n}$,

$$\begin{aligned} \langle \sigma([g]) \mid \lambda(\xi)\rangle &= \langle J_M([g]\cdot p) - \mathrm{Ad}^t_{[g]^{-1}}\, J_M(p) \mid \lambda(\xi)\rangle \\ &= \langle \mathrm{Ad}^t_{g^{-1}}\, j_M(p) \mid \xi\rangle - \langle j_M(g\cdot p) \mid \xi\rangle = \langle j_M \mid \mathrm{Ad}^t_{g^{-1}}\,\xi - \xi\rangle \\ &= \langle \mathrm{Ad}^t_{g^{-1}}\, j_M - j_M \mid \xi\rangle. \end{aligned}$$

Hence

$$\lambda^*\big(\sigma([g])\big) = \mathrm{Ad}^t_{g^{-1}}\, j_M - j_M. \tag{26}$$

■

Recall that $\mathfrak{n}$ is the Lie algebra of $N_M$. For each $\xi \in \mathfrak{n}$, the vector field $X^\xi$ is tangent to $M$. For each $p \in M$, let

$$\mathfrak{m}(p) = \{\xi \in \mathfrak{g} \mid X^\xi(p) \in T^\omega_p M\}, \tag{27}$$

where $T^\omega_p M$ is the symplectic annihilator of $T_pM$, see (7).

***Lemma 7.5*** *For each $p \in M$, $\mathfrak{m}(p)$ is independent of $p$ and*

$$\mathfrak{n} + \mathfrak{m}(p) = \mathfrak{g}.$$

**Proof** Recall that the tangent space $T_pM = T_pP_K$ consists of vectors $v \in T_pP$ which are invariant under the action $\Psi_p^K$ of $K$ on $T_pP$. In other words,

$$T_pM = \{v \in T_pP \mid \Psi_k(v) = \Psi_p^K(k, v) = T_p\Phi_k(v) = v \,\forall\, k \in K\}.$$

Moreover, for every $\xi \in \mathfrak{n}$ we have $X^\xi(p) \in T_pM$.

Since $\omega$ is $G$-invariant and $T_pM$ is $\Psi_k$-invariant, it follows that $T_p^\omega M$ is also $\Psi_k$-invariant. For every $u \in T_pP$, let $\overline{u}$ be the average of $u$ over $K$ (see (11)). Since $\overline{u}$ is $\Psi_k$-invariant, it belongs to $T_pM$. If $u \in T_p^\omega M$, then $\overline{u} \in T_p^\omega M$ because $T_p^\omega M$ is $\Psi_k$-invariant. Hence if $u \in T_p^\omega M$, it follows that $\overline{u} \in T_pM \cap T_p^\omega M = \{0\}$. Thus

$$T_p^\omega M = T_p^\perp P_K = \{u \in T_pP \mid \overline{u} = 0\}, \tag{28}$$

see (12).

For each $\xi \in \mathfrak{g}$, let

$$\overline{\xi} = \int_K T_eL_k(\xi)\, dk,$$

where $L_k\colon G \to G\colon g \mapsto kg$ is left translation by $k$. The map

$$\mathfrak{g} \to T_pP\colon \xi \mapsto X^\xi(p)$$

is equivariant, that is, $X^{T_eL_k\xi}(p) = T_p\Phi_k\big(X^\xi(p)\big)$. Since this map has kernel $\mathfrak{k}$, it follows that

$$\mathfrak{m}(p) = \{\xi \in \mathfrak{g} \mid \overline{\xi} \in \mathfrak{k}\}.$$

For every $\xi \in \mathfrak{g}$, we have $\xi = \overline{\xi} + (\xi - \overline{\xi})$, where $\overline{(\xi - \overline{\xi})} = 0$. This implies that $\mathfrak{m}(p)$ is independent of $p$. Since $T_eL_k\overline{\xi} = \overline{\xi}$ for all $k \in K$, it follows that $T_p\Phi_k\big(X^{\overline{\xi}}(p)\big) = X^{\overline{\xi}}(p)$ for $k \in K$. So $X^{\overline{\xi}}(p) \in T_pM$, that is, $\overline{\xi} \in \mathfrak{n}$. Moreover $\overline{(\xi - \overline{\xi})} = 0 \in \mathfrak{k}$, which implies that $\xi - \overline{\xi} \in \mathfrak{m}(p)$. Hence $\mathfrak{g} = \mathfrak{n} + \mathfrak{m}(p)$. ■

We continue with the proof of the second assertion of Theorem 7.1. If $p, p' \in J^{-1}(\mathcal{O}) \cap M$ then $J(p') = \mathrm{Ad}^t_{g^{-1}} J(p) = J(g\cdot p)$ for some $g \in N_M$. Since, $g\cdot p = [g]\cdot p$, where $[g]$ is the coset of $g$ in $G_M = N_M/K$, equation (23) yields

$$\begin{aligned}
\lambda^* \circ J_M(p') &= \nu^* \circ J(p') - j_M = \nu^* \circ \mathrm{Ad}^t_{g^{-1}} J(p) - j_M \\
&= \nu^* \circ J(g \cdot p) - j_M = \big(\lambda^* \circ J_M([g] \cdot p) + j_M\big) - j_M \\
&= \lambda^* \circ J_M([g] \cdot p) = \lambda^* \circ A_{[g]}\big(J_M(p)\big)\,.
\end{aligned}$$

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Since $\ker \lambda^* = 0$, it follows that

$$J_M(p') = A_{[g]}\big(J_M(p)\big).$$

This implies that $J_M(p')$ and $J_M(p)$ are in the same orbit $\mathcal{O}_M$ of the affine coadjoint action $A$ of $G_M$ on $\mathfrak{g}_M^*$ (see (25)), that is,

$$J^{-1}(\mathcal{O}) \cap M \subseteq J_M^{-1}(\mathcal{O}_M). \tag{29}$$

Conversely, if $p, p' \in J_M^{-1}(\mathcal{O}_M)$, then $J_M(p) = A_{[g]}\big(J_M(p')\big)$ where $g \in N_M$. Therefore,

$$\nu^* \circ J(p) = \nu^* \circ \mathrm{Ad}^t_{g^{-1}} J(p').$$

But $\nu^* \colon \mathfrak{g}^* \to \mathfrak{n}^*$ is the transpose of the inclusion mapping $\nu \colon \mathfrak{n} \to \mathfrak{g}$. So

$$\ker \nu^* = \mathfrak{n}^\circ = \{\alpha \in \mathfrak{g}^* \mid \langle \alpha \mid \xi \rangle = 0 \,\forall\, \xi \in \mathfrak{n}\}.$$

This implies that

$$J(p) - \mathrm{Ad}^t_{g^{-1}} J(p') \in \mathfrak{n}^\circ.$$

Hence for every $\xi \in \mathfrak{n}$, we have

$$\langle J(p) - \mathrm{Ad}^t_{g^{-1}} J(p') \mid \xi \rangle = 0.$$

On one hand, differentiating this equation in a direction $u$ tangent to $J_M^{-1}(\mathcal{O}_M)$ at $p$, we get

$$\Big\langle T_p\big(J(p) - \mathrm{Ad}^t_{g^{-1}} J(p')\big)(u) \mid \xi \Big\rangle = 0 \tag{30}$$

for every $u \in T_p J_M^{-1}(\mathcal{O}_M)$ and every $\xi \in \mathfrak{n}$. On the other hand, from (27) we see that $X^\xi(p) \in T_p^\omega M$ for $\xi \in \mathfrak{m}(p)$. But

$$\langle T_p J(u) \mid \xi \rangle = dJ^\xi(p)u = \omega_M(p)\big(X^\xi(p), u\big) = 0,$$

for all $u \in T_p J_M^{-1}(\mathcal{O}_M)$ and $\xi \in \mathfrak{m}(p)$. Therefore

$$\Big\langle T_p\big(J(p) - \mathrm{Ad}^t_{g^{-1}} J(p')\big)(u) \mid \xi \Big\rangle = 0 \tag{31}$$

for every $u \in T_p J_M^{-1}(\mathcal{O}_M)$ and every $\xi \in \mathfrak{m}(p)$. Since $\mathfrak{n} + \mathfrak{m}(p) = \mathfrak{g}$, equations (30) and (31) imply that $J(p) - \mathrm{Ad}^t_{g^{-1}} J(p')$ is independent of $p \in J_M^{-1}(\mathcal{O}_M)$. Moreover, $g \in N_M$ implies that $g \cdot p' = [g] \cdot p' \in J_M^{-1}(\mathcal{O}_M)$. Hence taking $p = g \cdot p'$, we get

$$J(p) - \mathrm{Ad}^t_{g^{-1}} J(p') = J(g \cdot p') - \mathrm{Ad}^t_{g^{-1}} J(p') = 0,$$

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because $J$ is coadjoint equivariant. Thus $J(p)$ and $J(p')$ are in the same coadjoint orbit $\mathcal{O}$. Therefore,

$$J_M^{-1}(\mathcal{O}_M) \subseteq J^{-1}(\mathcal{O}) \cap M.$$

Taking into account the inclusion (29) we obtain $J_M^{-1}(\mathcal{O}_M) = J^{-1}(\mathcal{O}) \cap M$. This completes the proof of Theorem 7.1. ∎

For another proof see Section 2.3 in Ortega [16]. We have included a complete proof because it introduces concepts and techniques we need in the paper.

Recall that $\overline{M} = M/G_M$ is the space of $G_M$-orbits on $M$ and that $\pi_M \colon M \to \overline{M} \colon p \mapsto G_M \cdot p$ the $G_M$-orbit map. Since the action of $G_M$ on $M$ is free and proper, $\overline{M}$ is a quotient manifold of $M$. Let $L$ be a connected component of $J^{-1}(\alpha) \cap M$ and $\overline{L}$ its projection to $\overline{M}$. Let $\iota_L \colon L \to M$ and $\iota_{\overline{L}} \colon \overline{L} \to \overline{M}$ be the inclusion maps and let $\pi_L \colon L \to \overline{L}$ be the map induced by the restriction of $\pi_M \colon M \to \overline{M}$ so that $\pi_L \circ \iota_L = \iota_{\overline{L}} \circ \pi_M$.

***Theorem 7.6*** *$\overline{L}$ is a connected submanifold of $\overline{M}$ endowed with a symplectic form $\omega_{\overline{L}}$ such $\pi_L^* \omega_{\overline{L}} = \iota_L^* \omega_M$.*

To prove Theorem 7.6 we need the following three lemmas.

***Lemma 7.7*** *For each $\alpha \in \mathfrak{g}^*$, every connected component of the set $\pi\big(J^{-1}(\alpha) \cap M\big)$ is of the form $\pi(L)$, where $L$ is a connected component of $J^{-1}(\alpha) \cap M$.*

**Proof** $J^{-1}(\alpha) \cap M = J_M^{-1}(\beta)$ for some $\beta \in \mathfrak{g}_M^*$. Hence, connected components of $J^{-1}(\alpha) \cap M$ are connected components of $J_M^{-1}(\beta)$. If $L$ and $L'$ are connected components of $J_M^{-1}(\beta)$ then $\pi_M(L)$ and $\pi_M(L')$ are connected. Suppose that $\pi_M(L) \cap \pi_M(L') \neq \varnothing$. Then there exist $p \in L$, $p' \in L'$ and $g \in G_M$ such that $p = g \cdot p'$. Let $L'' = g \cdot L'$. Then $L''$ is a connected component of $J_M^{-1}(\beta)$ and $p \in L'' \cap L$. Therefore, $L'' = L$ and $\pi_M(L)$ is a connected component of $\pi_M\big(J_M^{-1}(\beta)\big) \subseteq \overline{M}$. Since $\tau_M \colon \overline{M} \to \pi(M)$ is a diffeomorphism, it follows that a connected component of $\pi\big(J^{-1}(\alpha) \cap M\big) = \tau_M\Big(\pi_M\big(J_M^{-1}(\beta)\big)\Big)$ is of the form $\pi(L) = \tau_M\big(\pi_M(L)\big)$, where $L$ is a connected component of $J_M^{-1}(\beta) = J^{-1}(\alpha) \cap M$. ∎

***Lemma 7.8*** *$\overline{L}$ is a symplectic manifold. The ring $C^\infty(\overline{L})$ consists of functions $f_{\overline{L}} \colon \overline{L} \to \mathbb{R}$ such that $\pi_M^* \circ f_{\overline{L}} \in C^\infty\big(J_M^{-1}(\beta)\big)$. A symplectic form $\omega_{\overline{L}}$ on $\overline{L}$ is uniquely defined by $\pi_L^* \omega_{\overline{L}} = \iota_L^* \omega_M$.*

**Proof** Since the action of $G_M$ on $M$ is free, $\beta$ is a regular value of $J_M \colon M \to \mathfrak{g}_M^*$. Hence, $J_M^{-1}(\beta)$ is a closed submanifold of $M$.

Let $G_{M_\beta} \subseteq G_M$ be the isotropy group of $\beta$. The action of $G_M$ on $M$ restricted to $G_{M_\beta}$ induces an action of $G_{M_\beta}$ on $J_M^{-1}(\beta)$. Since the action of $G_M$ is free and proper, and $G_{M_\beta}$ and $J_M^{-1}(\beta)$ are closed, it follows that the action of $G_{M_\beta}$ on $J_M^{-1}(\beta)$ is free and proper. The regular reduction theorem for a momentum map equivariant with respect to an affine coadjoint action ensures that connected components of the orbit space $J_M^{-1}(\beta)/G_{M_\beta}$ are symplectic manifolds [11].

Points of $\pi_M\big(J_M^{-1}(\beta)\big)$ are $G_M$-orbits $G_M \cdot p$ through points $p \in J_M^{-1}(\beta)$. The map $\pi_M\big(J_M^{-1}(\beta)\big) \to J_M^{-1}(\beta)/G_{M_\beta} \colon G_M \cdot p \mapsto G_{M_\beta} \cdot p$ is a bijection. Hence, it induces in each connected component $\overline{L}$ of $\pi_M\big(J_M^{-1}(\beta)\big)$ the structure of a symplectic manifold. A function $f_{\overline{L}} \colon \overline{L} \to \mathbb{R}$ is in $C^\infty(\overline{L})$ if and only if $\pi_M^* \circ f_{\overline{L}} \in C^\infty\big(J_M^{-1}(\beta)\big)$. The symplectic form $\omega_{\overline{L}}$ on $\overline{L}$ is uniquely defined by $\pi_L^*\omega_{\overline{L}} = \iota_L^*\omega_M$. ■

***Lemma 7.9*** *For each $G_{M_\beta}$-invariant function $f \in C^\infty\big(J_M^{-1}(\beta)\big)$ and every $p \in L$, there exists a neighbourhood $U$ of $p$ in $M$ and a $G_M$-invariant function $f_M \in C^\infty(M)$ such that $f|L \cap U = f_M|L \cap U$.*

**Proof** Let $f \in C^\infty\big(J_M^{-1}(\beta)\big)$ be $G_{M_\beta}$-invariant and $p$ any point in $L$. Let $S_p$ be a slice at $p$ for the action of $G_M$ on $M$. Since the action of $G_M$ on $M$ is free, $\pi_M^{-1}\big(\pi_M(S_p)\big)$ is homeomorphic to $S_p \times G_M$, which is an open $G_M$-invariant neighbourhood of $p \in M$. We can choose $S_p$ so that $S_p \cap L$ is a slice at $p$ for the action of $G_{M_\beta}$ on $J_M^{-1}(\beta)$. Let $S'_p$ be an open subset of $S_p$ containing $p$ such that its closure $\mathrm{cl}(S'_p)$ is contained in $S_p$, and let $U = \pi_M^{-1}\big(\pi_M(S'_p)\big)$ which is homeomorphic to $S'_p \times G_M$.

Since $J_M \colon M \to \mathfrak{g}_M^*$ is continuous, it follows that $J_M^{-1}(\beta)$ is closed in $M$. Hence, $L$ is closed in $M$ as a connected component of $J_M^{-1}(\beta)$. Therefore, $S_p \cap L$ is closed in $S_p$.

Let $f_{S_p \cap L}$ be the restriction of $f$ to $S_p \cap L$. Then $f_{S_p \cap L}$ can be extended to a smooth function $f_{S_p}$ on $S_p$. We can extend $f_{S_p}$ to a $G$-invariant function $f_U$ on $U = \pi_M^{-1}\big(\pi_M(S'_p)\big)$. Since $f$ is $G_{M_\beta}$-invariant, it follows that $f_U|U \cap L = f|U \cap L$. Using a $G$-invariant partition of unity subordinate to the $G_M$-invariant covering $\Big\{\pi_M^{-1}\big(\pi_M(S'_p)\big), \pi_M^{-1}\Big(\pi_M\big(S_p \setminus \mathrm{cl}(S'_p)\big)\Big)\Big\}$, where $\mathrm{cl}(S'_p)$ is the closure of $S'_p$ in $S_p$, we can construct a smooth $G_M$-invariant function $f_M$ on $M$ such that $f_M|U \cap L = f|U \cap L$. ■

**Proof of Theorem 7.6** Lemma 7.7 implies that $\overline{L}$ is connected. From Lemma 7.8, it follows that $\overline{L}$ is a symplectic manifold. Following the argument given in the proof of Theorem 3.8, from Lemma 7.8 we obtain that the manifold differential structure of $\overline{L}$ coincides with the differential structure induced by the inclusion map $\iota_{\overline{L}} \colon \overline{L} \to \overline{M}$. Hence, $\overline{L}$ is a submanifold of $\overline{M}$. ■

We have obtained the following refinement

$$P = \bigcup_{K \text{ c.s } G} \ \bigcup_{M \text{ c.c. } P_K} \ \bigcup_{\alpha \in \mathfrak{g}^*} \ \bigcup_{L \text{ c.c. } M \cap J^{-1}(\alpha)} L \tag{32}$$

of the partition (13). Here $K$ c.s. $G$, $M$ c.c. $P_K$, and $L$ c.c. $M \cap J^{-1}(\alpha)$ mean that the union is taken over compact subgroups $K$ of $G$, connected components of sets $P_K$ of symmetry type $K$, and connected components $L$ of $M \cap J^{-1}(\alpha)$, respectively. Since the partition (32) is $G$-invariant, it induces a partition of the orbit space

$$\overline{P} = \bigcup \overline{L}, \tag{33}$$

where $\overline{L} = \pi(L)$, and $L$ is a connected component of the intersection of a level set $J^{-1}(\alpha)$ of the momentum map with a connected component $M$ of $P_K$.

***Remark 7.10*** Note that each $\overline{L}$ is a connected submanifold of $\overline{P}$. To see this observe that we have already shown that $\overline{L}$ is a connected submanifold of $\overline{M}$. Theorem 3.8 ensures that $\overline{M}$ is a submanifold of $\overline{P}$. Hence, $\overline{L}$ is a submanifold of $\overline{P}$.

We now investigate the geometry of the partition (32). For each function $f \in C^\infty(P)$, the Hamiltonian vector field $X_f$ is defined by $X_f \lrcorner\, \omega = df$. Noether's theorem implies that $X_f$ preserves the momentum map $J$ if and only if the function $f$ is $G$-invariant. Hence, the Hamiltonian vector fields of $G$-invariant functions are tangent to each $L$ making up the partition (32). Following the approach of [24] and [3] we are going to characterise each of these manifolds as an accessible set of the generalized distribution $E \subset TP$ spanned by Hamiltonian vector fields of $G$-invariant functions. In order to do so, we have first to review some of results of Stefan [26] and Sussmann [27].

## 8 Foliations with Singularities

Let $M$ be a finite dimensional paracompact smooth manifold. A subset $L$ of $M$ is said to be a *k-leaf* of $M$ if there is a differentiable structure on $L$ such that

1. with this differentiable structure $L$ is a connected $k$-dimensional immersed submanifold of $M$,
2. if $N$ is an arbitrary locally connected topological space and $\chi\colon N \to M$ is a continuous map such that $\chi(N) \subseteq L$, then the induced map $\chi\colon N \to L$ is continuous.

It follows from the properties of immersions that if $\chi\colon N \to M$ is a differentiable mapping of manifolds such that $\chi(N) \subseteq L$ then $\chi\colon N \to L$ is also differentiable. In particular, the differentiable structure on $L$ which makes $L$ into an immersed submanifold of $M$ is *unique*. Since $M$ is paracompact, every immersed connected submanifold of $M$ is separable. So $L$ does not admit a differentiable structure of a connected immersed submanifold of $M$ of dimension other than $k$.

A *smooth foliation with singularities* of a manifold $M$ is a partition of $M$ into smooth leaves such that, for every $p \in M$, there exists a local chart $\psi$ of $M$ with the following properties.

1. The domain of $\psi$ is of the form $U \times W$, where $U$ is an open neighbourhood of $0 \in \mathbb{R}^k$, $W$ is an open neighbourhood of $0 \in \mathbb{R}^{m-k}$, and $k$ is the dimension of the leaf $L_p$ through $p$ while $m = \dim M$.
2. $\psi(0, 0) = 0$.
3. If $L$ is a leaf of the foliation, then $L \cap \psi(U \times W) = \psi(U \times V_L)$, where $V_L = \{w \in W \mid \psi(0, w) \in L\}$.

A *generalized distribution* on $M$ is a subset $D \subseteq TM$ such that, there exists an open covering $\{U_\alpha\}$ of $M$ and smooth vector fields $X^1_{U_\alpha}, \dots, X^{k_\alpha}_{U_\alpha}$ on $U_\alpha$ which span the restriction of $D$ to $U_\alpha$. Note that the definition of a generalized distribution does not require that the vector fields $X^1_{U_\alpha}, \dots, X^{k_\alpha}_{U\alpha}$ be linearly independent.

An *accessible set* of a generalized distribution $D$ on $M$ is a maximal subset $L$ of $M$ such that every pair of points in $L$ can be joined by a piecewise integral curve of vector fields $\{X^1_{U_\alpha}, \dots, X^{k_\alpha}_{U_\alpha}\}$.

***Theorem 8.1*** *Accessible sets of a generalized distribution on $M$ form a smooth foliation with singularities on $M$. In particular, every accessible set of $D$ is a leaf of $M$ and thus it admits a unique differentiable structure of a connected immersed submanifold of $M$.*

**Proof** See [26] and [27].

***Corollary 8.2*** *Every pair of points in an accessible set of a generalized distribution $D$ on $M$ can be joined by a piece-wise integral curve of vector fields with values in $D$.*

In Section 7 we introduced a generalized distribution $E$ on $P$ locally spanned by the Hamiltonian vector fields of $G$-invariant functions. Theorem 8.1 ensures that accessible sets of $E$ define on $P$ a smooth foliation with singularities. In particular,

$$P = \bigcup_{L \text{ a.s. } E} L, \tag{34}$$

where $L$ a.s. $E$ means that the union is taken over accessible sets $L$ of $E$.

***Theorem 8.3*** *For each $p \in P$, the accessible set $L$ of $E$ through $p$ is the connected component of $M \cap J^{-1}(\alpha)$ containing $p$. Here $M$ is a connected component of $P_K$, $K$ is the isotropy group of $p$, and $\alpha = J(p)$.*

**Proof** It follows from Noether's theorem that, for each $G$-invariant function $f$ on $P$, the Hamiltonian vector field $X_f$ of $f$ preserves the momentum map $J$. In other words, $E \subseteq \ker dJ$. Moreover, it follows from Theorem 2.1 that the restriction $E_M$ of $E$ to $M$ is contained in $TM$. Hence, $E_M \subseteq TM \cap \ker dJ = \ker dJ_M$. To complete the proof of Theorem 8.3 we need the following:

***Lemma 8.4*** $E_M = \ker dJ_M$.

**Proof** The pull back $\omega_M$ of $\omega$ to $M$ is a symplectic form on $M$. For $p \in M$, consider $u \in T_pM \cap \ker dJ_M$. The covector $u \lrcorner \omega_M$ annihilates every vector tangent at $p$ to the orbit $G_M$.

Let $S_p$ be the slice at $p$ for the action of $G_M$ on $M$. Then $G_M \cdot S_p$ is a neighbourhood of the orbit $G_M \cdot p$ in $M$. From the definition of a slice it follows that $T_pM = T_p(G_M \cdot p) \oplus T_pS_p$. Since $u \lrcorner \omega_M$ annihilates $T_p(G_M \cdot p)$, it follows that $u \lrcorner \omega_M = \varphi$ for some $\varphi \in T^*_pS_p$. There exists a compactly supported $(G_M)_p$-invariant function $f_S$ on $S_p$ such that $\varphi = df_S(p)$. Let $f$ be a function on $M$ such that $f \mid (G_M \cdot S_p) = f_S$ and $f$ vanishes on the complement of $G_M \cdot S_p$ in $M$. Then, $f$ is $G_M$-invariant and $df(p) = \varphi = u \lrcorner \omega_M$. Hence, $u$ is the value at $p$ of the Hamiltonian vector field $X_f$ of $f$. ■

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Lemma 8.4 implies that connected components $L$ of $M \cap J^{-1}(\alpha)$ are accessible sets of the generalized distribution $E$ on $P$. This completes the proof of Theorem 8.3. ■

From Theorem 8.3 it follows that for a smooth proper Hamiltonian action of a Lie group $G$ on a symplectic manifold $(P, \omega)$ the two smooth foliations with singularities given by (32) and (34) coincide. To show that partition (32) is a smooth foliation with singularities we have used the hypotheses that the action of $G$ on $(P, \omega)$ is smooth, proper and Hamiltonian. On the other hand, the partition (34) is a smooth foliation with singularities provided the action of $G$ on $M$ is smooth and symplectic. Thus, it is well defined in the absence of a momentum map and for actions which are not proper.

## 9 Coadjoint Orbits

Let $\mathcal{O} \subseteq \mathfrak{g}^*$ be a coadjoint orbit. In this section we discuss the structure of $J^{-1}(\mathcal{O}) \subseteq P$ and $\pi\big(J^{-1}(\mathcal{O})\big) \subseteq \overline{P}$. Theorem 7.1 asserts that, for every connected component $M$ of $P_K$ and every coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$, there exists an orbit $\mathcal{O}_M \subseteq \mathfrak{g}^*_M$ of an affine coadjoint action of $G_M$ such that $J^{-1}(\mathcal{O}) \cap M = J_M^{-1}(\mathcal{O}_M)$. Here $J_M \colon M \to \mathfrak{g}^*_M$ is a momentum map for the free and proper action of $G_M$ on $M$.

***Proposition 9.1*** *Every connected component of an orbit $\mathcal{O}_M$ of an affine coadjoint action $G_M \times \mathfrak{g}^*_M \to \mathfrak{g}^*_M$ is a leaf of $\mathfrak{g}^*_M$. In particular, connected components of $\mathcal{O}_M$ are immersed submanifolds of $\mathfrak{g}^*_M$.*

**Proof** For each $\xi \in \mathfrak{g}_M$, let $X^\xi$ be the vector field on $\mathfrak{g}^*_M$ corresponding to the action of $\exp t\xi$. The vector fields $\{X^\xi \mid \xi \in \mathfrak{g}_M\}$ span a generalized distribution on $\mathfrak{g}^*_M$ with orbits $\mathcal{O}_M$ being accessible sets. Theorem 8.1 implies Proposition 9.1. ■

***Proposition 9.2*** *For each $G_M$-orbit $\mathcal{O}_M \subseteq \mathfrak{g}^*_M$, connected components of $J_M^{-1}(\mathcal{O}_M)$ are leaves of $M$. In particular, each connected component $Q$ of $J_M^{-1}(\mathcal{O}_M)$ has a unique differential structure of a smooth manifold of dimension* $\dim Q = \dim \mathcal{O}_M + \dim M - \dim \mathfrak{g}^*_M$ *such that the inclusion map $Q \hookrightarrow M$ is an immersion.*

**Proof** Since the action of $G_M$ on $M$ is free and proper, every point of $M$ is a regular point of $J_M$. Hence, $\dim \ker dJ_M = \dim M - \dim \mathfrak{g}^*_M$ is constant and $\ker dJ_M$ is an involutive distribution on $M$ giving rise to a foliation of $M$ by level sets of $J$. In particular, $\ker dJ_M$ is spanned locally by smooth $G_M$-invariant vector fields on $M$.

For every $G_M$-orbit $\mathcal{O}_M \subseteq \mathfrak{g}^*_M$,

$$J^{-1}(\mathcal{O}_M) = \bigcup_{\beta \in \mathcal{O}_M} G_M \cdot J_M^{-1}(\beta). \tag{35}$$

Hence, connected components of $J^{-1}(\mathcal{O}_M)$ are accessible sets of the generalized distribution spanned by $\ker dJ_M$ and the vector fields on $M$ which generate an action by one parameter subgroups of $G_M$. By Theorem 8.1, each connected component $Q$ of $J^{-1}(\mathcal{O}_M)$ is a leaf of $M$ having a unique differential structure of a smooth manifold of dimension $\dim \mathcal{O}_M + \dim M - \dim \mathfrak{g}_M^*$ such that the inclusion map $Q \hookrightarrow M$ is an immersion. ∎

A manifold $Q$ contained in a manifold $M$ carries two differential structures: the original manifold differential structure of $Q$, which we denote by $C_m^\infty(Q)$, and the differential structure $C_i^\infty(Q)$ induced by the inclusion map $\iota_Q\colon Q \hookrightarrow M$ described in Theorem 3.2. If the inclusion map $\iota_Q\colon Q \hookrightarrow M$ is an embedding, both differential structures coincide. If $\iota_Q\colon Q \hookrightarrow M$ is an immersion but not an embedding, then $C_i^\infty(Q)$ is a *proper* subset of $C_m^\infty(Q)$.

***Proposition 9.3*** *Let $Q$ be a connected component of $J^{-1}(\mathcal{O}_M)$. The restriction $\pi_Q\colon Q \to \overline{P}$ of the $G$-orbit map $\pi\colon P \to \overline{P}$ to $Q$ is smooth in both differential structures $C_m^\infty(Q)$ and $C_i^\infty(Q)$.*

**Proof** Let $\bar{f} \in C^\infty(\overline{P})$. Since $C_i^\infty(Q) \subseteq C_m^\infty(Q)$, it suffices to show that $\pi_Q^* \bar{f} = \bar{f} \circ \pi_Q \in C_i^\infty(Q)$. However, $\bar{f} \circ \pi_Q$ is the restriction to $Q$ of $f = \bar{f} \circ \pi \in C^\infty(P)$. Hence, $\bar{f} \circ \pi_Q$ is in $C_i^\infty(Q)$. ∎

***Proposition 9.4*** *Let $Q$ be a connected component of $J^{-1}(\mathcal{O}) \cap M$. Then $\pi(Q) = \pi(L)$ for some connected component $L$ of $J^{-1}(\alpha) \cap M$ contained in $Q$.*

**Proof** Theorem 7.1 ensures that there exists an orbit $\mathcal{O}_M \subset \mathfrak{g}_M^*$ such that $J^{-1}(\mathcal{O}) \cap M = J_M^{-1}(\mathcal{O}_M)$. As before, we denote by $\overline{M}$ the space of $G_M$-orbits on $M$, $\pi_M\colon M \to \overline{M}$ the orbit map, and $\bar{\iota}_M\colon \overline{M} \to \overline{P}$ the inclusion map. Equation (35) shows that $\pi_M\big(J_M^{-1}(\mathcal{O}_M)\big) = \pi_M\big(J_M^{-1}(\beta)\big)$ for any $\beta \in \mathcal{O}_M$.

By Remark 7.10 applied to the action of $G_M$ on $M$, connected components of $\pi_M\big(J_M^{-1}(\beta)\big)$ are of the form $\pi_M(L)$, where $L$ are connected components of $J_M^{-1}(\beta)$. Since $Q$ is connected, $\pi_M(Q)$ is connected. Hence, $\pi_M(Q) \subseteq \pi_M(L')$ for some connected component $L'$ of $J_M^{-1}(\beta)$. This implies that there exist $p \in Q$, $p' \in L$ and $g \in G_M$ such that $p = g \cdot p'$. Then $L = g \cdot L'$ is a connected component of $J_M^{-1}\big(A(g,\beta)\big) \subseteq J_M^{-1}(\mathcal{O}_M)$ such that $L \cap Q \neq \varnothing$, and $\rho_M(Q) \subseteq \rho_M(L)$. Since $Q$ is a connected component of $J_M^{-1}(\mathcal{O}_M)$ and $L$ is connected, $L \cap Q \neq \varnothing$ implies that $L \subseteq Q$. Hence, $\pi_M(L) \subseteq \pi_M(Q)$.

Since $\pi_M(L)$ is a subset of $\pi_M(Q)$ and vice versa, it follows that $\pi_M(Q) = \pi_M(L)$. Applying the map $\bar{\iota}_M\colon \overline{M} \to \overline{P}\colon G_M \cdot p \mapsto G \cdot p$ to both sides of this equality we get

$$\pi(Q) = \bar{\iota}_M\big(\pi_M(Q)\big) = \bar{\iota}_M\big(\pi_M(L)\big) = \pi(L),$$

which completes the proof. ∎

***Corollary 9.5*** *For each coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$, each compact subgroup $K$ of $G$, each connected component $M$ of $P_K$, and each connected component $Q$ of $J^{-1}(\mathcal{O}) \cap M$, $\pi(Q)$ is a symplectic submanifold of $\overline{P}$.*

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**Proof** There is a connected component $L$ of $J^{-1}(\alpha) \cap M$ such that $\pi(Q) = \pi(L)$, and $\pi(L)$ carries a symplectic form $\overline{\omega}_L$ which does not depend on the choice of $L$ such that $\pi(Q) = \pi(L)$. ■

It should be noted that Corollary 9.5 does not require that the orbit $\mathcal{O}$ be locally closed (see [20] for an example of a nonclosed coadjoint orbit).

## 10 Reduced Poisson Structure

As before, we consider a connected symplectic manifold $(P, \omega)$ with a proper Hamiltonian action $\Phi\colon G \times P \to P$ of a Lie group $G$ on $P$. The symplectic form $\omega$ on $P$ induces in $C^\infty(P)$ a Poisson bracket $\{\ ,\ \}$ such that

$$\{f_1, f_2\} = \omega(X_{f_1}, X_{f_2}) \tag{36}$$

for all $f_1, f_2 \in C^\infty(P)$. The Poisson bracket is antisymmetric, bilinear, satisfies the Jacobi identity

$$\big\{ f_1, \{f_2, f_3\}\big\} + \big\{ f_2, \{f_3, f_1\}\big\} + \big\{ f_3, \{f_1, f_2\}\big\} = 0, \tag{37}$$

and Leibniz' rule

$$\{f_1, f_2 f_3\} = f_2 \cdot \{f_1, f_3\} + \{f_1, f_2\} \cdot f_3 \tag{38}$$

for all $f_1, f_2, f_3 \in C^\infty(P)$. A commutative algebra endowed with a bilinear antisymmetric bracket operation which is a derivation and satisfies the Jacobi identity is called a *Poisson algebra*. The algebra $\big(C^\infty(P), \cdot\big)$ with the Poisson bracket (36) is called the Poisson algebra of $(P, \omega)$.

Since the action of $G$ on $P$ preserves $\omega$, it follows that the Poisson bracket $\{\ ,\ \}$ is $G$-invariant. In other words, if $f_1$ and $f_2$ are $G$-invariant, then $\{f_1, f_2\}$ is $G$-invariant. Hence, the algebra $C^\infty(P)^G$ of $G$-invariant functions on $P$ is a Poisson subalgebra of $C^\infty(P)$.

We denote by $\overline{P} = P/G$ the space of $G$-orbits with orbit map $\pi\colon P \to \overline{P}$. In Theorem 3.4 we have shown that the space $C^\infty(\overline{P})$ of all functions on $\overline{P}$ which pull back under the $G$-orbit map $\pi$ to a smooth $G$-invariant function on $P$ is a differential structure on $\overline{P}$.

***Proposition 10.1*** *The Poisson bracket $\{\ ,\ \}$ on $C^\infty(P)$ induces a bracket $\{\ ,\ \}_{\overline{P}}$ on $C^\infty(\overline{P})$ such that $\big(C^\infty(\overline{P}), \{\ ,\ \}_{\overline{P}}, \cdot\big)$ is a Poisson algebra.*

**Proof** The Poisson bracket $\{\ ,\ \}_{\overline{P}}$ on $C^\infty(\overline{P})$ is defined as follows. Let $\bar{f}, \overline{h} \in C^\infty(\overline{P})$. At each $p \in P$ let

$$\{\bar{f}, \overline{h}\}_{\overline{P}}\big(\pi(p)\big) = \{f, h\}(p),$$

where $\pi^*\bar{f} = f$, $\pi^*\overline{h} = h$ with $f, h \in C^\infty(P)^G$. Moreover, $\{\ ,\ \}$ is the usual Poisson bracket on the space of smooth functions on the symplectic manifold $(P, \omega)$. To see that the Poisson bracket $\{\ ,\ \}_{\overline{P}}$ is well defined, suppose that $\widetilde{f}$ is another smooth $G$-invariant function on $P$ which induces the function $\bar{f}$ on $\overline{P}$. Then

$$0 = \pi^* \bar{f} - \pi^* \bar{f} = f - \widetilde{f}$$

on $P$, since $\pi$ is surjective. Hence $\{f, h\} = \{\widetilde{f}, h\}$, which implies that $\{\bar{f}, \overline{h}\}_{\overline{P}}$ does not depend on the choice of representative of $\bar{f}$. Since $\{\ ,\ \}_{\overline{P}}$ is skew symmetric, the same argument shows that $\{\bar{f}, \overline{h}\}_{\overline{P}}$ does not depend on the choice of representative of $\overline{h}$ either. Hence $\{\ ,\ \}_{\overline{P}}$ is well defined.

From the fact that $\big(C^\infty(P)^G, \{\ ,\ \}, \cdot\big)$ is a Poisson algebra, it follows that $\big(C^\infty(\overline{P}), \{\ ,\ \}_{\overline{P}}, \cdot\big)$ is a Poisson algebra. ■

Let $L$ be an accessible set of the generalized distribution $E$ on $P$ spanned by the Hamiltonian vector fields of $G$-invariant functions and let $\iota_L \colon L \to P$ be the inclusion map. If $f$ is a $G$-invariant smooth function on $P$ then its Hamiltonian vector field $X_f$ is tangent to $L$. For every $h \in C^\infty(P)$, $\{f, h\} = \omega(X_f, X_h) = -X_f \mathbin{\lrcorner} dh$. Hence, the pull back $\iota_L^*\{f, h\}$ of $\{f, h\}$ to $L$ depends on $h$ only through $\iota_L^* h$.

***Proposition 10.2*** *For each leaf $L$ of the generalized distribution $E$ on $P$, the pull backs $\iota_L^* f$ of smooth $G$-invariant functions $f$ on $P$ to $L$ form a Poisson algebra on $L$ with Poisson bracket $\{\ ,\ \}_L^G$ defined by $\{\iota_L^* f_1, \iota_L^* f_2\}_L^G = \iota_L^*\{f_1, f_2\}$. The pull back map $f \to \iota_L^* f$ is a Poisson algebra homomorphism with kernel consisting of smooth $G$-invariant functions on $P$ which vanish on $L$.*

**Proof** Since

$$\iota_L^*\{f_1, f_2\} = -\iota_L^*(X_{f_1} \mathbin{\lrcorner} df_2) = \iota_L^*(X_{f_2} \mathbin{\lrcorner} df_1),$$

and $f_1, f_2 \in C^\infty(P)^G$, the argument before the statement of the proposition shows that $\iota_L^*\{f_1, f_2\}$ depends on $f_1$ and $f_2$ only through their pull backs to $L$. Hence $\{\iota_L^* f_1, \iota_L^* f_2\}_L^G$ is well defined. Clearly, it is bilinear and antisymmetric. Moreover, it satisfies the Jacobi identity (37) and Leibniz' rule (38) because they are satisfied by $\{\ ,\ \}$.

From the definition of the bracket $\{\ ,\ \}_L^G$ it follows that the restriction to $L$ of functions in $C^\infty(P)^G$ is a Poisson algebra homomorphism. Moreover, the kernel of the restriction to $L$ consists of functions which vanish on $L$. ■

Let $N_L = \{g \in G \mid g \cdot L = L\}$ be the stability group of $L$. The restrictions to $L$ of $G$-invariant functions on $P$ are $N_L$-invariant functions on $L$.

***Lemma 10.3*** *Every $N_L$-invariant smooth function $f_L$ on $L$ can be extended to a smooth $G$-invariant function $f$ on $P$.*

**Proof** Let $f_L \in C^\infty(L)$ be $N_L$-invariant. For each $p \in L \subseteq P$, let $S_p$ be a slice through $p$ for the action of $G$ on $P$. Since $L = M \cap J^{-1}(\alpha)$ is closed, its intersection with $S_p$ is closed in $S_p$. Hence, $f_L$ restricted to $S_p \cap L$ can be extended to a $K$-invariant function on $S_p$. We can construct a $G$-invariant neighbourhood $U_p$ of the orbit $G \cdot p$ and a smooth $G$-invariant function $f_{U_p}$ on $P$ such that $f_{U_p}|_{U_p \cap L} = f_L|_{U_p \cap L}$. Using a $G$-invariant partition of unity on $P$ (see [19]), we can construct a $G$-invariant smooth function $f$ on $P$ such that $f|_L = f_L$. ■

Let $\overline{L} = \pi(L)$ be the projection of $L$ to $\overline{P}$. By Remark 7.10, $\overline{L}$ is a submanifold of $\overline{P}$. Theorem 7.6 ensures that $\overline{L}$ is endowed with a symplectic form $\omega_{\overline{L}}$ such that $\pi_L^* \omega_{\overline{L}} = \iota_L^* \omega_M$, where $\pi_L \colon L \to \overline{L}$ is the projection and $\iota_L \colon L \to M$ is the inclusion map. Let $\{\ ,\ \}_{\overline{L}}$ be the Poisson bracket on $\overline{L}$ defined by the symplectic form $\omega_{\overline{L}}$. In other words, $\{\bar{f}_{\overline{L}}, \overline{h}_{\overline{L}}\}_{\overline{L}} = \omega_{\overline{L}}(X_{\bar{f}_{\overline{L}}}, X_{\overline{h}_{\overline{L}}})$ for every $\bar{f}_{\overline{L}}, \overline{h}_{\overline{L}}$ in $C^\infty(\overline{L})$.

***Proposition 10.4*** *The pull back of smooth functions on $\overline{L}$ by the projection map $\pi_L \colon L \to \overline{L}$ induces a Poisson algebra isomorphism*

$$\pi_L^* \colon \left(C^\infty(\overline{L}), \{\ ,\ \}_{\overline{L}}, \cdot\right) \to \left(C^\infty(L)^{N_L}, \{\ ,\ \}_L^G, \cdot\right).$$

*Here $C^\infty(L)^{N_L}$ is the space of $N^L$-invariant functions on $L$. Similarly, the pull back of smooth functions on $\overline{P}$ by the inclusion map $\iota_{\overline{L}} \colon \overline{L} \to \overline{P}$ induces a Poisson algebra homomorphism*

$$\iota_{\overline{L}}^* \colon \left(C^\infty(\overline{P}), \{\ ,\ \}_{\overline{P}}, \cdot\right) \to \left(C^\infty(\overline{L}), \{\ ,\ \}_{\overline{L}}, \cdot\right).$$

**Proof** For $\bar{f}_{\overline{L}}, \overline{h}_{\overline{L}}$ in $C^\infty(\overline{L})$ the pull backs $f_L = \pi_L^* \bar{f}_{\overline{L}}$ and $h_L = \pi_{\overline{L}}^* \overline{h}_{\overline{L}}$ are $N_L$-invariant functions in $C^\infty(L)$. By Lemma 10.3, they can be extended to $G$-invariant functions $f$ and $h$ on $P$. Let $\bar{f}$ and $\overline{h}$ denote the push forwards under $\pi$ of $f$ and $h$ to $\overline{P}$, respectively. Then, $\bar{f}_{\overline{L}} = \iota_{\overline{L}}^* \bar{f}$ and $\overline{h}_{\overline{L}} = \iota_{\overline{L}}^* \overline{h}$. Moreover, for each $p \in L$,

$$\begin{aligned}(\pi_L^* \{\bar{f}_{\overline{L}}, \overline{h}_{\overline{L}}\}_{\overline{L}})(p) &= \left(\pi_L^* \left(\omega_{\overline{L}}(X_{\bar{f}_{\overline{L}}}, X_{\overline{h}_{\overline{L}}})\right)\right)(p) = \pi_L^* \omega_{\overline{L}}\big(X_f(p), X_h(p)\big) \\ &= \{f_L, h_L\}_L^G(p) = \{\pi_L^* \bar{f}_{\overline{L}}, \pi_L^* \overline{h}_{\overline{L}}\}_L^G(p).\end{aligned}$$

Hence, $\pi_L^*$ is a homomorphism of Poisson algebras $\left(C^\infty(\overline{L}), \{\ ,\ \}_{\overline{L}}, .\right)$ and $\big(C^\infty(L)^{N_L}$, $\{\ ,\ \}_L^G, .\big)$. Since $\ker \pi_L^* = \{0\}$ and every function in $C^\infty(L)^{N_L}$ pushes forward to a function in $C^\infty(\overline{L})$, it follows that $\pi_L^*$ is an isomorphism.

Since $\iota_{\overline{L}} \circ \pi_L = \pi \circ \iota_L$,

$$\begin{aligned}\left(\pi_L^*(\iota_{\overline{L}}^*\{\bar{f}, \overline{h}\})\right)(p) &= \left(\iota_L^*(\pi^*\{\bar{f}, \overline{h}\})\right)(p) = \left(\iota_L^*(\{f, h\})\right)(p) \\ &= \{\iota_L^* f, \iota_L^* h\}_L^G(p) = \{f_L, h_L\}_L^G(p) = \{\pi_L^* \bar{f}_{\overline{L}}, \pi_L^* \overline{h}_{\overline{L}}\}(p) \\ &= (\pi_L^* \{\bar{f}_{\overline{L}}, \overline{h}_{\overline{L}}\}_{\overline{L}})(p) = (\pi_L^* \{\iota_{\overline{L}}^* \bar{f}, \iota_{\overline{L}}^* \overline{h}\}_{\overline{L}})(p).\end{aligned}$$

Therefore, $\pi_L^*(\iota_{\overline{L}}^*\{\bar{f}, \overline{h}\}) = \pi_L^*\{\iota_{\overline{L}}^* \bar{f}, \iota_{\overline{L}}^* \overline{h}\}_{\overline{L}}$ for every $\bar{f}, \overline{h} \in C^\infty(\overline{P})$. Since $\ker \pi_L^* = 0$, it follows that $\iota_{\overline{L}}^*\{\bar{f}, \overline{h}\} = \{\iota_{\overline{L}}^* \bar{f}, \iota_{\overline{L}}^* \overline{h}\}_{\overline{L}}$ for every $\bar{f}, \overline{h} \in C^\infty(\overline{P})$. Hence, $\iota_L^*$ is a homomorphism of Poisson algebras $\left(C^\infty(\overline{P}), \{\ ,\ \}_{\overline{P}}, \cdot\right)$ and $\left(C^\infty(\overline{L}), \{\ ,\ \}_{\overline{L}}, \cdot\right)$. ■

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In the approach to reduction via coadjoint orbits one considers the Poisson algebra structure on $J^{-1}(\mathcal{O})$ induced by the inclusion $J^{-1}(\mathcal{O}) \hookrightarrow P$ and the Poisson algebra structure on $\pi\big(J^{-1}(\mathcal{O})\big) = J^{-1}(\mathcal{O})/G$ induced by the orbit map $\pi$ [1]. Let $Q$ be a connected component of $J^{-1}(\mathcal{O}) \cap M$. By Proposition 9.2, $Q$ is an immersed submanifold of $M$, and therefore of $P$. It carries *two* differential structures: the manifold structure $C^\infty_m(Q)$ and the structure $C^\infty_i(Q)$ induced by the inclusion $\iota_Q \colon Q \to P$. In general, $C^\infty_i(Q)$ is a *proper* subset of $C^\infty_m(Q)$, since functions in $C^\infty_i(Q)$ need not extend to smooth functions on $P$ unless $Q$ is closed in $P$.

If $L$ is an accessible set of $E$ such that $E \cap Q \neq \varnothing$ then $L \subseteq Q$. Hence, for every $f_1, f_2 \in C^\infty(P)$ and $p \in Q$, $\{f_1, f_2\}(p)$ depends on $f_1$ and $f_2$ through their pull backs $\iota_Q^* f_1$ and $\iota_Q^* f_2$ to $Q$ (see the proof of Proposition 10.4). Hence, the map $\iota_Q^* \colon C^\infty(P) \to C^\infty_i(Q)$ enables us to push forward the Poisson bracket on $C^\infty(P)$ to a Poisson bracket on $\iota_Q^*\big(C^\infty(P)\big)$. The Poisson bracket at $p \in Q$ of two functions $\iota_Q^* f_1$ and $\iota_Q^* f_2$ in $\iota_Q^*\big(C^\infty(P)\big)$ depends only on their first jets $j^1_p(\iota_Q^* f_1)$ and $j^1_p(\iota_Q^* f_2)$ at $p$. However, $j^1_p\Big(\iota_Q^*\big(C^\infty(P)\big)\Big) = j^1_p\big(C^\infty_i(P)\big) = j^1_p\big(C^\infty_m(P)\big)$. Hence, we can extend the Poisson bracket on $\iota_Q^*\big(C^\infty(P)\big)$, induced by $\iota_Q^* \colon C^\infty(P) \to C^\infty_i(Q)$, to a Poisson bracket on $C^\infty_i(Q)$ and then to one on $C^\infty_m(Q)$ so that $\iota_Q^*\big(C^\infty(P)\big)$ is a Poisson subalgebra of $C^\infty_i(Q)$ and $C^\infty_i(Q)$ is a Poisson subalgebra of $C^\infty_m(Q)$. We shall denote this bracket by $\{\ ,\ \}_Q$.

Let $N_Q = \{g \in G \mid g \cdot Q = Q\}$ be the stability group of $Q$. The restrictions to $Q$ of $G$-invariant functions on $P$ are $N_Q$-invariant functions on $Q$. Since the Poisson bracket $\{\ ,\ \}_Q$ on $Q$ is $N_Q$-invariant, the spaces $C^\infty_i(Q)^{N_Q}$ and $C^\infty_m(Q)^{N_Q}$ of $N_Q$-invariant functions are Poisson subalgebras of $C^\infty_i(Q)$ and $C^\infty_m(Q)$, respectively. Let $\{\ ,\ \}^G_Q$ denote the restrictions of $\{\ ,\ \}_Q$ to $\iota_Q^*\big(C^\infty(P)^G\big)$, $C^\infty_i(Q)^{N_Q}$ and $C^\infty_m(Q)^{N_Q}$. According to Proposition 9.4, $\pi(Q) = \pi(L) = \overline{L}$ for an accessible set $L$ of $E$ contained in $Q$. We denote by $\pi_{Q\overline{L}} \colon Q \to \overline{L}$ the map such that $\iota_{\overline{L}} \circ \pi_{Q\overline{L}} = \pi_Q$, where $\pi_Q$ is the restriction of $\pi \colon P \to \overline{P}$ to $Q$ and $\iota_{\overline{L}} \colon \overline{L} \to \overline{P}$ is the inclusion map.

***Proposition 10.5*** *The pull back of smooth functions on $\overline{L}$ by the projection map $\pi_{Q\overline{L}} \colon Q \to \overline{L}$ is a Poisson algebra isomorphism*

$$\pi^*_{Q\overline{L}} \colon \big(C^\infty(\overline{L}), \{\ ,\ \}_{\overline{L}}, \cdot\big) \to \Big(\iota_Q^*\big(C^\infty(P)^G\big), \{\ ,\ \}^G_Q, \cdot\Big).$$

**Proof** For $\bar{f}_{\overline{L}} \in C^\infty(\overline{L})$ the pull back $f_L = \pi_L^* \bar{f}_{\overline{L}} \in C^\infty_L(L)^{N_L}$. By Lemma 8.4, it can be extended to $G$-invariant function $f$ on $P$. Let $f_Q$ be the restriction of $f$ to $Q$ and $\bar{f}$ the push forward of $f$ to $\overline{P}$. Then, $f_Q = \iota_Q^* f = \pi_Q^* \bar{f}$, and $\bar{f}_L = \iota_{\overline{L}}^* \bar{f}$. Hence, $\pi^*_{Q\overline{L}} \overline{f}_{\overline{L}} = \pi^*_{Q\overline{L}}(\iota^*_{\overline{L}} \bar{f}) = \pi_Q^* \bar{f} = f_Q$. So $f_Q \in \iota_Q^*\big(C^\infty(P)^G\big)$. Clearly, $f_Q = 0$ only if $\bar{f}_L = 0$. Hence, $\ker \pi^*_{Q\overline{L}} = 0$.

Conversely, let $f \in C^\infty(P)^G$. Then $f_L = \iota_L^* f \in C^\infty(L)^{N_L}$ pushes forward to $\bar{f}_{\overline{L}} \in C^\infty(\overline{L})$ such that $f_Q = \iota_Q^* f = \pi^*_{Q\overline{L}} \bar{f}_L$. Hence, $\pi^*_{Q\overline{L}}$ maps $C^\infty(\overline{L})$ onto $\iota_Q^*\big(C^\infty(P)^G\big)$. This implies that $\pi^*_{Q\overline{L}}$ is an isomorphism of the commutative algebras $\big(C^\infty(\overline{L}), \cdot\big)$ and $\iota_Q^*\big(C^\infty(P)^G, \cdot\big)$. An argument analogous to that in the proof of Proposition 10.4

implies that $\pi^*_{Q\overline{L}}$ preserves the Poisson bracket. Consequently, it is an isomorphism of Poisson algebras. ■

It follows from Proposition 10.5 that the coadjoint orbit approach to reduction does not introduce anything essentially new on the level of the reduced Poisson algebra, except for complications due to the existence of orbits which are not closed or even locally closed.

For each $\bar{f} \in C^\infty(\overline{P})$ gives rise to an inner derivation $\overline{Y}_{\bar{f}}$ of the Poisson algebra $\big(C^\infty(\overline{P}), \{\ ,\ \}_{\overline{P}}, \cdot\big)$ defined by

$$\overline{Y}_{\bar{f}}\overline{h} = \{\bar{f}, \overline{h}\}_{\overline{P}} \quad \text{for all } \overline{h} \in C^\infty(\overline{P}).$$

We can extend the notion of an accessible set of a generalized distribution to differential spaces. A curve $c\colon [t', t''] \to \overline{P}$ is an *integral curve an inner derivation* $\overline{Y}_{\bar{f}}$ if $\frac{\mathrm{d}}{\mathrm{d}t}\overline{h}\big(c(t)\big) = \overline{Y}_{\bar{f}}\overline{h}\big(c(t)\big)$ for every $t \in [t', t'']$ and every $\overline{h} \in C^\infty(\overline{P})$. We say that a continuous curve $c\colon [t', t''] \to \overline{P}$ is *piecewise an integral curve of inner derivations* if there is a partition of the interval $[t', t'']$ into a finite number of subintervals $[t_i, t_{i+1}]$, $i = 1, \ldots, n$, such that the restriction $c_i\colon [t_i, t_{i+1}] \to \overline{P}$ of the curve $c$ to $[t_i, t_{i+1}]$ is an integral curve of an inner derivation of $C^\infty(\overline{P})$. A subset of $\overline{P}$ is an *accessible set* of inner derivations if every pair of its points can be joined by a piecewise integral curve of inner derivations.

***Theorem 10.6*** *The subsets $\overline{L}$ of the decomposition (32) are accessible sets of inner derivations of the Poisson algebra $\big(C^\infty(\overline{P}), \{\ ,\ \}_{\overline{P}}, \cdot\big)$.*

**Proof** See [24].

Theorem 10.6 shows how the structure of $\overline{P}$ given by partition (32) is encoded in its Poisson algebra.

## 11 An Example

In this section we give an example illustrating the above theory.

Let $Q$ be the standard 2-sphere $S^2 = \{x \in \mathbb{R}^3 \mid \langle x, x\rangle = 1\}$ embedded in $\mathbb{R}^3$ with the standard Euclidean inner product $\langle\ ,\ \rangle$. Define an $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ action $\Phi\colon S^1 \times S^2 \to S^2$ on $S^2$ by restricting the linear orthogonal $S^1$ action

(39) $$\widetilde{\Phi}\colon S^1 \times \mathbb{R}^3 \to \mathbb{R}^3\colon (t, x) \to R_t x = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} x$$

to $S^2$. The action $\Phi$ is free except at the fixed points $(0, 0, \pm 1) \in S^2$.

To construct the orbit space of the action $\Phi$, we use invariant theory. The algebra of $\widetilde{\Phi}$-invariant polynomials on $\mathbb{R}^3$ is freely generated by

(40) $$\sigma_1 = x_3 \quad \text{and} \quad \sigma_2 = x_1^2 + x_2^2.$$

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The algebra of $\Phi$-invariant polynomials on $S^2$ is generated by $\sigma_1$ and $\sigma_2$ subject to the relation

$$\sigma_1^2 + \sigma_2 = 1, \quad \sigma_2 \geq 0, \tag{41}$$

which defines the orbit space $S^2/S^1 = \overline{S^2}$ as a semialgebraic variety in $\mathbb{R}^2$ (with coordinates $(\sigma_1, \sigma_2)$). The orbit map of the action $\Phi$ is

$$\pi\colon S^2 \to \overline{S^2}\colon x \mapsto \big(\sigma_1(x), \sigma_2(x)\big)\,. \tag{42}$$

The orbit space $\overline{S^2}$ is a differential space with differential structure $C^\infty(\overline{S^2})$ given by restricting smooth $\widetilde{\Phi}$-invariant functions to $S^2$. Using a theorem of Schwarz [21], it follows that $\pi$ is a smooth map between differential spaces. In addition, $\overline{S^2}$ is homeomorphic to $[-1, 1]$.

The lift of the action $\Phi$ to the tangent bundle

$$TS^2 = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle x, x\rangle = 1, \langle x, y\rangle = 0\}$$

of $S^2$ is the action

$$\Psi\colon S^1 \times TS^2 \to TS^2\colon (t, x, y) \mapsto (R_t x, R_t, y) \tag{43}$$

The lifted action preserves the 1-form $\vartheta = \langle y, dx\rangle|_{TS^2}$ on $TS^2$ and hence the symplectic form $\Omega = -d\vartheta$. Moreover, $\Psi$ is a Hamiltonian action on $(TS^2, \Omega)$ with momentum

$$J\colon TS^2 \to \mathbb{R}\colon (x, y) \to x_1 y_2 - x_2 y_1, \tag{44}$$

since $X \mathbin{\lrcorner} \vartheta = J$, where $X = (x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2})|_{TS^2}$ is the infinitesimal generator of the action $\Phi$.

We now give a decomposition of $TS^2$ into a fiber product of $\Psi$-invariant vertical and horizontal differential spaces. We begin by defining the vertical subspace. At every $\overline{x} \in \overline{S^2}$, the fiber $\pi^{-1}(\overline{x})$ of the $\Phi$-orbit map is the $S^1$-orbit $\mathcal{O}_x = \{R_t x \mid t \in S^1\}$ of the action $\Phi$ through $x$. Let $T_x\mathcal{O}_x$ the *vertical subspace* of $T_x S^2$. The collection of vertical subspaces is

$$\text{ver}\, TS^2 = \big\{ (x, y) \in TS^2 \mid y \in \text{span}\{(-x_2, x_1, 0)\}\big\}\,. \tag{45}$$

To specify the horizontal subspace at $x$, we endow $S^2$ with the Riemannian metric, which is the pull back of the Euclidean metric on $\mathbb{R}^3$ (coming from the Euclidean inner product) by the inclusion map. The *horizontal subspace* of $T_x S^2$ at $x$ is the orthogonal complement $T_x^\perp \mathcal{O}_x$ to $T_x\mathcal{O}_x$ with respect the Riemannian metric on $S^2$. The set of all horizontal subspaces is

(46)

$$\begin{aligned}\text{hor}\, TS^2 &= \big\{ (x, y) \in S^2 \times \mathbb{R}^3 \mid y \in (\text{span}\{x\})^\perp \cap \big(\text{span}\{(-x_2, x_1, 0)\}\big)^\perp \big\} \\ &= \left\{ (x, y) \in TS^2 \mid y \in \begin{cases} \text{span}\big\{ \big(x_1 x_3, x_2 x_3, -(x_1^2 + x_2^2)\big) \big\}, x_3 \neq \pm 1 \\ \text{span}\{(1, 0, 0), (0, 1, 0)\}, x_3 = \pm 1. \end{cases} \right\}\end{aligned}$$

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Both ver $TS^2$ and hor $TS^2$ are differential spaces with differential structure induced by the inclusion mappings $j_{\mathrm{ver}}\colon \mathrm{ver}\, TS^2 \hookrightarrow TS^2$ and $j_{\mathrm{hor}}\colon \mathrm{hor}\, TS^2 \hookrightarrow TS^2$, respectively. Let $\tau\colon TS^2 \to S^2\colon (x,y) \to x$ be the tangent bundle projection. Then the projection maps $\tau_{\mathrm{ver}} = \tau \circ j_{\mathrm{ver}}\colon \mathrm{ver}\, TS^2 \to S^2$ and $\tau_{\mathrm{hor}} = \tau \circ j_{\mathrm{hor}}\colon \mathrm{hor}\, TS^2 \to S^2$ are smooth maps between differential spaces. Thus we obtain the fiber product decomposition

$$TS^2 = \mathrm{ver}\, TS^2 \times_{S^2} \mathrm{hor}\, TS^2. \tag{47}$$

To construct the orbit space of the lifted action $\Psi$, we again use invariant theory. The algebra of polynomials which are invariant under the lifted action $\Psi$ (43) is generated by

$$\begin{aligned} \sigma_1 &= x_3 & \sigma_4 &= y_1^2 + y_2^2 + y_3^2 \\ \sigma_2 &= x_1^2 + x_2^2 & \sigma_5 &= x_1 y_2 - x_2 y_1 \\ \sigma_3 &= y_3 & \sigma_6 &= x_1 y_1 + x_2 y_2 \end{aligned} \tag{48}$$

subject to the relations

$$\begin{gathered} 1 = \sigma_1^2 + \sigma_2 \\ 0 = \sigma_6 + \sigma_1 \sigma_3 \\ \sigma_5^2 + \sigma_6^2 = \sigma_2(\sigma_4 - \sigma_3^2), \quad \sigma_2 \geq 0, (\sigma_4 - \sigma_3^2) \geq 0. \end{gathered} \tag{49}$$

Equation (49) defines the orbit space $\overline{TS^2} = (TS^2)/S^1$ as a semialgebraic variety in $\mathbb{R}^6$ (with coordinates $(\sigma_1, \ldots, \sigma_6)$). The orbit map of the lifted action $\Psi$ is

$$\rho\colon TS^2 \to \overline{TS^2} \subseteq \mathbb{R}^6\colon (x,y) \to \big(\sigma_1(x,y), \ldots, \sigma_6(x,y)\big). \tag{50}$$

The orbit space $\overline{TS^2}$ is a differential space and the orbit map $\rho$ is a smooth map between differential spaces.

Since both ver $TS^2$ and hor $TS^2$ are $\Psi$-invariant, the decomposition (47) gives rise to the decomposition

$$\overline{TS^2} = (\mathrm{ver}\, TS^2)/S^1 \times_{\overline{S^2}} (\mathrm{hor}\, TS^2)/S^1. \tag{51}$$

The differential stuctures on $(\mathrm{ver}\, TS^2)/S^1$ and $(\mathrm{hor}\, TS^2)/S^1$ are induced by the inclusion maps $\iota_{\mathrm{ver}}\colon (\mathrm{ver}\, TS^2)/S^1 \to (TS^2)/S^1$ and $\iota_{\mathrm{hor}}\colon (\mathrm{hor}\, TS^2)/S^1 \to (TS^2)/S^1$, respectively. Since the tangent bundle projection $\tau$ intertwines the lifted action $\Psi$ and the action $\Phi$, that is, $\tau\big(\Psi_t(x,y)\big) = \Phi\big(\tau(x,y)\big)$, it induces a smooth mapping $\overline{\tau}\colon \overline{TS^2} \to \overline{S^2}$. Consequently, the projection maps $\pi_{\mathrm{ver}} = \overline{\tau} \circ \iota_{\mathrm{ver}}\colon (\mathrm{ver}\, TS^2)/S^1 \to \overline{S^2}$ and $\pi_{\mathrm{hor}} = \overline{\tau} \circ \iota_{\mathrm{hor}}\colon (\mathrm{hor}\, TS^2)/S^1 \to \overline{S^2}$ are smooth.

We would like to give a geometric description of the decomposition (51). We begin by describing the orbit space $\overline{TS^2}$. Eliminating the variables $\sigma_2$ and $\sigma_6$ from

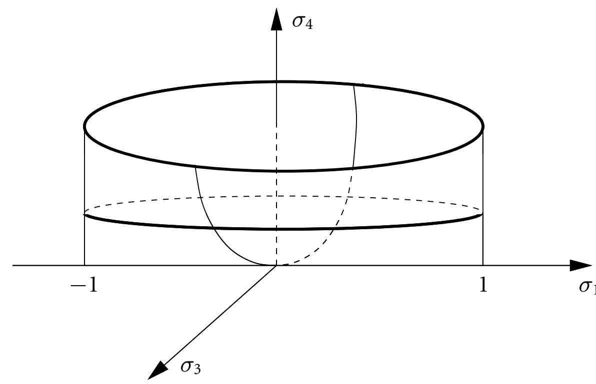


*Figure 1*: The solid canoe $\mathcal{V}$.

the third equation in (49), we see that $\overline{TS^2}$ is the semialgebraic variety in $\mathbf{R}^4$ (with coordinates $(\sigma_1, \sigma_3, \sigma_4, \sigma_5)$) defined by

$$(52) \qquad \sigma_3^2 + \sigma_5^2 = (1 - \sigma_1^2)\sigma_4, \quad |\sigma_1| \leq 1, \sigma_4 \geq 0.$$

To visualize the orbit space $\overline{TS^2}$, consider the $\mathbb{Z}_2$-action generated by

$$(53) \qquad (\sigma_1, \sigma_3, \sigma_4, \sigma_5) \mapsto (\sigma_1, \sigma_3, \sigma_4, -\sigma_5).$$

The algebra of $\mathbb{Z}_2$-invariant polynomials on $\overline{TS^2}$ is generated by

$$\sigma_1, \sigma_3, \sigma_4 \text{ and } \tau = \sigma_5^2.$$

The orbit space[1] $\mathcal{V} = (\overline{TS^2})/\mathbb{Z}_2$ is the semialgebraic variety in $\mathbb{R}^4$ (with coordinates $(\sigma_1, \sigma_3, \sigma_4, \tau)$)

$$(54) \qquad \tau + \sigma_3^2 = (1 - \sigma_1^2)\sigma_4, \quad |\sigma_1| \leq 1, \sigma_4 \geq 0, \tau \geq 0.$$

The boundary $\partial\mathcal{V}$ of $\mathcal{V}$ is the semialgebraic variety $\mathcal{C}$ in $\mathbb{R}^4$

$$(55) \qquad \sigma_3^2 = (1 - \sigma_1^2)\sigma_4 \quad |\sigma_1| \leq 1, \sigma_4 \geq 0, \tau = 0,$$

which we call the *canoe*. The canoe is homeomorphic to $\mathbb{R}^2$ with conical singular points at $(\pm 1, 0, 0, 0)$. We will refer to the orbit space $\mathcal{V}$ as the *solid canoe* (see Figure 1). The solid canoe is homeomorphic to a closed half space in $\mathbb{R}^3$ with conical

[1]Another way to obtain $\mathcal{V}$ is the following. Consider the O(2)-action on $S^2$ generated by the $SO(2) = S^1$-action $\Phi$ and the reflection $(x_1, x_2, x_3) \to (x_1, -x_2, x_3)$. Lift this action to an O(2)-action $\widehat{\Psi}$ on $TS^2$. The space $(TS^2)/O(2)$ of O(2)-orbits on $TS^2$ is precisely $\mathcal{V}$. Note that the action $\widehat{\Psi}$ is Hamiltonian on $(TS^2, \Omega)$ with momentum $\widehat{J} = J^2$.

singular points $(\pm 1, 0, 0, 0)$ on its boundary. By construction $\overline{TS^2}$ is a twofold covering of the solid canoe $\mathcal{V}$, which is branched along the canoe. Thus $\overline{TS^2}$ is homeomorphic to $\mathbb{R}^3$, being the union of two closed half spaces glued together along their common boundary by the identity map. $\overline{TS^2}$ has conical singular points $(\pm 1, 0, 0, 0)$.

Next we describe $(\operatorname{hor} TS^2)/S^1$. First we determine the image of $\operatorname{hor} TS^2$ under the orbit map $\rho$ (50). Suppose that $x_3 \neq \pm 1$, then using the definitions of $\operatorname{hor} TS^2$ and the map $\rho$, we find that

$$\rho\big(x_1, x_2, x_3, x_1x_3, x_2x_3, -(x_1^2 + x_2^2)\big) = (\sigma_1, \sigma_2, -\sigma_2, \sigma_2, 0, \sigma_1\sigma_2).$$

Hence $\rho(\operatorname{hor} TS^2 \setminus \{x_3 = \pm 1\})$ lies in the subvariety $V$ of $\overline{TS^2}$ defined by

$$\sigma_3^2 = (1 - \sigma_1^2)\sigma_4, \quad |\sigma_1| < 1, \sigma_4 \geq 0, \sigma_4 = -\sigma_3, \sigma_5 = 0.$$

Topologically $V$ is $(-1, 1) \times \mathbb{R}$ and is (Zariski) open subset of $(\operatorname{hor} TS^2)/S^1$. When $x_3 = \pm 1$,

$$\rho(0, 0, \pm 1, y_1, y_2, 0) = (\pm 1, 0, 0, \sigma_4, 0, 0).$$

Hence the image of $(\operatorname{hor} TS^2) \cap \{x_3 = \pm 1\}$ under $\rho$ is the subvariety $W$ of $\overline{TS^2}$ defined by

$$\sigma_3^2 = (1 - \sigma_1^2)\sigma_4, \quad \sigma_1 = \pm 1, \sigma_4 \geq 0, \sigma_5 = 0.$$

Topologically, $W$ is the union of two half lines $\{(\pm 1, 0, \sigma_4, 0) \mid \sigma_4 \geq 0\}$. Thus $(\operatorname{hor} TS^2)/S^1$ is the canoe $\mathcal{C}$ (55). Note that the Zariski tangent space to $\mathcal{C}$ at the singular points $(\pm 1, 0, 0, 0)$ is $\{0\}$; whereas the tangent cone at $(\pm 1, 0, 0, 0)$ is the *half line* $\{(\pm 1, 0, \sigma_4, 0) \mid \sigma_4 \geq 0\}$. Thus $(\operatorname{hor} TS^2)/S^1$ is a geometric realization of the bundle of inner tangent vectors to the orbit space $\overline{S^2}$, see [15].

To describe $(\operatorname{ver} TS^2)/S^1$ geometrically, we will use the Lie algebra of the gauge group $\operatorname{Gauge}(S^2)$ of the fibration $\pi\colon S^2 \to \overline{S^2}$. Recall that a smooth map $\widetilde{\varphi}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is *equivariant* under the $S^1$-action $\widetilde{\Phi}$ (39) if and only if

$$\widetilde{\varphi}\big(\widetilde{\Phi}_t(x)\big) = \widetilde{\Phi}_t\big(\widetilde{\varphi}(x)\big). \tag{56}$$

If $\widetilde{\varphi}$ restricts to a diffeomorphism $\varphi$ of $S^2$, which induces the identity map on $\overline{S^2}$, then $\varphi$ is a *gauge transformation*. The collection of all gauge transformations forms a group $\operatorname{Gauge}(S^2)$ called the *gauge group*. We now determine the gauge group. Infinitesimalizing (56) gives

$$0 = \frac{d}{dt}\bigg|_{t=0} \widetilde{\varphi}(x) = \frac{d}{dt}\bigg|_{t=0} \widetilde{\Phi}_{-t} \circ \widetilde{\varphi} \circ \widetilde{\Phi}_t(x) = \frac{d}{dt}\bigg|_{t=0} \widetilde{\Phi}_t^*\big(Y(x)\big),$$

thinking of the mapping $\widetilde{\varphi}$ as an $S^1$-invariant vector field $Y$ on $\mathbb{R}^3$. Thus

$$0 = L_X Y = [X, Y],$$

where $X(x) = \frac{d}{dt}|_{t=0} R_t(x) = -x_2 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$ is the infinitesimal generator of the $S^1$-action $\widetilde{\Phi}$. A straightforward calculation shows that the vector field $Y$ can be written as

$$f_1(x_3, x_1^2 + x_2^2)\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right) + f_2(x_3, x_1^2 + x_2^2)\left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}\right) + f_3(x_3, x_1^2 + x_2^2)\frac{\partial}{\partial x_3},$$

for some $f_i \in C^\infty(\mathbb{R}^2)$ for $i = 1, 2, 3$. As an equivariant mapping $\widetilde{\varphi}$ of $\mathbb{R}^3$ into itself, the vector field $Y$ is

$$\widetilde{\varphi}(x) = (x_1 f_1 + x_2 f_2, x_2 f_1 - x_1 f_2, f_3).$$

In order that $\widetilde{\varphi}$ induce a map $\varphi$ of $S^2$ into itself, we must have

$$1 = (x_1^2 + x_2^2)(f_1^2 + f_2^2) + f_3^2.$$

A short calculation shows that $\varphi$ induces the map

$$\overline{\varphi}\colon \overline{S^2} \to \overline{S^2}\colon (\sigma_1, \sigma_2) \to (\widetilde{\sigma}_1, \widetilde{\sigma}_2) = \left(\bar{f}_3(\sigma_1, \sigma_2), \sigma_2(\bar{f}_1^2 + \bar{f}_2^2)(\sigma_1, \sigma_2)\right),$$

where $\overline{f_i} = \pi^* \bar{f}_i$. The map $\overline{\varphi}$ is the identity map on $\overline{S^2}$ if and only if $\overline{f_3} = \sigma_1$ and $(\bar{f}_1^2 + \bar{f}_2^2)(\sigma_1, \sigma_2)$. Thus the gauge group is

$$\{\varphi \in \mathrm{Diff}(S^2) \mid \varphi(x) = (x_1 f_1 + x_2 f_2, x_2 f_1 - x_1 f_2, x_3)$$
$$\text{where } f_i = g_i|S^2, g_i = g_i(x_3, x_1^2 + x_2^2) \in C^\infty(\mathbb{R}^2) \text{ and } f_1^2 + f_2^2 = 1\}.$$

Set $f_1 = \cos\theta$ and $f_2 = \sin\theta$, where $\theta = \Theta|S^2$ and $\Theta = \Theta(x_3, x_1^2 + x_2^2) \in C^\infty(\mathbb{R}^2)$. Then we can write $\varphi \in \mathrm{Gauge}(S^2)$ as $\varphi(x) = R_{\theta(x)} x$. From this representation one easily sees that the gauge group is abelian. Since a one parameter subgroup of $\mathrm{Gauge}(S^2)$ is given by $\varphi_t(x) = R_{t\theta(x)} x$, its infinitesimal generator is the smooth vector field

$$Z(x) = \frac{d}{dt}\bigg|_{t=0} \varphi_t(x) = \theta(x) \frac{d}{dt}\bigg|_{t=0} R_t x = \theta(x) X(x). \tag{57}$$

Note that $Z(0, 0, \pm 1) = 0$. Thus the Lie algebra $\mathrm{gauge}(S^2)$ of the gauge group is the subalgebra of the Lie algebra $\mathfrak{X}(S^2)$ of smooth vector fields on $S^2$ which satisfy (57). In fact, every infinitesimal gauge transformation $Z$ is $S^1$-invariant, since

$$(\Phi_t^* Z)(x) = T\Phi_{-t} Z\big(\Phi_t(x)\big) = \theta\big(\Phi_t(x)\big)\, \Phi_{-t} X\big(\Phi_t(x)\big) = \theta(x) X(x) = Z(x).$$

Therefore each $Z \in \mathrm{gauge}(S^2)$ corresponds to a unique $S^1$-invariant section of the bundle $\mathrm{ver}\, TS^2 \to S^2$. Thus $Z$ induces the smooth mapping

$$\overline{Z}\colon \overline{S^2} \to (\mathrm{ver}\, TS^2)/S^1\colon (\sigma_1, \sigma_2) \mapsto \left(\sigma_1, 0, \overline{\theta}^2(\sigma_1, \sigma_2)\sigma_2, \overline{\theta}(\sigma_1, \sigma_2)\sigma_2\right),$$

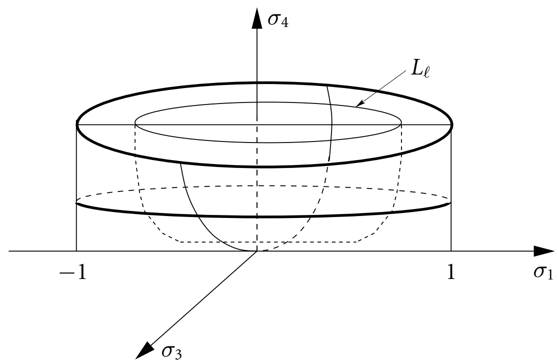


*Figure 2*: The solid canoe as the sum of the slit canoe and the canoe.

where $\pi^*\overline{\theta} = \theta$. Hence the image of $\overline{Z}$ is contained in the semialgebraic subvariety $U$ of $\overline{TS^2}$ defined by

$$\sigma_5^2 = (1 - \sigma_1^2)\sigma_4, \quad |\sigma_1| \le 1, \sigma_3 = 0, \sigma_4 \ge 0.$$

Since $\overline{Z}(\pm 1, 0) = (\pm 1, 0, 0, 0, 0, 0)$, the only part of the half lines $\{(\pm 1, 0, \sigma_4, 0) \mid \sigma_4 \ge 0\}$ of $U$ which lie in the image of $\overline{Z}$ are the points $\{(\pm 1, 0, 0, 0)\}$. Because every point of $W = U \setminus \{(\pm 1, 0, \sigma_4, 0) \mid \sigma_4 > 0\}$ lies in the image of $\overline{Z}$ for some $Z \in \text{gauge}(S^2)$, we may identify $(\text{ver}\, TS^2)/S^1$ with $W$. Geometrically, $(\text{ver}\, TS^2)/S^1$ is a *slit canoe*, namely, the canoe (55) with its bow and stern cut out.

We now give a visualization of the decomposition

$$\overline{TS^2} = (\text{ver}\, TS^2)/S^1 \times_{\overline{S^2}} (\text{hor}\, TS^2)/S^1. \tag{58}$$

We apply the $\mathbb{Z}_2$-action (53) to the decomposition (58). The $\mathbb{Z}_2$-orbit space of $\overline{TS^2}$ is the solid canoe $\mathcal{V}$ (54). Every point in the interior of $\mathcal{V}$ lies on a leaf $L_\ell$

$$\sigma_3^2 + \ell^2 = (1 - \sigma_1^2)\sigma_4, \quad |\sigma_1| < 1, \sigma_4 \ge 0\ \tau = \ell^2,$$

which is the image of the space $J^{-1}(\ell)/S^1$ of orbits of the action $\Phi$ of angular momentum $\ell$ under the $\mathbb{Z}_2$ orbit map.[2] The image of $(\text{ver}\, TS^2)/S^1$ under the $\mathbb{Z}_2$-orbit map is the union of $(\pm 1, 0, 0, 0)$ and

[2]Or what is the same thing, the space $\widehat{J}^{-1}(\ell)/S^1$ of the O(2)-orbits of angular momentum $\ell^2$ (see footnote 1).

$$0 = (1 - \sigma_1^2)\sigma_4 \quad |\sigma_1| < 1, \sigma_4 \geq 0,$$

which is the center section of the solid canoe omitting its bow and stern. In other words, it is the slit canoe. Thus every point in the solid canoe can be written as the sum of a point in the canoe and a point in the slit canoe. This is the desired geometric realization of the decomposition (58), see Figure 2.

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