Ergod. Th. & Dynam. Sys. (1986), **6**, 485–488
Printed in Great Britain

Perron units which are not Mahler measures

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(*Received* 18 *October* 1986)

Abstract. The Mahler measure $M(\alpha)$ of an algebraic integer α is the product of the absolute value of the conjugates of α which lie outside the unit circle. The quantity $\log M(\alpha)$ occurs in ergodic theory as the entropy of an endomorphism of the torus. Adler and Marcus showed that if $\beta = M(\alpha)$ then β is a Perron number which is a unit if α is a unit. They asked whether the Perron number β whose minimal polynomial is $t^m - t - 1$ is the measure of any algebraic integer. We show here that the answer is negative for all $m > 3$.

Introduction

Let α be an algebraic integer with conjugates $\alpha = \alpha_1, \ldots, \alpha_n$ over the rationals. The *Mahler measure* $M(\alpha)$ of α is defined by

$$M(\alpha) = \prod_i \max(|\alpha_i|, 1).$$

If we number the α_i so that

$$|\alpha_1| \geq \cdots \geq |\alpha_\nu| > 1 \geq |\alpha_{\nu+1}| \geq \cdots \geq |\alpha_n|$$

then $M(\alpha) = \beta = u\alpha_1 \cdots \alpha_\nu$ where $u = \pm 1$. Thus β is itself an algebraic integer. We wish to consider the inverse question of deciding if a given algebraic integer β is $M(\alpha)$ for some α, in which case we say that *β is a measure.*

The quantity $\log M(\alpha)$ occurs in ergodic theory as the entropy of an automorphism of the torus. In [**1**], Adler and Marcus observed that if $\beta = M(\alpha)$ then β is a Perron number, i.e. if γ is a conjugate of β different from β then $|\gamma| < \beta$. Furthermore, if α is a unit then so is β. They asked [**1**, p. 80] whether the positive zero β of $t^m - t - 1$ can be a measure for any $m > 3$.

Here we will present two additional requirements which must be satisfied by a measure and apply our results to show, in particular, that the above β is not a measure for any $m > 3$. This may be regarded as indirect evidence in favour of Lehmer's conjecture [**5**] that the set of measures is bounded away from 1 since the above $\beta \to 1$ as $m \to \infty$.

Our interest in these questions was awakened by the interesting lecture of Boyle [**4**]. This research was supported in part by a grant from NSERC.

THEOREM 1. *Suppose α is an algebraic integer and that $\beta = M(\alpha) = u\alpha_1 \cdots \alpha_\nu$. Then:*

(a) *All conjugates $\gamma \neq \beta$ of β lie in the annulus $\beta^{-1} \leq |\gamma| < \beta$. If $|\gamma| = \beta^{-1}$ then $\gamma = \pm\beta^{-1}$.*

https://doi.org/10.1017/S0143385700003643 Published online by Cambridge University Press

(b) *If* $\deg \alpha = n$, $\deg \beta = m$ *and* α *has* ν *conjugates in* $|z| > 1$, *then* $m\nu/n = r$ *is an integer and* $N(\beta) = u^m N(\alpha)^r$.

In particular, if $N(\alpha) = 1$ and m is even then $N(\beta) = 1$.

Proof. (a) Let I denote a subset of $[1, n] = \{1, 2, \ldots, n\}$ of cardinality $|I|$ and let J be its complement. Write

$$\alpha(I) = \prod \{\alpha_k : k \in I\}.$$

Then each conjugate γ of β is of the form

$$\gamma = u\alpha(I), \qquad \text{with } |I| = \nu.$$

Since $N(\alpha) = \alpha_1 \cdots \alpha_n = \alpha(I)\alpha(J)$, we have

$$\gamma^{-1} = u\alpha(J)/N(\alpha).$$

Clearly $\beta = M(\alpha) = \max\{|\alpha(I)| : I \subset [1, n]\}$. If α is non-reciprocal then $|\alpha_{\nu+1}| < 1$ so equality holds only if $I = [1, \nu]$. If α is non-reciprocal then $|\alpha_k| = 1$ for $\nu + 1 \le k \le n - \nu$ so equality holds only if $[1, \nu] \subset \dot{I} \subset [1, n - \nu]$.

Thus, if $\gamma \neq \beta$ then

$$|\gamma| = |\alpha(I)| < \beta.$$

Also $|\gamma^{-1}| = |\alpha(J)|/|N(\alpha)| \le \beta$. Equality here requires $N(\alpha) = \pm 1$ and $J = [1, n - \nu]$. Thus, either α is reciprocal and $\gamma^{-1} = \beta$ or else α is non-reciprocal, $2\nu = n$ and $\gamma^{-1} = N(\alpha)\beta = \pm\beta$. This completes the proof of (a).

(b) Let G be the Galois group of $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ over $\mathbb{Q}$ represented as a permutation group on $[1, n]$. Then G acts on the ν-subsets of $[1, n]$ in the obvious way.

Let $O = \{I_1, \ldots, I_M\}$ denote the orbit of $I_1 = [1, \nu]$ under G. The conjugates of $\beta = u\alpha(I_1)$ are thus $\{u\alpha(I_1), \ldots, u\alpha(I_M)\}$. Each of the m conjugates of β appears, say, s times in the list $u\alpha(I_1), \ldots, u\alpha(I_M)$ with $M = ms$. But here $\beta = u\alpha(I_1) > |u\alpha(I_j)|, j \neq 1$ so $s = 1$ and thus $M = m$.

Consider $S_j = \{I_k \mid j \in I_k\}$ for $j = 1, \ldots, n$. We claim that $|S_j| = r$ is independent of j. For G is transitive and hence for any i, j there is a π in G with $j = \pi(i)$. The action of π on O defines a one-one correspondence between S_i and S_j so $|S_i| = |S_j|$. The orbit O thus consists of m ν-sets which together contain the n conjugates of α r times each. Hence $m\nu = nr$ and

$$N(\beta) = u^m \prod_{k=1}^{m} \alpha(I_k) = u^m(\alpha_1 \cdots \alpha_n)^r = u^m N(\alpha)^r.$$

COROLLARY. *Let* $\theta_0 > 1$ *solve* $t^3 - t - 1 = 0$. *If* $\beta < \theta_0$, β *is a measure and* $\deg \beta$ *is even, then* $N(\beta) = 1$.

Proof. By a theorem of Smyth [7]; if $M(\alpha) < \theta_0$ then α is reciprocal, i.e. α is an algebraic integer with α^{-1} a conjugate of α. Then $N(\alpha) = +1$ and by (b), $N(\beta) = +1$ if m is even.

PROPOSITION. *Let* r *be as in* (b) *of theorem* 1. *Then* $M(\beta) \le \beta^r$. *If* $r = 1$ *then* β *is a Pisot or Salem number.*

https://doi.org/10.1017/S0143385700003643 Published online by Cambridge University Press

Proof. $M(\beta)$ is a product of certain $|\alpha(I_k)|$ with I_k in O. Since each α_j occurs at most r times in the disjoint union of the I_k in O we can estimate this product by

$$M(\beta) \leq |\alpha_1|^r \cdots |\alpha_\nu|^r \cdot 1^r \cdots 1^r = M(\alpha)^r = \beta^r.$$

If $r=1$ then $M(\beta)=\beta$ so $\beta>1$ and has all its other conjugates in $|z|\leq 1$, thus is a Pisot or Salem number.

THEOREM 2. *Let $m>3$ and let β be the positive zero of t^m-t-1. Then β is not a measure.*

Proof. The complex zeros of $P(z)=z^m-z-1$ are discussed by Selmer **[6]** in his proof that P is irreducible. They clearly lie on the curve C_m defined by $|z|^m=|z+1|$. If $z=r\,e^{i\phi}$ then C_m has the polar equation $r^{2m}=r^2-2r\cos\phi+1$. This has a unique positive solution $r=f(\phi)$ where f is even and strictly decreasing on $0\leq\phi\leq\pi$. Clearly $\beta=f(0)$.

A given $r\geq 0$ will be in the range of f if and only if the circles $|z|=r$ and $|z+1|=r^m$ intersect, i.e. if $r^m+r\geq 1$. In particular, if $r=1/\beta$ then $r^m+r=1/(\beta+1)+1/\beta\geq 1$ since $\beta^2-\beta-1\leq\beta^m-\beta-1=0$. Let $\beta^{-1}\,e^{i\phi_0}$ denote the point of intersection of $|z|=\beta^{-1}$ and $|z+1|=\beta^{-m}$ with $0<\phi_0<\pi$. If $m>2$ then $\phi_0<\pi$, and then $f(\phi)<\beta^{-1}$ for $\phi_0<\phi\leq\pi$.

To show that $P(z)=0$ has a zero γ with $|\gamma|<\beta^{-1}$ we must show that it has a zero $re^{i\phi}$ with $\phi_0<\phi<\pi$. If $m>2$ is even this is clear since $-f(\pi)$ is such a zero.

If m is odd then the argument principle shows that $P(z)=0$ has two zeros in the sector $|\arg z-\pi|<\pi/m$. If $\pi/m<\pi-\phi_0$ then P has a zero with $\phi_0<\phi<\pi$.

Since $P(1+1/m)>0>P(1)$, it follows that $1<\beta<1+1/m$ so, for $m\geq 5$, $\beta<1.2$ and

$$\pi-\phi_0=\cos^{-1}\left(\frac{\beta}{2}(1+\beta^{-2}-(1+\beta)^{-2})\right)$$
$$>\cos^{-1}\left(\frac{1.2}{2}(1+(1.2)^{-2}-(2.2)^{-2})\right)=\delta,$$

where $\pi/\delta>6.7$. Thus P has a zero with $|\gamma|<\beta^{-1}$ if $m\geq 7$.

If $m=5$, a numerical calculation shows that $\beta=1.16730\cdots$ and that there is a zero γ with $|\gamma|=0.84219\cdots<1/\beta$.

Applying theorem 1(a), β is not a measure for $m\geq 4$.

Remarks. (1) For even m, the corollary also applies since $\beta<\theta_0$ but $N(\beta)=-1$.

(2) The argument principle shows that z^m-z-1 has exactly one zero in each of the sectors

$$S_k=\left\{\left|\arg z-\frac{2k\pi}{m-1}\right|\leq\frac{\pi}{m}\right\}\qquad\text{for } k=0,1,\ldots,m-2;\ k\neq(m-1)/2.$$

Since the ϕ_0 of the proof tends to $\pi-\cos^{-1}(7/8)$ as $m\to\infty$, asymptotically $m\cos^{-1}(7/8)/\pi$ of the zeros of z^m-z-1 satisfy $|z|<1/\beta$.

(3) The positive zero β of z^6-z^5-1 satisfies theorem 1(a), but not 1(b) hence is not a measure.

(4) The polynomial $z^5 - z^2 - 1$ has zeros $\beta = 1.1938\ldots$, $|\beta_2| = |\beta_3| = 1.0864\cdots$ and $|\beta_4| = |\beta_5| = 0.8423\cdots > 1/\beta$ hence does not violate either part of theorem 1. It seems unlikely that β is a measure.

(5) There are reciprocal α for which $M(\alpha)$ is non-reciprocal. We presented some examples of degree 6 in [**2**]. In particular if the minimal polynomial of α is $P(t) = t^6 + t^5 + 2t^4 + 3t^3 + 2t^2 + t + 1$ then $\beta = M(\alpha)$ has minimal polynomial $t^3 - t^2 - t - 1$. The explanation depends on the Galois group G of P. Recently [**3**] we have constructed such examples of every degree $n = 2(2k+1) \geq 6$.

(6) A complete characterization of the set of measures would be very desirable but seems difficult.

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