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UNBOUNDED FREDHOLM MODULES AND SPECTRAL FLOW

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ABSTRACT. An *odd unbounded* (respectively, p*-summable*) *Fredholm module* for a unital Banach $*$-algebra, A, is a pair (H, D) where A is represented on the Hilbert space, H, and D is an unbounded self-adjoint operator on H satisfying:

(1) $(1 + D^2)^{-1}$ is compact (respectively, $\mathrm{Trace}\Big((1 + D^2)^{-(p/2)}\Big) < \infty$), and

(2) $\{a \in A \mid [D, a] \text{ is bounded}\}$ is a dense $*-$subalgebra of A.

If u is a unitary in the dense $*-$subalgebra mentioned in (2) then

$$uDu^* = D + u[D, u^*] = D + B$$

where B is a bounded self-adjoint operator. The path

$$D^u_t := (1 - t)D + tuDu^* = D + tB$$

is a "continuous" path of unbounded self-adjoint "Fredholm" operators. More precisely, we show that

$$F^u_t := D^u_t\Big(1 + (D^u_t)^2\Big)^{-\frac{1}{2}}$$

is a norm-continuous path of (bounded) self-adjoint Fredholm operators. The *spectral flow* of this path $\{F^u_t\}$ (or $\{D^u_t\}$) is roughly speaking the net number of eigenvalues that pass through 0 in the positive direction as t runs from 0 to 1. This integer,

$$\mathrm{sf}(\{D^u_t\}) := \mathrm{sf}(\{F^u_t\}),$$

recovers the pairing of the K-homology class $[D]$ with the K-theory class $[u]$.

We use I. M. Singer's idea (as did E. Getzler in the θ-summable case) to consider the operator B as a parameter in the Banach manifold, $B_{\mathrm{sa}}(H)$, so that spectral flow can be exhibited as the integral of a closed 1-form on this manifold. Now, for B in our manifold, any $X \in T_B\Big(B_{\mathrm{sa}}(H)\Big)$ is given by an X in $B_{\mathrm{sa}}(H)$ as the derivative at B along the curve $t \longmapsto B + tX$ in the manifold. Then we show that for m a sufficiently large half-integer:

$$\alpha(X) = \frac{1}{\tilde{C}_m}\,\mathrm{Tr}\bigg(X\Big(1 + (D + B)^2\Big)^{-m}\bigg)$$

is a closed 1-form. For any piecewise smooth path $\{D_t = D + B_t\}$ with D_0 and D_1 unitarily equivalent we show that

$$\mathrm{sf}(\{D_t\}) = \frac{1}{\tilde{C}_m}\int_0^1 \mathrm{Tr}\Big(\frac{d}{dt}(D_t)(1 + D_t^2)^{-m}\Big)\,dt$$

the integral of the 1-form α. If D_0 and D_1 are not unitarily equivalent, we must add a pair of correction terms to the right-hand side. We also prove a bounded finitely summable version of the form:

$$\mathrm{sf}(\{F_t\}) = \frac{1}{C_n}\int_0^1 \mathrm{Tr}\Big(\frac{d}{dt}(F_t)(1 - F_t^2)^n\Big)\,dt$$

for $n \geq \frac{p-1}{2}$ an integer. The unbounded case is proved by reducing to the bounded case via the map $D \longmapsto F = D(1 + D^2)^{-\frac{1}{2}}$. We prove simultaneously a type II version of our results.

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Introduction. Spectral flow was invented by Atiyah and Lusztig to handle smooth paths of self-adjoint elliptic operators. In the work of Atiyah-Patodi-Singer [APS], integral formulas for spectral flow were lurking in the background at the outset. In particular, Singer indicated how, in the case of certain geometric operators, spectral flow could be interpreted as the integral of a 1-form obtained as the exterior derivative of the eta invariant on a certain manifold of operators [Si, p. 190]. Douglas-Hurder-Kaminker in their study of the eta invariant used Singer's formula in an essential way [DHK; H; Kam]. Moreover, the idea of type II spectral flow arose naturally in their work as well as in the paper of Mathai [M].

Recent work by one of us [P1, 2] has clarified the purely functional analytic aspects of spectral flow in both the type I and type II cases. We felt that it was now both possible and desirable to obtain analogous integral formulas for spectral flow in much more general settings. In fact, two such formulas already exist in particular cases: [G] and [P2]. This paper grew out of an attempt to find a deeper connection between these two integral formulas beyond mere analogy.

To discuss this further, we need to set some terminology and notation. If A is a Banach $*$-algebra, then a (*bounded*, *odd*) *Fredholm module for A* is a pair (H, F) where H is a Hilbert space on which A is represented and F is a self-adjoint operator on H satisfying $F^2 = 1$ and $[F, a]$ is compact for all $a \in A$. An *unbounded* (*odd*) *Fredholm module for A* is a pair (H, D) where, again, H is a Hilbert space on which A acts and D is an unbounded self-adjoint operator on H satisfying $(1{+}D^2)^{-1}$ is compact and $\{a \in A \mid [D, a]$ is bounded$\}$ is dense in A. At this level of generality, the mapping $(H, D) \longmapsto (H, F)$ where $F = \operatorname{sign}(D)$ produces a bounded Fredholm module from an unbounded one. One can impose summability conditions more stringent than compactness in the above definitions and thereby axiomatize concrete cycles for cyclic cohomology or entire cyclic cohomology [C1,2]. In this specialized setting, the properties of the mapping $(H, D) \longmapsto (H, F)$ are more subtle. Returning to the general case, if u is a unitary in A, then $F - uFu*$ is compact, and the straight line path $\{F_t\}$ from F to $uFu*$ is a continuous path of self-adjoint Fredholm operators. The *spectral flow* of this path, $\mathrm{sf}\{F_t\}$, is roughly, the net number of eigenvalues that pass through 0 in the positive direction as t runs from 0 to 1. If P is the projection on the positive spectral subspace for F, then PuP is a Fredholm operator on $P(H)$ and $\operatorname{ind}(PuP) = \mathrm{sf}\{F_t\}$. Similarly, $\{D_t := (1-t)D + t\, uDu*\}$ is a "continuous" path of self-adjoint "unbounded Fredholm operators" and we have $\operatorname{ind}(PuP) = \mathrm{sf}\{D_t\}$ where P is the projection on the nonnegative spectral subspace for D.

In order to obtain explicit integral formulas for spectral flow, it is clear that some sort of summability conditions need to be imposed. To this end, Ezra Getzler in [G] outlined a method of exhibiting spectral flow as the integral of a 1-form in the context of unbounded *θ-summable* Fredholm modules ($\mathrm{Tr}(e^{-tD^2}) < \infty$ for all $t > 0$). In particular, if $\{D_t\}$ is the path mentioned above he indicated how to prove that

$$\mathrm{sf}\{D_t\} = \frac{1}{\sqrt{\pi}} \int_0^1 \mathrm{Tr}(D_t' e^{-D_t^2})\, dt$$

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in the θ-summable context. Then, in [P2] one of us showed that for (H,F) a bounded p-summable Fredholm module $\big(\mathrm{Tr}(|[F,a]|^p) < \infty\big)$ and n a sufficiently large positive integer,

$$\mathrm{sf}(\{F_t\}) = \frac{1}{C_n}\int_0^1 \mathrm{Tr}\Big(F_t'(1-F_t^2)^n\Big)\,dt,$$

where $\{F_t\}$ is as above. These are the two formulas that we want to reconcile.

Our path seemed clear. One, there should be a finitely summable unbounded version of Getzler's formula of the form:

$$\mathrm{sf}\{D_t\} = \frac{1}{\tilde{C}_m}\int_0^1 \mathrm{Tr}\Big(D_t'(1+D_t^2)^{-m}\Big)\,dt$$

(here $\mathrm{Tr}\big((1+D^2)^{-p}\big) < +\infty$ for some $p > 0$), where the D_t vary in $D + \mathcal{B}(H)_{\mathrm{sa}}$. Two, the finitely summable bounded version should be more general than the computation done in [P2]: that is, the $\{F_t\}$ should vary freely in a manifold, $F + \mathcal{L}_{\mathrm{sa}}$ where $\mathcal{L}$ is some Schatten-like class in $\mathcal{B}(H)$. Third, we should be able to obtain the unbounded from the bounded case via the map $D \longmapsto F_D = D(1+D^2)^{-\frac{1}{2}}$. Of course, F_D is a smooth approximation of sign(D) and the corresponding pair (H, F_D) is called a *pre*-Fredholm module. The technical obstacles to this program are legion, especially since we include the type II situation at all stages.

In chapter one we anticipate our needs and consider finitely summable *pre*-Fredholm modules (actually, Breuer-Fredholm modules) (N, F_0) for a Banach $*$-algebra A. That is, $1-F_0^2$ is $\frac{p}{2}$-summable and $[F_0, a]$ is p-summable in the semifinite factor N for a dense set of a's in A. Our manifold of allowable perturbations of F_0 is precisely the subspace of $F_0 + \mathcal{L}^p_{\mathrm{sa}}$ which retains these two summability criteria. We denote this manifold by $F_0 + \mathcal{L}^{p,\frac{p}{2}}_{\mathrm{sa}}$. For F in this manifold and $X \in \mathcal{L}^{p,\frac{p}{2}}_{\mathrm{sa}}$ a tangent vector (at F), let $\alpha(X) = \frac{1}{C_n}\mathrm{Tr}\big(X(1-F^2)^n\big)$ where $n \geq \frac{p-1}{2}$ is an integer and C_n is a normalizing constant. Then α is a *closed* 1-form; that is, $d\alpha = 0$ (Proposition 1.3). By a version of the Poincaré Lemma (Proposition 1.4), α is *exact*; that is, $\alpha = d\theta$ where $\theta(F)$ is the line integral of α along the segment $[F_0 \longrightarrow F]$. A simple argument (Proposition 1.5) shows that the integral of α along a piecewise-C^1 path Γ depends only on the endpoints of Γ. Now, if Γ is such a path from F_1 to F_2 we can extend it to a path Γ' from sign(F_1) to sign(F_2). By Theorem 3.1 of [P2], the integral of α along Γ' is, therefore, sf(Γ'). The formula for sf(Γ) with natural correction terms follows (Theorem 1.7).

In chapter two we study p-summable *unbounded* modules by reducing to the bounded case. Our main technical tools are the functional calculus and an integral formula for $(1+D^2)^{-\frac{1}{2}}$. We first show that if (N, D_0) is p-summable then (N, F_{D_0}) is q-summable for $q > p$ (Proposition 2.4). This result may be known but it does not appear to be in the literature: Connes discusses the *even* case in [C1] and does not use the map $D \longmapsto F_D$. We believe that $q > p$ is necessary. We then show that $D \longmapsto F_D$ mapping $D_0 + N_{\mathrm{sa}}$ to $F_{D_0} + \mathcal{L}^{q,\frac{q}{2}}_{\mathrm{sa}}$ is continuous for $q > p$ (Corollary 2.8). Next, we show that if $t \longmapsto D_t$ is C^1 in operator norm then $t \longmapsto F_t = F_{D_t}$ is C^1 in the norm of $\mathcal{L}^q_{\mathrm{sa}}$ (Proposition 2.10). This is sufficient by Remark 1.8. We also obtain an integral formula for $\frac{d}{dt}(F_t)$ in Proposition 2.10. Using this formula and the trace property we show that

$$\mathrm{Tr}\Big(F_t'(1-F_t^2)^n\Big) = \mathrm{Tr}\Big(D_t'(1+D_t^2)^{-(n+\frac{3}{2})}\Big).$$

The extra $\frac{3}{2}$ is exactly what is expected from *formally* differentiating $F_t = D_t(1+D_t^2)^{-\frac{1}{2}}$ (Proposition 2.12). Finally, we deduce that if (N, D_0) is p-summable and $m = n+\frac{3}{2}$ where $n > \frac{p-1}{2}$ is an integer, *then* for $D \in D_0 + N_{\mathrm{sa}}$ and $X \in N_{\mathrm{sa}}$ a tangent vector (at D) the 1-form, $\frac{1}{\tilde{C}_m}\mathrm{Tr}\Big(X(1+D^2)^{-m}\Big)$ is *exact* and integrating this 1-form along a path $\{D_t\}$ yields $\mathrm{sf}\{D_t\}$ (modulo natural correction terms).

The operator norm estimates and trace norm estimates needed in Chapter 2 are of independent interest and are contained in Appendix A and Appendix B, respectively. In Appendix C we present some examples.

We are currently working on a sequel where we study the θ-summable situation in both the bounded and unbounded cases for types I and II.

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1. Spectral flow as the integral of a 1-form—Finitely summable bounded Fredholm modules (Types I and II).

DEFINITION 1.1. Let A be a unital Banach $*$-algebra and $p > 0$, then an *odd p-summable pre-Breuer-Fredholm module for A* is a pair (N, F_0) where N is a semifinite factor (on a separable Hilbert space), A is unitally $*$-represented in N, F_0 in N is a self-adjoint operator satisfying

(1) $1 - F_0^2$ is $\frac{p}{2}$-summable, and

(2) $[F_0, a]$ is p-summable for all a in a dense $*$-subalgebra $\mathcal{A}$ of A (see [P2]).

We observe that if χ is the characteristic function of $[0,\infty)$ then

$$\tilde{F}_0 = 2\chi(F_0) - 1 = \mathrm{sign}(F_0)$$

is self-adjoint, in N, and satisfies $\tilde{F}_0^2 = 1$. Moreover,

$$1 - F_0^2 = \tilde{F}_0^2 - F_0^2 = (\tilde{F}_0 - F_0)(\tilde{F}_0 + F_0)$$

and since $(\tilde{F}_0 + F_0)$ is invertible in N, we see that

$$(\tilde{F}_0 - F_0) = (1 - F_0^2)(\tilde{F}_0 + F_0)^{-1}$$

is $\frac{p}{2}$-summable. This, of course, implies that $[\tilde{F}_0, a]$ is p-summable whenever $[F_0, a]$ is p-summable. Thus, we can obtain a genuine p-summable Breuer-Fredholm module $(N, \tilde{F}_0)$. However, we want the greater generality when we come to apply these results to unbounded Breuer-Fredholm modules.

In general, we denote the $*$-ideal of n-summable operators in N by $\mathcal{L}^n(N)$ or $\mathcal{L}^n$. We denote by $\mathcal{L}^n_{\mathrm{sa}}(N)$ or $\mathcal{L}^n_{\mathrm{sa}}$ the real subspace of self-adjoints in $\mathcal{L}^n(N)$.

We let $P = \chi(F_0)$ so that $\tilde{F}_0 = 2P - 1$ and hence, relative to the decomposition, $1 = P + (1-P)$, $\tilde{F}_0$ has the matrix, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. For $a \in \mathcal{A}$ we let $a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ relative to this decomposition, so that $[\tilde{F}_0, a] \in \mathcal{L}^p(N)$ implies that a_{12} and a_{21} are in $\mathcal{L}^p(N)$ (and conversely). If $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ is a unitary in $\mathcal{A}$, then

$$u\tilde{F}_0u^* - \tilde{F}_0 = \left[\begin{array}{c|c} -2u_{12}u_{12}^* & u_{11}u_{21}^* - u_{12}u_{22}^* \\ \hline u_{21}u_{11}^* - u_{22}u_{12}^* & 2u_{21}u_{21}^* \end{array}\right]$$

is a self-adjoint element in the $*$-algebra,

$$\left[\begin{array}{c|c} P\mathcal{L}^{\frac{p}{2}}P & P\mathcal{L}^pP^\perp \\ \hline P^\perp\mathcal{L}^pP & P^\perp\mathcal{L}^{\frac{p}{2}}P^\perp \end{array}\right].$$

This is easily seen to be a Banach $*$-algebra in the norm

$$\|(a_{i,j})\| = \|a_{11}\|_{p/2} + \|a_{12}\|_p + \|a_{21}\|_p + \|a_{22}\|_{p/2} + \sum_{i,j} \|a_{ij}\|_\infty.$$

If N is a type I factor, the operator-norm terms are not necessary [CP, Proposition A-1]. We denote this Banach $*$-algebra by $\mathcal{L}^{p,\frac{p}{2}}$ and its real self-adjoint subspace by $\mathcal{L}^{p,\frac{p}{2}}_{\mathrm{sa}}$.

Now,

$$uF_0u^* - F_0 = u(F_0 - \tilde{F}_0)u^* + (u\tilde{F}_0u^* - \tilde{F}_0) + (\tilde{F}_0 - F_0)$$

which is in

$$\mathcal{L}^{\frac{p}{2}}_{\mathrm{sa}} + \mathcal{L}^{p,\frac{p}{2}}_{\mathrm{sa}} + \mathcal{L}^{\frac{p}{2}}_{\mathrm{sa}} = \mathcal{L}^{p,\frac{p}{2}}_{\mathrm{sa}}.$$

Thus, uF_0u^* is in the affine space $F_0 + \mathcal{L}^{p,\frac{p}{2}}_{\mathrm{sa}}$, as are $\tilde{F}_0$ and $u\tilde{F}_0u^*$ for all $u \in U(\mathcal{A})$.

LEMMA 1.2. *Let (N, F_0) be an odd p-summable pre-Breuer-Fredholm module for the unital Banach $*$-algebra, A. Let $\tilde{F}_0 = 2\chi(F_0) - 1$ and let*

$$\mathcal{A} = \{a \in A \mid [F_0, a] \in \mathcal{L}^p(N)\}.$$

Then,

(1) F_0, $\tilde{F}_0$, uF_0u^*, $u\tilde{F}_0u^*$ are in the affine space $F_0 + \mathcal{L}^{p,\frac{p}{2}}_{\mathrm{sa}}$ for all $u \in U(\mathcal{A})$, and

(2) for all F in $F_0 + \mathcal{L}^{p,\frac{p}{2}}_{\mathrm{sa}}$, $(1 - F^2) \in \mathcal{L}^{\frac{p}{2}}_{\mathrm{sa}}$, and

$$F \longmapsto (1 - F^2) \colon F_0 + \mathcal{L}^{p,\frac{p}{2}}_{\mathrm{sa}} \longrightarrow \mathcal{L}^{\frac{p}{2}}_{\mathrm{sa}}$$

is continuous.

PROOF. It remains to show (2). By (1) $F_0 - \tilde{F}_0$ is in $\mathcal{L}^{p,\frac{p}{2}}_{\mathrm{sa}}$, so any F in $F_0 + \mathcal{L}^{p,\frac{p}{2}}_{\mathrm{sa}}$ is of the form $F = \tilde{F}_0 + k$ with $k \in \mathcal{L}^{p,\frac{p}{2}}_{\mathrm{sa}}$. Thus,

$$\begin{aligned} 1 - F^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\left[\begin{array}{c|c} 1 + k_{11} & k_{12} \\ \hline k_{12}^* & -1 + k_{22} \end{array}\right]\right)^2 \\ &= -\left[\begin{array}{c|c} k_{11}^2 + 2k_{11} + k_{12}k_{12}^* & k_{11}k_{12} + k_{12}k_{22} \\ \hline k_{12}^*k_{11} + k_{22}k_{12}^* & k_{22}^* - 2k_{22} + k_{12}^*k_{12} \end{array}\right] \end{aligned}$$

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is in $L_{\mathrm{sa}}^{\frac{p}{2}}$ as claimed. ∎

NOTES. (1) The affine space $F_0+L_{\mathrm{sa}}^{p,\frac{p}{2}}$ appears to have a distinguished point, namely F_0. However, if $F_1 \in F_0 + L_{\mathrm{sa}}^{p,\frac{p}{2}}$ is any other point in this space and we use the positive and negative spectral subspaces of F_1 to define a new space, say $(L_{\mathrm{sa}}^{p,\frac{p}{2}})'$ then, in fact:

(a) $(L_{\mathrm{sa}}^{p,\frac{p}{2}})' = L_{\mathrm{sa}}^{p,\frac{p}{2}}$
(b) $L_{\mathrm{sa}}^{p,\frac{p}{2}} = \{X \in L_{\mathrm{sa}}^{p} \mid 1-(F_0+X)^2 \in L^{\frac{p}{2}}\}$
(c) $F_1 + L_{\mathrm{sa}}^{p,\frac{p}{2}} = F_0 + L_{\mathrm{sa}}^{p,\frac{p}{2}}$.

PROOF. To see (b), first note that by part (2) of the previous lemma we get: $\subseteq$. On the other hand if $X \in L_{\mathrm{sa}}^{p}$ and $1-(F_0+X)^2 \in L^{\frac{p}{2}}$ then letting $Y_0 = F_0 - \tilde{F}_0 \in L_{\mathrm{sa}}^{\frac{p}{2}}$, we have $1-\bigl(\tilde{F}_0+(Y_0+X)\bigr)^2 \in L^{\frac{p}{2}}$

$$\begin{aligned}&\Rightarrow (Y_0+X)^2 + \tilde{F}_0Y_0 + \tilde{F}_0X + X\tilde{F}_0 + Y_0\tilde{F}_0 \in L^{\frac{p}{2}}\\ &\Rightarrow \tilde{F}_0X + X\tilde{F}_0 \in L^{\frac{p}{2}} \Rightarrow X \in L_{\mathrm{sa}}^{p,\frac{p}{2}}.\end{aligned}$$

So (b) holds.

To see (a) let $X \in L_{\mathrm{sa}}^{p,\frac{p}{2}}$ then $X \in L_{sa}^{p}$ and $1-F_1^2 = 1-(F_0+X)^2 \in L^{\frac{p}{2}}$. But, by hypothesis $1-(F_1-X)^2 = 1-F_0^2 \in L^{\frac{p}{2}}$, so by part (b) applied to F_1 we get $(-X) \in (L_{\mathrm{sa}}^{p,\frac{p}{2}})'$ and so $X \in (L_{\mathrm{sa}}^{p,\frac{p}{2}})'$. The reverse inequality is proved similarly.

Part (c) is trivial. ∎

(2) For further evidence that these slightly exotic spaces are the correct ones to use in this context, we show in Chapter 2 that they are exactly the receptacles for the transformation $D \longmapsto F_D = D(1+D^2)^{-\frac{1}{2}}$ from the unbounded to the bounded set-up.

We now fix A and (N, F_0) as above and consider the space $M = F_0 + L_{\mathrm{sa}}^{p,\frac{p}{2}}$ as a real Banach manifold. Let n be a positive integer with $n \geq \frac{p-1}{2}$. We define a 1-form α on M via

$$\alpha(X) = \frac{1}{C_n}\operatorname{Tr}\bigl(X(1-F^2)^n\bigr)$$

where

$$\begin{gathered}F \in M = F_0 + L_{\mathrm{sa}}^{p,\frac{p}{2}},\\ X \in T_F(M) = L_{\mathrm{sa}}^{p,\frac{p}{2}} \subseteq L_{\mathrm{sa}}^{n}, \quad \text{and}\\ C_n = \int_{-1}^{1}(1-t^2)^n\,dt = \frac{n!\,2^{n+1}}{1\cdot 3\cdots(2n+1)}.\end{gathered}$$

By the lemma, α is certainly a real-valued function of X and F and for fixed $F \in M$ it is a bounded linear function of X. It is probably true that

$$\alpha\colon T(M) = L_{\mathrm{sa}}^{p,\frac{p}{2}} \times M \longrightarrow \mathbf{R}$$

is continuously differentiable in the strongest sense: *i.e.*, for each (X,F) in $T(M)$, $D\alpha(X,F)$ is in $L(L_{\mathrm{sa}}^{p,\frac{p}{2}} \times M, \mathbf{R})$ and $(X,F) \longmapsto D\alpha(X,F)$ is norm-continuous. Since we are only using the language and ideas of (elementary) differential geometry as motivation

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and none of the big theorems we do not need this. We only need the weaker result that $d\alpha$, the exterior derivative of α, makes sense and $d\alpha = 0$.

To this end, we use the invariant definition of exterior differentiation [S, p. 292]. For F in M, we have X, Y in $T_F(M) = L^{p,\frac{p}{2}}_{\mathrm{sa}}$ realized as tangent vectors at F by differentiating the curves $F + sX$ and $F + sY$ at $s = 0$. That is, we consider X and Y also as the canonical vector fields on M (or flows on M) given by flowing in the X direction or Y direction. Then, by definition:

$$d\alpha(X, Y) = X \cdot \big(\alpha(Y)\big) - Y \cdot \big(\alpha(X)\big) - \alpha([X, Y]).$$

Since X and Y commute *as flows* the last term is 0 and so drops from our calculation.

PROPOSITION 1.3. *Let N be a semifinite factor and F_0 a self-adjoint element in N satisfying $1 - F_0^2$ is $\frac{p}{2}$-summable for some $p > 0$. Let M be the real Banach manifold $F_0 + L^{p,\frac{p}{2}}_{\mathrm{sa}}$. Let $n \geq \frac{p-1}{2}$ be a positive integer and let $\alpha(X) = \frac{1}{C_n}\operatorname{Tr}\big(X(1 - F^2)^n\big)$. Then α is a closed 1-form on M; that is, $d\alpha = 0$.*

(NOTE. (N, F_0) is an odd p-summable pre-Breuer-Fredholm module for the C^*-algebra $A = \mathbf{C}1$, but the particular algebra is unnecessary here.)

PROOF. Fix $F \in M$ and $X, Y \in T_F(M)$, then

$$\begin{aligned}(d\alpha)_F(X, Y) &= X_F \cdot \big(\alpha(Y)\big) - Y_F \cdot \big(\alpha(X)\big)\\ &= \frac{d}{ds}\Big|_{s=0}\Big[\frac{1}{C_n}\Big\{\operatorname{Tr}\Big(Y\big(1 - (F + sX)^2\big)^n\Big) - \operatorname{Tr}\Big(X\big(1 - (F + sY)^2\big)^n\Big)\Big\}\Big]\end{aligned}$$

so it suffices to see that

$$\frac{d}{ds}\Big|_{s=0}\Big(\operatorname{Tr}\Big(Y\big(1 - (F + sX)^2\big)^n\Big)\Big) = \frac{d}{ds}\Big|_{s=0}\Big(\operatorname{Tr}\Big(X\big(1 - (F + sY)^2\big)^n\Big)\Big).$$

We simplify our notation a little by using $F_s = F + sX$ for $s > 0$.

$$\begin{aligned}&\frac{d}{ds}\Big|_{s=0}\Big(\operatorname{Tr}\big(Y(1 - F_s^2)^n\big)\Big)\\ &\quad= \lim_{s\to 0}\operatorname{Tr}\Big(Y\frac{1}{s}[(1 - F_s^2)^n - (1 - F^2)^n]\Big)\\ &\quad= \lim_{s\to 0}\operatorname{Tr}\Big(Y\frac{1}{s}\Big[\sum_{k=0}^{n-1}(1 - F_s^2)^k\big((1 - F_s^2) - (1 - F^2)\big)(1 - F^2)^{n-k-1}\Big]\Big)\\ &\quad= \lim_{s\to 0}\operatorname{Tr}\Big(Y\Big[\sum_{k=0}^{n-1}(1 - F_s^2)^k(-FX - XF - sX^2)(1 - F^2)^{n-k-1}\Big]\Big)\\ &\quad= \operatorname{Tr}\Big(Y\Big[\sum_{k=0}^{n-1}(1 - F^2)^k(-FX - XF)(1 - F^2)^{n-k-1}\Big]\Big)\end{aligned}$$

where $(1-F_s^2)^k \longrightarrow (1-F^2)^k$ in $L^{\frac{p}{2k}}$ and $sX^2 \longrightarrow 0$ in $L^{\frac{p}{2}}$ as X^2 is in $L^{\frac{p}{2}}$, so that $XF+FX$ is in $L^{\frac{p}{2}}$ and by the Hölder inequality [*D*], the sum converges in $L^{\frac{p}{2n}}$. Since $Y \in L^{p,\frac{p}{2}} \subseteq L^p$ and $\frac{1}{p} + \frac{2n}{p} = \frac{2n+1}{p} \geq 1$, the traces converge as claimed. Thus,

$$\frac{d}{ds}\Big|_{s=0}\Big(\operatorname{Tr}\big(Y(1 - F_s^2)^n\big)\Big) = -\sum_{k=0}^{n-1}\operatorname{Tr}\big(Y(1 - F^2)^k(FX + XF)(1 - F^2)^{n-k-1}\big)$$

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Now,

$$Y(1-F^2)^k \in L^{\frac{p}{2k+1}} \quad \text{and} \quad X(1-F^2)^{n-k-1} \in L^{\frac{p}{2(n-k-1)+1}}$$

by the Hölder inequality, so that

$$Y(1-F^2)^k FX(1-F^2)^{n-k-1} \quad \text{and} \quad Y(1-F^2)^k XF(1-F^2)^{n-k-1}$$

are both summable. Thus,

$$\begin{aligned}
&\frac{d}{ds}\Big|_{s=0}\Big(\mathrm{Tr}\big(Y(1-F_s^2)^n\big)\Big)\\
&\quad = -\sum_{k=0}^{n-1}\Big[\mathrm{Tr}\big(Y(1-F^2)^k FX(1-F^2)^{n-k-1}\big) + \mathrm{Tr}\big(Y(1-F^2)^k XF(1-F^2)^{n-k-1}\big)\Big]\\
&\quad = -\sum_{k=0}^{n-1}\Big[\mathrm{Tr}\big(X(1-F^2)^{n-k-1} Y(1-F^2)^k F\big) + \mathrm{Tr}\big(XF(1-F^2)^{n-k-1} Y(1-F^2)^k\big)\Big]\\
&\quad = -\sum_{k=0}^{n-1}\Big[\mathrm{Tr}\big(X(1-F^2)^{n-k-1} YF(1-F^2)^k\big) + \mathrm{Tr}\big(X(1-F^2)^{n-k-1} FY(1-F^2)^k\big)\Big]
\end{aligned}$$

which, after changing the variables $j = n-k-1$ is precisely

$$\frac{d}{ds}\Big|_{s=0}\Big(\mathrm{Tr}\Big(X\big(1-(F+sY)^2\big)^n\Big)\Big). \qquad \blacksquare$$

Now, since α is closed and M is convex, one would expect from some version of Poincaré's Lemma that α is, in fact, exact. This is the case, and we prove it below. Fixing (N, F_0) we define $\theta \colon M \longrightarrow \mathbf{R}$ via

$$\theta(F) = \frac{1}{C_n}\int_0^1 \mathrm{Tr}\big((F-F_0)(1-F_t^2)^n\big)\,dt$$

where $F_t = F_0 + t(F-F_0)$ for t in $[0,1]$. Since the integrand is clearly continuous, θ is well-defined. By definition,

$$d\theta_F(X) = \frac{d}{ds}\Big|_{s=0}\big(\theta(F+sX)\big).$$

PROPOSITION 1.4. *With the assumptions of Proposition 1.3 and the above definition of θ we have that $d\theta = \alpha$.*

PROOF. Fix $F_1 \in M = F_0 + L_{\mathrm{sa}}^{p,\frac{p}{2}}$ and let $Y = (F_1 - F_0)$ so that

$$\theta(F_1) = \frac{1}{C_n}\int_0^1 \mathrm{Tr}\big(Y(1-F_t^2)^n\big)\,dt$$

and

$$\theta(F_1+sX) = \frac{1}{C_n}\int_0^1 \mathrm{Tr}\Big((Y+sX)\big(1-(F^s)_t^2\big)^n\Big)\,dt$$

where

$$(F^s)_t = F_0 + t(F_1 + sX - F_0) = F_0 + t(Y+sX)$$

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for real s. Now, by the product rule

$$\begin{aligned}\frac{d}{ds}&\Big(\mathrm{Tr}\Big((Y+sX)\big(1-(F^s)_t^2\big)^n\Big)\Big)\\ &=\mathrm{Tr}\Big(X\big(1-(F^s)_t^2\big)^n\Big)+\mathrm{Tr}\Big((Y+sX)\frac{d}{ds}\big(1-(F^s)_t^2\big)^n\Big)\\ &=\mathrm{Tr}\Big(X\big(1-(F^s)_t^2\big)^n\Big)\\ &\qquad-\sum_{k=0}^{n-1}\Big\{\mathrm{Tr}\Big((Y+sX)\big(1-(F^s)_t^2\big)^k\big(t(F_0+tY)X+tX(F_0+tY)+t^2 2sX^2\big)\\ &\qquad\qquad\cdot\big(1-(F^s)_t^2\big)^{n-k-1}\Big)\Big\}\end{aligned}$$

by a calculation very similar to that of Proposition 1.3. Now, as $s\longrightarrow 0$ this real-valued function converges to

$$\frac{d}{ds}\Big|_{s=0}\Big(\mathrm{Tr}\Big((Y+sX)\big(1-(F^s)_t^2\big)^n\Big)\Big)$$

uniformly in t. By the Mean Value Theorem this shows that the difference quotients for

$$\frac{d}{ds}\Big|_{s=0}\Big(\mathrm{Tr}\Big((Y+sX)\big(1-(F^s)_t^2\big)^n\Big)\Big)$$

converge uniformly in t. That is, we can pass $\frac{d}{ds}|_{s=0}$ through the integral and obtain

$$\begin{aligned}&d\theta_{F_1}(X)\\ &=\frac{d}{ds}\Big|_{s=0}\big(\theta(F_1+sX)\big)=\frac{1}{C_n}\int_0^1\frac{d}{ds}\Big|_{s=0}\mathrm{Tr}\Big((Y+sX)\big(1-(F^s)_t^2\big)^n\Big)\,dt\\ &=\frac{1}{C_n}\int_0^1\Big\{\mathrm{Tr}\big(X(1-F_t^2)^n\big)+\mathrm{Tr}\Big(Y\frac{d}{ds}\Big|_{s=0}\big(1-(F^s)_t^2\big)^n\Big)\Big\}\,dt\\ &=\frac{1}{C_n}\int_0^1\Big\{\mathrm{Tr}\big(X(1-F_t^2)^n\big)-\sum_{k=0}^{n-1}\Big[\mathrm{Tr}\Big(Y(1-F_t^2)^k(tF_tX+tXF_t)(1-F_t^2)^{n-k-1}\Big)\Big]\Big\}\,dt\\ &=\frac{1}{C_n}\int_0^1\Big\{\mathrm{Tr}\big(X(1-F_t^2)^n\big)-\sum_{k=0}^{n-1}\Big[\mathrm{Tr}\Big(X(1-F_t^2)^k(F_ttY+tYF_t)(1-F_t^2)^{n-k-1}\Big)\Big]\Big\}\,dt\\ &=\frac{1}{C_n}\int_0^1\Big\{\mathrm{Tr}\big(X(1-F_t^2)^n\big)+t\frac{d}{dt}\,\mathrm{Tr}\big(X(1-F_t^2)^n\big)\Big\}\,dt\\ &=\frac{1}{C_n}\int_0^1\frac{d}{dt}\Big[t\,\mathrm{Tr}\big(X(1-F_t^2)^n\big)\Big]\,dt=\frac{1}{C_n}\,\mathrm{Tr}\big(X(1-F_1^2)^n\big)\end{aligned}$$

which is $\alpha_{F_1}(X)$ as required. ∎

Now, if we had used F_0' in place of F_0 we would get θ' in place of θ and $d\theta'=\alpha=d\theta$ so that $d(\theta'-\theta)=0$. The usual argument shows that $\theta=\theta'+C$ for a constant C. Evaluating at F_0 gives $\theta(F_0')=\theta'(F_0')+C=C$. That is, $\theta(F)=\theta'(F)+\theta(F_0')$. In other words, the integral

$$\frac{1}{C_n}\int_\Gamma\mathrm{Tr}\Big(\frac{d}{dt}(F_t)(1-F_t^2)^n\Big)\,dt$$

is independent of the piecewise linear paths Γ from F_0 to F given by

$$\Gamma_1 \colon F_0 \bullet\!\!-\!\!\bullet F \quad \text{and} \quad \Gamma_2 \colon F_0 \bullet\!\!-\!\!\bullet F_0' \bullet\!\!-\!\!\bullet F.$$

By a very easy induction argument, this can be extended to show that

$$\frac{1}{C_n}\int_\Gamma \operatorname{Tr}\Big(\frac{d}{dt}(F_t)(1-F_t^2)^n\Big)\,dt$$

depends only on the endpoints of any piecewise-linear path Γ. By an approximation argument, this conclusion can be extended to paths which are piecewise continuously differentiable (see Remark 1.8). Thus, we have proved

PROPOSITION 1.5. *The integral of the* 1*-form* α *along a piecewise continuously differentiable path* Γ *in* M *depends only on the endpoints of the path* Γ.

DEFINITION 1.6. Let $F \in M = F_0 + L_{\mathrm{sa}}^{p,\frac{p}{2}}$ and let $\tilde F = 2\chi(F) - 1$ the corresponding symmetry. Then, as before $\tilde F \in M$ as well. Let $\{F_t\}_{t\in[0,1]}$ a piecewise smooth path in M beginning at F and ending at $\tilde F$. For example, we could choose $F_t = F + t(\tilde F - F)$ as our path. Let n be a positive integer, $n \geq \frac{p-1}{2}$. We define

$$\beta_n(F) = \frac{1}{C_n}\int_0^1 \operatorname{Tr}\Big(\frac{d}{dt}(F_t)(1-F_t^2)^n\Big)\,dt.$$

It is clear by considering the linear path that if F_1 and F_2 are unitarily equivalent in M then $\beta_n(F_1) = \beta_n(F_2)$.

THEOREM 1.7. *Let* (N, F_0) *be an odd* p*-summable pre-Breuer-Fredholm module for a unital Banach* $*$*-algebra* A*, let* $n \geq \frac{p-1}{2}$ *be a positive integer and let* $M = F_0 + L_{\mathrm{sa}}^{p,\frac{p}{2}}$. *Let* $F_1, F_2 \in M$ *and let* $\{F_t\} = \Gamma$ *be any piecewise continuously differentiable path in* M *from* F_1 *to* F_2*. Then the spectral flow from* F_1 *to* F_2 *in* M *is given by:*

$$\mathrm{sf}(F_1, F_2) = \frac{1}{C_n}\int_\Gamma \operatorname{Tr}\Big(\frac{d}{dt}(F_t)(1-F_t^2)^n\Big)\,dt + \beta_n(F_2) - \beta_n(F_1).$$

PROOF. The formula on the right is just the integral of α along a curve in M from $\tilde F_1$ to $\tilde F_2$. Thus, it is equal to the integral of α along the straight line from $\tilde F_1$ to $\tilde F_2$. But, $\tilde F_1 - \tilde F_2 \in L_{\mathrm{sa}}^{p,\frac{p}{2}} \subseteq L_{\mathrm{sa}}^p$ and so Theorem 3.1 of [P2] applies (we note that Theorem 3.1 of [P2] is true for $2n \geq p-1$ by an appeal to Theorem 4.1 of [ASS]). Hence the formula gives the spectral flow of the straight line path from $\tilde F_1$ to $\tilde F_2$. But, then the two piecewise linear paths from $\tilde F_1$ to $\tilde F_2$ indicated below are clearly homotopic in M and so yield the same spectral flow:

$$\tilde F_1 \bullet\!\!-\!\!\bullet \tilde F_2 \quad \text{and} \quad \tilde F_1 \bullet\!\!-\!\!\bullet F_1 \bullet\!\!-\!\!\bullet F_2 \bullet\!\!-\!\!\bullet \tilde F_2.$$

That is,

$$\mathrm{sf}(\tilde F_1, \tilde F_2) = \mathrm{sf}(\tilde F_1, F_1) + \mathrm{sf}(F_1, F_2) + sf(F_2, \tilde F_2).$$

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But, since χ is constant on the path from $\tilde{F}_1$ to F_1, $\mathrm{sf}(\tilde{F}_1, F_1) = 0$ [P2, Remark 2.3]. Similarly, $\mathrm{sf}(F_2, \tilde{F}_2) = 0$. Hence,

$$\begin{aligned}\mathrm{sf}(F_1, F_2) &= \mathrm{sf}(\tilde{F}_1, \tilde{F}_2)\\ &= \frac{1}{C_n}\int_\Gamma \mathrm{Tr}\Big(\frac{d}{dt}(F_t)(1-F_t^2)^n\Big)\,dt + \beta_n(F_2) - \beta_n(F_1)\end{aligned}$$

as claimed. ∎

REMARK 1.8. We can relax the hypotheses in Theorem 1.7 to the following:

(1) $t \longmapsto F_t$ is continuous in $M = F_0 + L_{\mathrm{sa}}^{p,\frac{p}{2}}$, and

(2) piecewise, $\frac{d}{dt}(F_t)$ exists in L^{2n+1} as a $(2n+1)$-norm limit and $t \longmapsto \frac{d}{dt}(F_t)$ is piecewise continuous in $(2n+1)$-norm, for $n \geq \frac{p-1}{2}$.

It is clear that the integral exists in this generality, as the integrand is piecewise continuous and trace-class by Hölder's inequality. By a standard continuity and compactness argument we can approximate $\{F_t\}$ by a continuous piecewise linear path *in* $M = F_0 + L_{\mathrm{sa}}^{p,\frac{p}{2}}$ with the same end points so that $\|F_t - G_t\|_M$ and $\|F_t' - G_t'\|_{2n+1}$ are uniformly small. As

$$F \longmapsto (1-F^2)^n \colon F_0 + L_{\mathrm{sa}}^{p,\frac{p}{2}} \longrightarrow L^{\frac{p}{2n}} \hookrightarrow L^{\frac{2n+1}{2n}}$$

is continuous this will imply that $\|(1-F_t^2)^n - (1-G_t^2)^n\|_{\frac{2n+1}{2n}}$ is uniformly small. Hence $\|F_t'(1-F_t^2)^n - G_t'(1-G_t^2)^n\|_1$ will be uniformly small.

Thus,

$$\int_0^1 \mathrm{Tr}\Big(\frac{d}{dt}(F_t)(1-F_t^2)^n\Big)\,dt$$

is close to

$$\int_0^1 \mathrm{Tr}\Big(\frac{d}{dt}(G_t)(1-G_t^2)^n\Big)\,dt.$$

Since $F_0 = G_0$ and $F_1 = G_1$ we get that

$$\mathrm{sf}(F_0, F_1) = \mathrm{sf}(G_0, G_1) = \frac{1}{C_n}\int_0^1 \mathrm{Tr}\Big(\frac{d}{dt}(G_t)(1-G_t^2)^n\Big)\,dt + \beta_n(G_1) - \beta_n(G_0)$$

which we can choose to be arbitrarily close to

$$\frac{1}{C_n}\int_0^1 \mathrm{Tr}\Big(\frac{d}{dt}(F_t)(1-F_t^2)^n\Big)\,dt + \beta_n(F_1) - \beta_n(F_0).$$

Hence, they are equal. ∎

THEOREM 1.9. *Let (N, F_0) be an odd p-summable pre-Breuer-Fredholm module for the unital Banach $*$-algebra $\mathcal{A}$. Let $P = \chi(F_0)$ and let n be a positive integer $n \geq \frac{p-1}{2}$. For each $u \in U(\mathcal{A})$ with $[F, u]$ p-summable, the path $F_t^u = F_0 + t(uF_0u^* - F_0)$ lies in $M = F_0 + L_{\mathrm{sa}}^{p,\frac{p}{2}}$ and*

$$\mathrm{ind}(PuP) = \mathrm{sf}(\{F_t^u\}) = \frac{1}{C_n}\int_0^1 \mathrm{Tr}\Big(\frac{d}{dt}(F_t^u)\big(1-(F_t^u)^2\big)^n\Big)\,dt.$$

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PROOF. The second equality follows from the previous theorem and the fact that $\beta_n(uF_0u^*) = \beta_n(F_0)$. The first equality follows from Theorem 3.3 of [P2] (again, we use the improved version with $2n \geq p-1$, courtesy of Theorem 4.1 of [ASS]) since

$$\begin{aligned}\operatorname{ind}(PuP) &= \operatorname{sf}(\tilde{F}_0, u\tilde{F}_0u^*) = \operatorname{sf}\bigl(\tilde{F}_0, (uF_0u^*)^{\sim}\bigr)\\ &= \operatorname{sf}(F_0, uF_0u^*) = \operatorname{sf}(\{F_t^u\}).\end{aligned}$$

■

2. **Spectral flow as the integral of a 1-form for finitely summable unbounded Fredholm modules (Types I and II).** The idea of this chapter is to prove a spectral flow formula as indicated in the title by passing to the bounded case (Chapter 1) *via* the transformation $D \longmapsto F = D(1+D^2)^{-\frac{1}{2}}$. Beneath this simple idea lurks a plethora of technical difficulties. Many of these technicalities have been shunted to the appendices; indeed, it is the problems raised in this chapter which made the appendices necessary. In order to make the material more digestible, we have broken down the chapter into subheadings A, B, *etc.* with self-explanatory titles. A quick perusal of these titles by the reader would be an excellent overview for the chapter.

We begin with:

DEFINITION 2.1. Let A be a unital Banach $*$-algebra and $p > 0$, then an *odd p-summable unbounded Breuer-Fredholm module for A* is a pair (N, D) where N is a semifinite factor (on a separable Hilbert space), A is unitally $*$-represented in N, D is an unbounded self-adjoint operator affiliated with N satisfying

(1) $(1+D^2)^{-1}$ is $\frac{p}{2}$-summable, and
(2) $\mathcal{A} := \{a \in A \mid a(\operatorname{dom} D) \subseteq \operatorname{dom} D \text{ and } [D,a] \text{ is bounded}\}$ is a dense $*$-subalgebra of A.

If $(1+D^2)^{-1}$ is in the ideal $\mathcal{K}_N$ but not necessarily finitely summable, we still call (N, D) an *unbounded Breuer-Fredholm module for A*.

A. **If (N, D) is an odd, p-summable, unbounded Fredholm module, then $\bigl(N, D(1+D^2)^{-\frac{1}{2}}\bigr)$ is an odd q-summable pre-Fredholm module for $q > p$.**

In [C1], Connes discusses the *even* case of Proposition 2.4. In the *odd* case, he only discusses bounded Fredholm modules. In order to take $q = p$ one is forced either to consider weak $\mathcal{L}^p$ spaces as in [C2, IV.2] or, as hinted at in [C2, IV.8, Remark 5], to assume that the commutators, $[|D|, a]$ are bounded. We may take this up in the sequel.

LEMMA 2.2. *If $\lambda \longmapsto S(\lambda)\colon (a,b) \longrightarrow N$ is operator-norm continuous and*

$$T = \int_a^b S(\lambda)\, d\lambda$$

converges in operator norm, then for any $p \geq 1$,

$$\|T\|_p \leq \int_a^b \|S(\lambda)\|_p\, d\lambda$$

(where, of course, $\|X\|_p^p = \mathrm{Tr}(|X|^p)$*).*

PROOF. First note that $\lambda \longmapsto |S(\lambda)|^p$ is operator-norm continuous, so that $\lambda \longmapsto \|S(\lambda)\|_p$ is lower semicontinuous. We can assume that the Lebesgue integral

$$\int_a^b \|S(\lambda)\|_p \, d\lambda$$

is finite, otherwise there is nothing to prove. In this case the integral is equal to the lower Riemann integral and hence by a judicious choice of a sequence of partitions $\{P_k\}$ of $[a,b]$ we can assume:

(1) $T = \|\cdot\| - \lim_{k\to\infty}\big(\sum_{P_k} S(\lambda_{k,i})\Delta_{k,i}(\lambda)\big)$ and

(2) $\int_a^b \|S(\lambda)\|_p \, d\lambda = \lim_{k\to\infty}\big(\sum_{P_k} \|S(\lambda_{k,i})\|_p \Delta_{k,i}(\lambda)\big)$.

Let $T_k = \sum_{P_k} S(\lambda_{k,i})\Delta_{k,i}(\lambda)$ so that $T = \|\cdot\| - \lim_{k\to\infty} T_k$ and hence

$$|T|^p = \|\cdot\| - \lim_{k\to\infty} |T_k|^p.$$

Thus,

$$\begin{aligned}\|T\|_p &\le \liminf_{k\to\infty} \|T_k\|_p \le \liminf_{k\to\infty} \sum_{P_k} \|S(\lambda_{k,i})\|_p \Delta(\lambda_{k,i}) \\ &= \int_a^b \|S(\lambda)\|_p \, d\lambda.\end{aligned}$$

∎

LEMMA 2.3. *Let D be an unbounded self-adjoint operator and let a be a bounded operator satisfying $a(\operatorname{dom} D) \subseteq \operatorname{dom} D$ so that $[D,a]$ is densely defined on $\operatorname{dom} D$. Then for each $x > 0$ we have:*

$$[a,(x+D^2)^{-1}] = D(x+D^2)^{-1}[D,a](x+D^2)^{-1} + (x+D^2)^{-1}[D,a]D(x+D^2)^{-1}$$

as everywhere-defined operators.

PROOF. We first note that this would be the usual resolvent calculation if $a(\operatorname{dom} D^2) \subseteq \operatorname{dom} D^2$. We must be more subtle.

$$\begin{aligned}[a,(x+D^2)^{-1}] &= a(x+D^2)^{-1} - (x+D^2)^{-1}a \cdot 1_H \\ &= a(x+D^2)^{-1} - (x+D^2)^{-1}a(x+D^2)(x+D^2)^{-1} \\ &= \big(a - (x+D^2)^{-1}a(x+D^2)\big)(x+D^2)^{-1} \\ &= \big(a - (x+D^2)^{-1}xa - (x+D^2)^{-1}aD^2\big)(x+D^2)^{-1} \\ &= \Big(a - \big(1 - D^2(x+D^2)^{-1}\big)a - (x+D^2)^{-1}aD^2\Big)(x+D^2)^{-1} \\ &= D^2(x+D^2)^{-1}a(x+D^2)^{-1} - (x+D^2)^{-1}aD^2(x+D^2)^{-1}.\end{aligned}$$

Now, $D^2(x+D^2)^{-1} = D(x+D^2)^{-1}D$ *on* $\operatorname{dom} D$ and since

$$\operatorname{range}(x+D^2)^{-1} = \operatorname{dom} D^2 \subseteq \operatorname{dom} D$$

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which is left invariant by a, we see that

$$D^2(x+D^2)^{-1}a(x+D^2)^{-1} = D(x+D^2)^{-1}Da(x+D^2)^{-1}$$

on *all* of H. Similarly,

$$D(x+D^2)^{-1}aD(x+D^2)^{-1} = (x+D^2)^{-1}DaD(x+D^2)^{-1}$$

on *all* of H. Thus,

$$\begin{aligned}[a,(x+D^2)^{-1}] &= D(x+D^2)^{-1}Da(x+D^2)^{-1} - D(x+D^2)^{-1}aD(x+D^2)^{-1}\\ &\quad + (x+D^2)^{-1}DaD(x+D^2)^{-1} - (x+D^2)^{-1}aD^2(x+D^2)^{-1}\\ &= D(x+D^2)^{-1}[D,a](x+D^2)^{-1} + (x+D^2)^{-1}[D,a]D(x+D^2)^{-1}\end{aligned}$$

as claimed. ∎

With even more care, this lemma can be proved under the weaker assumption that $\{\xi \in \operatorname{dom} D \mid a\xi \in \operatorname{dom} D\}$ is dense in $\operatorname{dom} D$ in the graph norm. One then needs to replace $[D,a]$ with its closure $\overline{[D,a]}$ for the conclusion to make sense. It is possible (likely?) that most of the theory of unbounded Fredholm modules can be pushed through in this generality. We do not attempt this here, but note that [C1] uses the stronger domain invariance condition while the more expository [C2] is sometimes a little vague on the meaning of "$[D,a]$ is bounded".

PROPOSITION 2.4. *Let (N,D) be an odd p-summable unbounded Breuer-Fredholm module for the Banach $*$-algebra A and let $F = D(1+D^2)^{-\frac{1}{2}}$. Then (N,F) is an odd q-summable pre-Breuer-Fredholm module for A for any $q > p$.*

PROOF. $1 - F^2 = (1+D^2)^{-1}$ which is $\frac{p}{2}$-summable and hence $\frac{q}{2}$-summable for any $q \geq p$.

Now, for $a \in A$ with $[D,a]$ bounded, one checks that

$$[F,a] = [D,a](1+D^2)^{-\frac{1}{2}} + D[(1+D^2)^{-\frac{1}{2}},a]$$

so it suffices to see that $D[(1+D^2)^{-\frac{1}{2}},a]$ is q-summable for $q > p$. By Remark 3 of Appendix A,

$$(1+D^2)^{-\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(1+D^2+\lambda)^{-1}\,d\lambda$$

converges in operator norm. So, by Lemma 2.3

$$\begin{aligned}D\Big[(1+D^2)^{-\frac{1}{2}},a\Big] = D\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\{&(1+D^2+\lambda)^{-1}[a,D]D(1+D^2+\lambda)^{-1}\\ &+ D(1+D^2+\lambda)^{-1}[a,D](1+D^2+\lambda)^{-1}\}\,d\lambda.\end{aligned}$$

Since D is a closed operator and the integral converges in norm and hence pointwise on H, we can pass D through the integral provided the resulting integral also converges at least pointwise on H. In fact, the resulting integral

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\Big(&D(1+D^2+\lambda)^{-1}[a,D]D(1+D^2+\lambda)^{-1}\\ &+ D^2(1+D^2+\lambda)^{-1}[a,D](1+D^2+\lambda)^{-1}\Big)\,d\lambda\end{aligned}$$

converges in norm using the estimates of Remark 5 of Appendix A (and $\|D^2(1+D^2+\lambda)^{-1}\| \le 1$). Since $q > p$ we can write $p = (1-\epsilon)q$. By Lemma 2.2,

$$\begin{aligned}
&\|D[(1+D^2)^{-\frac{1}{2}}, a]\|_q \\
&\quad \le \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\Big(\|D(1+D^2+\lambda)^{-1}[a,D]D(1+D^2+\lambda)^{-(\frac{1+\epsilon}{2})}\|\,\|(1+D^2+\lambda)^{-(\frac{1-\epsilon}{2})}\|_q \\
&\qquad + \|D^2(1+D^2+\lambda)^{-1}[a,D](1+D^2+\lambda)^{-(\frac{1+\epsilon}{2})}\|\,\|(1+D^2+\lambda)^{-(\frac{1-\epsilon}{2})}\|_q\Big)\,d\lambda \\
&\quad \le \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\Big(\frac{1}{2\sqrt{\lambda+1}}\|[a,D]\|\,\|D(1+D^2+\lambda)^{-(\frac{1+\epsilon}{2})}\|\,\|(1+D^2+\lambda)^{-\frac{1}{2}}\|_p^{\frac{p}{q}} \\
&\qquad + 1\cdot\|[a,D]\|\,\|(1+D^2+\lambda)^{-(\frac{1+\epsilon}{2})}\|\,\|(1+D^2+\lambda)^{-\frac{1}{2}}\|_p^{\frac{p}{q}}\Big)\,d\lambda.
\end{aligned}$$

Now, $\frac{p}{q} = 1-\epsilon$ and $(1+D^2+\lambda)^{-\frac{1}{2}} \le (1+D^2)^{-\frac{1}{2}}$ so that

$$\|(1+D^2+\lambda)^{-\frac{1}{2}}\|_p^{\frac{p}{q}} \le \|(1+D^2)^{-\frac{1}{2}}\|_p^{(1-\epsilon)}.$$

By the spectral theorem and a little estimating we get

$$\|D(1+D^2+\lambda)^{-(\frac{1+\epsilon}{2})}\| \le (1+\lambda)^{-\frac{\epsilon}{2}}$$

and

$$\|(1+D^2+\lambda)^{-(\frac{1+\epsilon}{2})}\| \le (1+\lambda)^{-(\frac{1+\epsilon}{2})}.$$

Finally, we get

$$\begin{aligned}
&\|D[(1+D^2)^{-\frac{1}{2}}, a]\|_q \\
&\quad \le \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\Big(\frac{1}{2\sqrt{1+\lambda}}(1+\lambda)^{-\frac{\epsilon}{2}} + (1+\lambda)^{-(\frac{1+\epsilon}{2})}\Big)\,d\lambda\|[a,D]\|\,\|(1+D^2)^{-\frac{1}{2}}\|_p^{1-\epsilon} \\
&\quad = \frac{3}{2}\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda)^{-(\frac{1+\epsilon}{2})}\,d\lambda\|[a,D]\|\,\|(1+D^2)^{-\frac{1}{2}}\|_p^{1-\epsilon} \\
&\quad < +\infty.
\end{aligned}$$

∎

COROLLARY 2.5. *Let (N,D) be an odd p-summable unbounded Breuer-Fredholm module for the Banach $*$-algebra A, and let $F_0 = 2\chi(D)-1$ where χ is the characteristic function of $\mathbf{R}^+$. Then, (N,F_0) is an odd q-summable Breuer-Fredholm module for any $q > p$.*

PROOF. This follows immediately from Proposition 2.4, the remarks after Definition 1.1 and the fact that $\chi(D) = \chi\big(D(1+D^2)^{-\frac{1}{2}}\big)$. ∎

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B. **If (N, D) is an odd, p-summable, unbounded Fredholm module and $t \longmapsto D_t = D + A_t \in D + N_{sa}$ is operator-norm continuous, then for $q > p$, $t \longmapsto F_t = D_t(1 + D_t^2)^{-\frac{1}{2}}$ is continuous in $F + L_{sa}^{q,\frac{q}{2}}$.**

LEMMA 2.6. *Let D be an unbounded self-adjoint operator, $\lambda \geq 0$ and $\frac{1}{2} \geq \epsilon \geq 0$. Then*

(1) $\|D(1 + D^2)^{\frac{1}{2}-\epsilon}(1 + D^2 + \lambda)^{-1}\| \leq (1 + \lambda)^{-\epsilon}$ *and*

(2) $\|(1 + D^2)^{\frac{1}{2}-\epsilon}(1 + D^2 + \lambda)^{-1}\| \leq (1 + \lambda)^{-(\frac{1}{2}+\epsilon)}$.

PROOF. We prove (1). The proof of (2) follows the same plan. It suffices by the functional calculus to prove the following numerical inequality:

$$\left|\frac{x(1 + x^2)^{\frac{1}{2}-\epsilon}}{1 + x^2 + \lambda}\right| \leq (1 + \lambda)^{-\epsilon} \quad \text{for all } x \in \mathbf{R}.$$

We break this into two cases: if $x^2 \leq \lambda$ then

$$\left|\frac{x(1 + x^2)^{\frac{1}{2}-\epsilon}}{1 + x^2 + \lambda}\right| \leq \frac{\sqrt{\lambda}(1 + \lambda)^{\frac{1}{2}-\epsilon}}{1 + \lambda} \leq \frac{(1 + \lambda)^{\frac{1}{2}}(1 + \lambda)^{\frac{1}{2}-\epsilon}}{1 + \lambda} = (1 + \lambda)^{-\epsilon}.$$

If $x^2 \geq \lambda$ then

$$\left|\frac{x(1 + x^2)^{\frac{1}{2}-\epsilon}}{1 + x^2 + \lambda}\right| \leq \frac{(1 + x^2)^{\frac{1}{2}}(1 + x^2)^{\frac{1}{2}-\epsilon}}{1 + x^2} = (1 + x^2)^{-\epsilon} \leq (1 + \lambda)^{-\epsilon}. \qquad \blacksquare$$

The following lemma is crucial for the remainder of this chapter and indeed for the sequel to this paper on the θ-summable case.

LEMMA 2.7. *Let D_0 be an unbounded self-adjoint operator affiliated with the semifinite factor N. Let $A \in N_{sa}$ and let $D_1 = D_0 + A$. For $i = 0, 1$ let $F_{D_i} = D_i(1 + D_i^2)^{-\frac{1}{2}}$. Then for fixed ϵ, $0 < \epsilon < \frac{1}{2}$, $F_{D_1} - F_{D_0} = B_\epsilon(1 + D_0^2)^{-(\frac{1}{2}-\epsilon)}$ where*

$$\begin{aligned} B_\epsilon = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}[&(1 + \lambda)(1 + D_1^2 + \lambda)^{-1}A(1 + D_0^2)^{\frac{1}{2}-\epsilon}(1 + D_0^2 + \lambda)^{-1} \\ &- D_1(1 + D_1^2 + \lambda)^{-1}AD_0(1 + D_0^2)^{\frac{1}{2}-\epsilon}(1 + D_0^2 + \lambda)^{-1}]\, d\lambda \end{aligned}$$

converges in operator norm and $\|B_\epsilon\| \leq C(\epsilon)\|A\|$.

PROOF. The norm-convergence of the integral and the final estimate both follow from the previous lemma and Remark 5 of Appendix A.

Now, by Appendix A, Lemma 4 we have for all $\xi \in \operatorname{dom} D_0 = \operatorname{dom} D_1$:

$$F_{D_1}\xi - F_{D_0}\xi = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}[D_1(1 + D_1^2 + \lambda)^{-1} - D_0(1 + D_0^2 + \lambda)^{-1}]\xi\, d\lambda$$

where the integral is norm convergent in H. However, by Appendix A, Lemma 6, part (2) the integral converges in operator norm and so

$$F_{D_1} - F_{D_0} = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}[D_1(1 + D_1^2 + \lambda)^{-1} - D_0(1 + D_0^2 + \lambda)^{-1}]\, d\lambda.$$

Now,

$$
\begin{aligned}
D_1(1+&D_1^2+\lambda)^{-1} - D_0(1+D_0^2+\lambda)^{-1} \\
&= A(1+D_0^2+\lambda)^{-1} + D_1[(1+D_1^2+\lambda)^{-1} - (1+D_0^2+\lambda)^{-1}] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{which by Lemma 2.9 of the next section} \\
&= A(1+D_0^2+\lambda)^{-1} - D_1[D_1(1+D_1^2+\lambda)^{-1}A(1+D_0^2+\lambda)^{-1} \\
&\qquad + (1+D_1^2+\lambda)^{-1}AD_0(1+D_0^2+\lambda)^{-1}] \\
&= A(1+D_0^2+\lambda)^{-1} - D_1^2(1+D_1^2+\lambda)^{-1}A(1+D_0^2+\lambda)^{-1} \\
&\qquad - D_1(1+D_1^2+\lambda)^{-1}AD_0(1+D_0^2+\lambda)^{-1} \\
&= A(1+D_0^2+\lambda)^{-1} - [1-(1+\lambda)(1+D_1^2+\lambda)^{-1}]A(1+D_0^2+\lambda)^{-1} \\
&\qquad - D_1(1+D_1^2+\lambda)^{-1}AD_0(1+D_0^2+\lambda)^{-1} \\
&= (1+\lambda)(1+D_1^2+\lambda)^{-1}A(1+D_0^2+\lambda)^{-1} \\
&\qquad - D_1(1+D_1^2+\lambda)^{-1}AD_0(1+D_0^2+\lambda)^{-1} \\
&= [\cdots](1+D_0^2)^{\frac{1}{2}-\epsilon}(1+D_0^2)^{-\frac{1}{2}+\epsilon} \\
&= [(1+\lambda)(1+D_1^2+\lambda)^{-1}A(1+D_0^2)^{\frac{1}{2}-\epsilon}(1+D_0^2+\lambda)^{-1} \\
&\qquad - D_1(1+D_1^2+\lambda)^{-1}A(1+D_0^2)^{\frac{1}{2}-\epsilon}(1+D_0^2+\lambda)^{-1}](1+D_0^2)^{-\frac{1}{2}+\epsilon}.
\end{aligned}
$$

Hence, as $(1+D_0^2)^{-\frac{1}{2}+\epsilon}$ is bounded:

$$
\begin{aligned}
F_{D_1} - F_{D_0} = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}[(1+\lambda)(1+D_1^2+\lambda)^{-1}A(1+D_0^2)^{\frac{1}{2}-\epsilon}(1+D_0^2+\lambda)^{-1} \\
- D_1(1+D_1^2+\lambda)^{-1}A(1+D_0^2)^{\frac{1}{2}-\epsilon}(1+D_0^2+\lambda)^{-1}]\,d\lambda(1+D_0^2)^{-\frac{1}{2}+\epsilon}
\end{aligned}
$$

as claimed. ∎

COROLLARY 2.8. *Let (N, D_0) be p-summable and let $D_t = D_0 + A_t \in D_0 + N_{\mathrm{sa}}$ be an operator-norm continuous path. Then,*

$$F_t = D_t(1+D_t^2)^{-\frac{1}{2}} \in F_0 + \mathcal{L}_{\mathrm{sa}}^{q,\frac{q}{2}}$$

is continuous for $q > p$.

PROOF. Let $q > p$. Then there exists $\epsilon > 0$ so that $p = (1-2\epsilon)q$. By Lemma 2.7,

$$F_t - F_0 = B_\epsilon(1+D_0^2)^{-(\frac{1}{2}-\epsilon)}$$

and since $(\frac{1}{2}-\epsilon)q = \frac{p}{2}$, we have $F_t - F_0 \in \mathcal{L}_{\mathrm{sa}}^q$. Now, by Appendix B, Lemma 6 we see that $(1+D_{t_0}^2)^{-(\frac{1}{2}-\epsilon)} \in \mathcal{L}_{\mathrm{sa}}^q$ for each t_0, and by Lemma 2.7 above

$$F_t - F_{t_0} = B_\epsilon(t)(1+D_{t_0}^2)^{-(\frac{1}{2}-\epsilon)}$$

where

$$\|B_\epsilon(t)\| \le C(\epsilon)\|A_t - A_{t_0}\|.$$

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Thus, $t \longmapsto F_t \in F_0 + \mathcal{L}^q_{\mathrm{sa}}$ is q-norm continuous. Letting $\tilde{F}_0 = 2\chi(F_0) - 1$ we see by the discussion after Definition 1.1 that

$$F_t \in \tilde{F}_0 + \mathcal{L}^q_{\mathrm{sa}} = F_0 + \mathcal{L}^q_{\mathrm{sa}}.$$

That is, $F_t = \tilde{F}_0 + X_t$ where $t \longmapsto X_t$ is continuous in $\mathcal{L}^q_{\mathrm{sa}}$. Let $\tilde{F}_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and relative to this decomposition, let $X_t = \begin{bmatrix} A_t & B_t \\ B_t^* & C_t \end{bmatrix}$ where each entry is continuous in $\mathcal{L}^q$, and A_t, C_t are in $\mathcal{L}^q_{\mathrm{sa}}$.

Now, $1 - F_t^2 = (1 + D_t^2)^{-1}$ is continuous in $\mathcal{L}^{\frac{q}{2}}_{\mathrm{sa}}$ by Proposition 10 of Appendix B (with $r = 1$ and $\frac{q}{2}$ in place of n). Hence, $t \longmapsto \frac{1}{2}(1 - F_t^2 + X_t^2)$ is also continuous in $\mathcal{L}^{\frac{q}{2}}$. One easily calculates that

$$\frac{1}{2}(1 - F_t^2 + X_t^2) = -\frac{1}{2}(X_t\tilde{F}_0 + \tilde{F}_0 X_t) = \begin{bmatrix} -A_t & 0 \\ 0 & C_t \end{bmatrix}.$$

That is,

$$X_t = \begin{bmatrix} A_t & B_t \\ B_t^* & C_t \end{bmatrix}$$

is in $\mathcal{L}^{q,\frac{q}{2}}_{\mathrm{sa}}$ and furthermore, $t \longmapsto B_t$ is continuous in $\mathcal{L}^q$ and $t \longmapsto A_t$, $t \longmapsto C_t$ are continuous in $\mathcal{L}^{\frac{q}{2}}_{\mathrm{sa}}$.

Finally since $t \longmapsto F_t$ is operator-norm continuous by Theorem 8 of Appendix A, we see that $t \longmapsto F_t \in (\tilde{F}_0 + \mathcal{L}^{q,\frac{q}{2}}_{sa}) = (F_0 + \mathcal{L}^{q,\frac{q}{2}}_{\mathrm{sa}})$ is continuous in the Banach space norm of $F_0 + \mathcal{L}^{q,\frac{q}{2}}_{\mathrm{sa}}$ as claimed. ∎

C. **If** (N, D) **is an odd unbounded Fredholm module and** $t \longmapsto D_t = D + A_t \in N_{\mathrm{sa}}$ **is** C^1 **in operator norm, then for** $q > p$ $t \longmapsto F_t = D_t(1 + D_t^2)^{\frac{1}{2}}$ **is a path in** $\mathcal{L}^{q,\frac{q}{2}}_{\mathrm{sa}}$ **which is** C^1 **in the norm of** $\mathcal{L}^q_{\mathrm{sa}}$**.**

LEMMA 2.9. *If D_0 is an unbounded self-adjoint operator on H, A is a bounded self-adjoint operator on H and $D = D_0 + A$, then for $x > 0$*

$$(D^2 + x)^{-1} - (D_0^2 + x)^{-1} = -D_0(D_0^2 + x)^{-1}A(D^2 + x)^{-1} - (D_0^2 + x)^{-1}AD(D^2 + x)^{-1}.$$

PROOF. We first assume that $A(\operatorname{dom} D_0) \subseteq \operatorname{dom} D_0$ so that (with a little thought) all of the domain difficulties disappear in the resolvent calculation:

$$\begin{aligned}(D^2 + x)^{-1} - (D_0^2 + x)^{-1} &= (D_0^2 + x)^{-1}[(D_0^2 + x) - (D^2 + x)](D^2 + x)^{-1} \\ &= (D_0^2 + x)^{-1}[-A^2 - D_0A - AD_0](D^2 + x)^{-1} \\ &= (D_0^2 + x)^{-1}[-D_0A - AD](D^2 + x)^{-1} \\ &= -D_0(D_0^2 + x)^{-1}A(D^2 + x)^{-1} - (D_0^2 + x)^{-1}AD(D^2 + x)^{-1}.\end{aligned}$$

We then apply the trick in the proof of Lemma 6 of Appendix B to get the result for general A. ∎

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PROPOSITION 2.10. *If (N, D_0) is an odd unbounded p-summable Breuer-Fredholm module (for $\mathbf{C}$, say) and $t \longmapsto A_t$ is a C^1 path in N_{sa}, then letting $D_t = D_0 + A_t$, we have that $t \longmapsto F_t = D_t(1 + D_t^2)^{-\frac{1}{2}}$ is a path of Breuer-Fredholm operators in $F_0 + \mathcal{L}_{\mathrm{sa}}^{q,\frac{q}{2}}$ for $q > p$ which is C^1 in the Banach space norm of $\mathcal{L}_{\mathrm{sa}}^{q}$. Moreover,*

$$\frac{d}{dt}(F_t) = \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\Big[(1 + D_t^2 + \lambda)^{-1}(1 + \lambda)\frac{d}{dt}(A_t)(1 + D_t^2 + \lambda)^{-1} - D_t(1 + D_t^2 + \lambda)^{-1}\frac{d}{dt}(A_t)D_t(1 + D_t^2 + \lambda)^{-1}\Big]\, d\lambda$$

where the integral converges both in the q-norm and the operator norm.

PROOF. We observe that by Remark 5 of Appendix A, for each fixed t the above integrand is an operator-norm continuous function of λ and that as $\lambda \longrightarrow +\infty$ the operator norm of the integrand approaches $\frac{C_1}{\sqrt{\lambda}(1+\lambda)}$ while as $\lambda \longrightarrow 0$ the operator norm of the integrand approaches $\frac{C_2}{\sqrt{\lambda}}$ so that the integral does indeed converge in operator norm.

On the other hand, letting $p = (1-2\epsilon)q$ we have by Hölder's inequality and Lemma 2.6 that

$$\begin{aligned}\|(1 + D_t^2 + \lambda)^{-1}\|_q &\le \|(1 + D_t^2)^{\frac{1}{2}-\epsilon}(1 + D_t^2 + \lambda)^{-1}\|\,\|(1 + D_t^2)^{-\frac{1}{2}+\epsilon}\|_q\\ &\le (1 + \lambda)^{-(\frac{1}{2}+\epsilon)}\|(1 + D_t^2)^{-\frac{1}{2}+\epsilon}\|_q\end{aligned}$$

and similarly,

$$\|D_t(1 + D_t^2 + \lambda)^{-1}\|_q \le (1 + \lambda)^{-\epsilon}\|(1 + D_t^2)^{-\frac{1}{2}+\epsilon}\|_q.$$

So, that combining these estimates with Remark 5 of Appendix A, we see that the q-norm of the integrand approaches $\frac{C_1}{\sqrt{\lambda}(1+\lambda)^{\frac{1}{2}+\epsilon}}$ as $\lambda \longrightarrow \infty$ and approaches $\frac{C_2}{\sqrt{\lambda}}$ as $\lambda \longrightarrow 0$. Thus, the integral will be seen to converge in the q-norm once we know that the integrand is q-norm continuous. To see this it suffices to see that $\lambda \longmapsto (1+D_t^2+\lambda)^{-1}$ and $\lambda \longmapsto D_t(1+D_t^2+\lambda)^{-1}$ are q-norm continuous since the other terms are operator-norm continuous by Remark 5 of Appendix A. We show continuity of $\lambda \longmapsto (1 + D_t^2 + \lambda)^{-1}$ as the other term is similar:

$$\begin{aligned}&\|(1 + D_t^2+\lambda)^{-1} - (1 + D_t^2 + \gamma)^{-1}\|_q\\ &\quad\le \|(1 + D_t^2)^{\frac{1}{2}-\epsilon}[(1 + D_t^2 + \lambda)^{-1} - (1 + D_t^2 + \gamma)^{-1}]\|\,\|(1 + D_t^2)^{-\frac{1}{2}+\epsilon}\|_q\\ &\quad= \|(1 + D_t^2)^{\frac{1}{2}-\epsilon}(1 + D_t^2 + \lambda)^{-1}(\gamma - \lambda)(1 + D_t^2 + \gamma)^{-1}\|\,\|(1 + D_t^2)^{-\frac{1}{2}+\epsilon}\|_q\\ &\quad\le (1 + \lambda)^{-(\frac{1}{2}-\epsilon)}|\gamma - \lambda|(1 + \gamma)^{-1}\|(1 + D_t^2)^{-\frac{1}{2}+\epsilon}\|_q\end{aligned}$$

which $\longrightarrow 0$ as $|\gamma - \lambda| \longrightarrow 0$. Thus the integrand is q-norm continuous and the integral converges in q-norms for each fixed t.

Now, to calculate $\frac{d}{dt}(F_t)$ we observe that since each D_t defines an unbounded p-summable Breuer-Fredholm module (Lemma 6, Appendix B) we can assume $t = 0$ and $A_0 = 0$. At $t = 0$, the purported derivative is:

$$\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\Big[(1 + D_0^2 + \lambda)^{-1}(1 + \lambda)A_0'(1 + D_0^2 + \lambda)^{-1} - D_0(1 + D_0^2 + \lambda)^{-1}A_0'D_0(1 + D_0^2 + \lambda)^{-1}\Big]\, d\lambda$$

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which equals:

$$\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\Big[(1+D_0^2+\lambda)^{-1}(1+\lambda)A_0'(1+D_0^2)^{\frac{1}{2}-\epsilon}(1+D_0^2+\lambda)^{-1} \\ - D_0(1+D_0^2+\lambda)^{-1}A_0'D_0(1+D_0^2)^{\frac{1}{2}-\epsilon}(1+D_0^2+\lambda)^{-1}\Big]\,d\lambda(1+D_0^2)^{-\frac{1}{2}+\epsilon}$$

where the new integrand converges in operator norm by the estimates of Lemma 2.7. But, by Lemma 2.7 the difference quotient $\frac{1}{t}(F_t - F_0)$ equals:

$$\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\Big[(1+D_t^2+\lambda)^{-1}(1+\lambda)\frac{1}{t}A_t(1+D_0^2)^{\frac{1}{2}-\epsilon}(1+D_0^2+\lambda)^{-1} \\ - D_t(1+D_t^2+\lambda)^{-1}\frac{1}{t}A_tD_0(1+D_0^2)^{\frac{1}{2}-\epsilon}(1+D_0^2+\lambda)^{-1}\Big]\,d\lambda(1+D_0^2)^{-\frac{1}{2}+\epsilon}.$$

The q-norm difference between these two operators can thus be estimated by:

$$\frac{1}{\pi}\int_0^\infty\lambda^{-\frac{1}{2}}\Big[\Big\|(1+D_t^2+\lambda)^{-1}\frac{1}{t}A_t-(1+D_0^2+\lambda)^{-1}A_0'\Big\|(1+\lambda)(1+\lambda)^{-(\frac{1}{2}+\epsilon)} \\ +\Big\|D_t(1+D_t^2+\lambda)^{-1}\frac{1}{t}A_t-D_0(1+D_0^2+\lambda)^{-1}A_0'\Big\|(1+\lambda)^{-\epsilon}\Big]\,d\lambda\|(1+D_0^2)^{-\frac{1}{2}+\epsilon}\|_q.$$

By Remark 5 and Lemma 6 of Appendix A,

$$\begin{aligned}\Big\|(1+D_t^2+\lambda)^{-1}\frac{1}{t}A_t-(1+D_0^2+\lambda)^{-1}A_0'\Big\| &\\ \le \|(1+D_t^2+\lambda)^{-1}-(1+D_0^2+\lambda)^{-1}\|\,\Big\|\frac{1}{t}A_t\Big\| &\\ +\Big\|\frac{1}{t}A_t-A_0'\Big\|\,\|(1+D_0^2+\lambda)^{-1}\| &\\ \le (1+\lambda)^{-\frac{3}{2}}\|A_t\|\,\Big\|\frac{1}{t}A_t\Big\|+\Big\|\frac{1}{t}A_t-A_0'\Big\|(1+\lambda)^{-1}.&\end{aligned}$$

Similarly,

$$\begin{aligned}\Big\|D_t(1+D_t^2+\lambda)^{-1}\frac{1}{t}A_t-D_0(1+D_0^2+\lambda)^{-1}A_0'\Big\| &\\ \le (1+\lambda)^{-1}\|A_t\|\,\Big\|\frac{1}{t}A_t\Big\|+\Big\|\frac{1}{t}A_t-A_0'\Big\|\frac{1}{2}(1+\lambda)^{-\frac{1}{2}}.&\end{aligned}$$

Integrating, we see that the q-norm of the difference between the difference quotient and the claimed derivative is less than or equal to $C_1\|A_t\|\,\|\frac{1}{t}A_t\|+C_2\|\frac{1}{t}A_t-A_0'\|$ where C_1 and C_2 are positive constants independent of t. As $t\longrightarrow 0$ both of these terms go to 0 (recall $A_0=0$). Thus, $\frac{d}{dt}(F_t)$ is the claimed integral. Similar (but slightly easier) calculations show that the limit of the difference quotients in the operator norm exists and is the aforementioned integral, also.

To see the q-norm continuity of $t \longmapsto F'_t$ we use the integral formula for $F'_s - F'_t$ and estimate. For example, we need:

$$\begin{aligned}
(1+\lambda)\|(1+&D_s^2+\lambda)^{-1}A'_s(1+D_s^2+\lambda)^{-1} - (1+D_t^2+\lambda)^{-1}A'_t(1+D_t^2+\lambda)^{-1}\|_q \\
&\leq (1+\lambda)\{\|(1+D_s^2+\lambda)^{-1}\|_q\|A'_s\|\,\|(1+D_s^2+\lambda)^{-1} - (1+D_t^2+\lambda)^{-1}\| \\
&\quad + \|(1+D_s^2+\lambda)^{-1}\|_q\|A'_s - A'_t\|\,\|(1+D_t^2+\lambda)^{-1}\| \\
&\quad + \|(1+D_s^2+\lambda)^{-1} - (1+D_t^2+\lambda)^{-1}\|\,\|A'_t\|\,\|(1+D_t^2+\lambda)^{-1}\|_q\} \\
&\leq (1+\lambda)\{(1+\lambda)^{-(\frac{1}{2}+\epsilon)}\|(1+D_s^2)^{-\frac{1}{2}+\epsilon}\|_q\|A'_s\|(1+\lambda)^{-\frac{3}{2}}\|A_s - A_t\| \\
&\quad + (1+\lambda)^{-(\frac{1}{2}+\epsilon)}\|(1+D_s^2)^{-\frac{1}{2}+\epsilon}\|_q\|A'_s - A'_t\|(1+\lambda)^{-1} \\
&\quad + (1+\lambda)^{-\frac{3}{2}}\|A_s - A_t\|\,\|A'_t\|(1+\lambda)^{-(\frac{1}{2}+\epsilon)}\|(1+D_t^2)^{-\frac{1}{2}+\epsilon}\|_q\} \\
&\leq \{(1+\lambda)^{-(1+\epsilon)}C\cdot\|(1+D_0^2)^{-\frac{1}{2}+\epsilon}\|_q\|A'_s\|\,\|A_s - A_t\| \\
&\quad + (1+\lambda)^{-(\frac{1}{2}+\epsilon)}C\cdot\|(1+D_0^2)^{-\frac{1}{2}+\epsilon}\|_q\|A'_s - A'_t\| \\
&\quad + (1+\lambda)^{-(1+\epsilon)}C\cdot\|A_s - A_t\|\,\|A'_t\|\,\|(1+D_0^2)^{-\frac{1}{2}+\epsilon}\|_q\}
\end{aligned}$$

where we have used Remark 5 and Lemma 6 of Appendix A, the q-norm estimates of $(1+D_t^2+\lambda)^{-1}$ given earlier and Corollary 8, part (1) of Appendix B since

$$\sup_t \|A_t\| < +\infty.$$

We get similar estimates for the other part of the integrand. Integrating these estimates we get:

$$\|F'_s - F'_t\|_q \leq [C_1\|A_s - A_t\| + C_2\|A'_s - A'_t\|]\,\|(1+D_0^2)^{-\frac{1}{2}+\epsilon}\|_q$$

where C_1 and C_2 are positive constants independent of s and t. Thus, $t \longmapsto F_t$ is C^1 in q-norm.

By similar (but easier) estimates we can show that $t \longmapsto F_t$ is also C^1 in operator norm, and hence in the Banach space norm of $F_0 + \mathcal{L}^q_{sa}$. ∎

D. **If (N, D) is p-summable and $t \longmapsto D_t$ is C^1 in operator norm, then for $n > \frac{p-1}{2}$,**

$$\operatorname{Tr}\Big(\frac{d}{dt}(F_t)(1-F_t^2)^n\Big) = \operatorname{Tr}\Big(\frac{d}{dt}(D_t)(1+D_t^2)^{-(n+\frac{3}{2})}\Big).$$

LEMMA 2.11. *If D is an unbounded self-adjoint operator, then*

$$(1+D^2)^{-\frac{3}{2}} + (1+D^2)^{-\frac{1}{2}} = \frac{2}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda)(1+D^2+\lambda)^{-2}\,d\lambda$$

where the integral converges in operator norm.

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PROOF. Since

$$\|(1 + D^2 + \lambda)^{-2}\| \leq \Big(\frac{1}{1+\lambda}\Big)^2$$

the integral converges in operator norm. Letting $B = (1 + D^2)^{-1}$, it is equivalent to showing that

$$B^{\frac{3}{2}} + B^{\frac{1}{2}} = \frac{2}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda)B^2(1+\lambda B)^{-2}\,d\lambda$$

for a bounded positive operator, $B \leq 1$. It is an easy calculus exercise to show that this equation holds for nonnegative constants. Now, let $\{E_t\}$ be the spectral measure for the operator B and let

$$A = \frac{2}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda)B^2(1+\lambda B)^{-2}\,d\lambda.$$

For each $\xi \in H$,

$$\begin{aligned}
\langle A\xi, \xi\rangle &= \frac{2}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda)\langle B^2(1+\lambda B)^{-2}\xi,\xi\rangle\,d\lambda\\
&= \frac{2}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda)\Big(\int_0^1 t^2(1+\lambda t)^{-2}\,d\langle E_t\xi,\xi\rangle\Big)\,d\lambda\\
&= \int_0^1\Big(\frac{2}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(1+\lambda)t^2(1+\lambda t)^{-2}\,d\lambda\Big)\,d\langle E_t\xi,\xi\rangle\\
&= \int_0^1 (t^{\frac{3}{2}} + t^{\frac{1}{2}})\,d\langle E_t\xi,\xi\rangle = \langle (B^{\frac{3}{2}} + B^{\frac{1}{2}})\xi,\xi\rangle,
\end{aligned}$$

so that $A = B^{\frac{3}{2}} + B^{\frac{1}{2}}$ as claimed. The interchange of the order of integration is justified by Tonelli's Theorem, as the two nonnegative measures, $d\lambda$ and $d\langle E_t\xi,\xi\rangle$ are both σ-finite, and the nonnegative integrand is continuous and hence product measurable. ∎

PROPOSITION 2.12. *If (N, D) is an odd p-summable unbounded Breuer-Fredholm module (for $\mathbf{C}$, say) and $t \longmapsto A_t$ is a C^1-path in N_{sa}, then letting $F_t = D_t(1+D_t^2)^{-\frac{1}{2}}$ where $D_t = D + A_t$, we have for $n > \frac{p-1}{2}$ that*

$$\mathrm{Tr}\Big(\frac{d}{dt}(F_t)(1-F_t^2)^n\Big) = \mathrm{Tr}\Big(\frac{d}{dt}(A_t)(1+D_t^2)^{-(n+\frac{3}{2})}\Big).$$

PROOF. Since $2n + 1 > p$, $\frac{d}{dt}(F_t) \in \mathcal{L}^{2n+1}$ by Proposition 2.10. Moreover,

$$(1 - F_t^2)^n = (1 + D_t^2)^{-n} \in \mathcal{L}^{\frac{p}{2n}} \subseteq \mathcal{L}^{\frac{2n+1}{2n}}$$

so that the left-hand side of the equation is well-defined by Hölder's inequality. Also, by

Proposition 2.10,

$$
\begin{aligned}
&\operatorname{Tr}\Big(\frac{d}{dt}(F_t)(1-F_t^2)^n\Big)\\
&\qquad = \operatorname{Tr}\Big(\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}[(1+D_t^2+\lambda)^{-1}(1+\lambda)A_t'(1+D_t^2+\lambda)^{-1}\\
&\qquad\qquad - D_t(1+D_t^2+\lambda)^{-1}A_t'D_t(1+D_t^2+\lambda)^{-1}]\,d\lambda(1+D_t^2)^{-n}\Big)
\end{aligned}
$$

(where the integrand is $(2n+1)$-norm continuous and converges in $(2n+1)$-norm)

$$
= \operatorname{Tr}\Big(\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}[\cdots](1+D_t^2)^{-n}\,d\lambda\Big)
$$

(the integrand is trace-norm continuous and converges in trace-norm)

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\operatorname{Tr}\big([\cdots](1+D_t^2)^{-n}\big)\,d\lambda\\
&= \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(\operatorname{Tr}[(1+D_t^2+\lambda)^{-1}(1+\lambda)A_t'(1+D_t^2+\lambda)^{-1}(1+D_t^2)^{-n}]\\
&\qquad - \operatorname{Tr}[D_t(1+D_t^2+\lambda)^{-1}A_t'D_t(1+D_t^2+\lambda)^{-1}(1+D_t^2)^{-n}])\,d\lambda
\end{aligned}
$$

since both parts are trace-class by the estimates in the proof of Proposition 2.10. Using the trace property on each piece and recombining, this equals

$$
\begin{aligned}
&\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}\operatorname{Tr}[A_t'(1+\lambda-D_t^2)(1+D_t^2+\lambda)^{-2}(1+D_t^2)^{-n}]\,d\lambda\\
&\qquad = \operatorname{Tr}\Big(\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}[A_t'(1+\lambda-D_t^2)(1+D_t^2+\lambda)^{-2}(1+D_t^2)^{-n}]\,d\lambda\Big)
\end{aligned}
$$

(this new integrand is easily seen to converge in trace-norm). Now, this integrand is also convergent in operator-norm to the same (trace-class) operator because both imply strong-operator convergence. Thus, it suffices to see that the integral (in operator norm convergence) equals

$$
A_t'(1+D_t^2)^{-(n+\frac{3}{2})}.
$$

But,

$$
\begin{aligned}
&\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}[A_t'(1+\lambda-D_t^2)(1+D_t^2+\lambda)^{-2}(1+D_t^2)^{-n}]\,d\lambda\\
&\qquad = A_t'\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}[2(1+\lambda)-(1+D_t^2+\lambda)](1+D_t^2+\lambda)^{-2}d\lambda(1+D_t^2)^{-n}\\
&\qquad = A_t'\Big[\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}2(1+\lambda)(1+D_t^2+\lambda)^{-2}\,d\lambda\\
&\qquad\qquad - \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(1+D_t^2+\lambda)^{-1}\,d\lambda\Big](1+D_t^2)^{-n}\\
&\qquad = A_t'[(1+D_t^2)^{-\frac{3}{2}}+(1+D_t^2)^{-\frac{1}{2}}-(1+D_t^2)^{-\frac{1}{2}}](1+D_t^2)^{-n}\\
&\qquad = A_t'(1+D_t^2)^{-(n+\frac{3}{2})}
\end{aligned}
$$

by Lemma 2.11. ∎

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E. **If** (N, D_0) **is** p**-summable,** $n > \frac{p-1}{2}$ **is an integer,** $M_0 = D_0 + N_{\mathrm{sa}}$**, then** $\mathrm{Tr}\big(X(1+D^2)^{-(n+\frac{3}{2})}\big)$ **is an exact** 1**-form on** M_0**.**

DEFINITION 2.13. Let (N, D_0) be an odd p-summable Breuer-Fredholm module (for $\mathbf{C}$, say), let $n > \frac{p-1}{2}$ be an integer, and let $m = n + \frac{3}{2}$ and let $M_0 = D_0 + N_{\mathrm{sa}}$. Then for $D \in M_0$ we define:

$$\gamma(D) := \frac{1}{\tilde{C}_m} \int_0^1 \mathrm{Tr}\Big(\frac{d}{dt}(D_t)(1+D_t^2)^{-m}\Big)\, dt$$

where $\{D_t\}$ is any piecewise C^1, continuous path in M_0 from D_0 to D and

$$\tilde{C}_m = \int_{-\infty}^{\infty} (1+x^2)^{-m}\, dx$$

$(= C_n = \int_{-1}^{1}(1-s^2)^n\, ds$ where $m = n + \frac{3}{2})$.

THEOREM 2.14. *γ is well-defined and*

$$d\gamma_D(X) = \frac{1}{\tilde{C}_m} \mathrm{Tr}\big(X(1+D^2)^{-m}\big)$$

for

$$D \in M_0 = D_0 + N_{\mathrm{sa}} \quad \text{and} \quad X \in T_D(M_0) = N_{sa}.$$

Therefore, the latter is an exact (and hence closed) 1*-form on* M_0*.*

PROOF. Fix q so that $2n+1 > q > p$. Let $\{D_t\}$ be any piecewise C^1, continuous path from D_0 to D so that $F_t = D_t(1+D_t^2)^{-\frac{1}{2}}$ is a piecewise q-norm-C^1 continuous path from $F_0 = D_0(1+D_0^2)^{-\frac{1}{2}}$ to $F_1 = D(1+D^2)^{-\frac{1}{2}}$ in $F_0 + L_{\mathrm{sa}}^{q,\frac{q}{2}}$ by parts B and C. Moreover, by part D,

$$\frac{1}{\tilde{C}_m} \int_0^1 \mathrm{Tr}\Big(\frac{d}{dt}(D_t)(1+D_t^2)^{-m}\Big)\, dt = \frac{1}{C_n} \int_0^1 \mathrm{Tr}\Big(\frac{d}{dt}(F_t)(1-F_t^2)^n\Big)\, dt$$

which by Theorem 1.7 and Remark 1.8 is equal to

$$\mathrm{sf}(F_0, F_1) - \beta_n(F_1) + \beta_n(F_0)$$

and this only depends on the end points D_0 and $D_1 = D$. Thus, γ is well-defined.

Now, for $X \in T_D(M_0) = N_{\mathrm{sa}}$, we have

$$d\gamma_D(X) = \frac{d}{ds}\Big|_{s=0} \big(\gamma(D+sX)\big).$$

Since γ is independent of path we can choose our path from D_0 to $D+sX$ to pass through D (*e.g.*, our path can be linear from D_0 to D and then linear from D to $D+sX$). Then,

$$\gamma(D+sX) = \frac{1}{\tilde{C}_m} \int_0^{\frac{1}{2}} \mathrm{Tr}\Big(\frac{d}{dt}(D_t)(1+D_t^2)^{-m}\Big)\, dt + \frac{1}{\tilde{C}_m} \int_{\frac{1}{2}}^1 \mathrm{Tr}\Big(\frac{d}{dt}(D_t)(1+D_t^2)^{-m}\Big)\, dt.$$

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Since the first half of the path (from D_0 to D) does not depend on s, we get

$$d\gamma_D(X) = \frac{d}{ds}\Big|_{s=0}\left(\frac{1}{\tilde{C}_m}\int_{\frac{1}{2}}^{1}\mathrm{Tr}\Big(\frac{d}{dt}(D_t)(1+D_t^2)^{-m}\Big)\,dt\right)$$

where D_t for $t \in [\frac{1}{2}, 1]$ is the line from D to $D+sX$. That is, $D_t = D+(2t-1)sX$. Hence

$$\frac{d}{dt}(D_t) = 2sX$$

and so

$$d\gamma_D(X) = \lim_{s\to 0}\int_{\frac{1}{2}}^{1}\mathrm{Tr}\Big(2X\Big(1+\big(D+(2t-1)sX\big)^2\Big)^{-m}\Big)\,dt.$$

Now, by Proposition 11 of Appendix B we get

$$\begin{aligned}
&\Big\|\Big(1+\big(D+(2t-1)sX\big)^2\Big)^{-m}-(1+D^2)^{-m}\Big\|_1\\
&\quad\le 2^{\frac{1}{m+\frac{1}{2}}}\big(2\|(2t-1)sX\|\big)^{\frac{m}{m+\frac{1}{2}}}\Big[f\big(\|(2t-1)sX\|\big)\Big]^m(m+1)\|(1+D^2)^{-m}\|_1\\
&\quad\le 4(s\|X\|)^{\frac{m}{m+\frac{1}{2}}}[f(\|X\|)]^m(m+1)\|(1+D^2)^{-m}\|_1 \quad (\text{for } |s|\le 1)\\
&\quad\longrightarrow 0 \text{ uniformly in } t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d\gamma_D(X) &= \frac{1}{\tilde{C}_m}\int_{\frac{1}{2}}^{1}\mathrm{Tr}\big(2X(1+D^2)^{-m}\big)\,dt\\
&= \frac{1}{\tilde{C}_m}\mathrm{Tr}\big(X(1+D^2)^{-m}\big)
\end{aligned}$$

as claimed. ∎

F. **The Theorems.**

DEFINITION 2.15. If (N, D_0) is an odd unbounded Breuer-Fredholm module (for $\mathbf{C}$, say) and $\{D_t\}_{t\in[a,b]}$ is any norm continuous path in $M_0 = D_0 + N_{\mathrm{sa}}$ then we *define* the *spectral flow of the path* $\{D_t\}$, $\mathrm{sf}(D_a, D_b)$ to be the spectral flow of the norm-continuous path $\{F_t = D_t(1+D_t^2)^{-\frac{1}{2}}\}$ of self-adjoint Breuer-Fredholm operators, [P2]. This is well-defined by Theorem 8 of Appendix A, and the fact that $1-F_t^2 = (1+D_t^2)^{-1}$ is in K_N for each t.

THEOREM 2.16. *Let (N, D_0) be an odd p-summable unbounded Breuer-Fredholm module (for $\mathbf{C}$) and let $M_0 = D_0 + N_{\mathrm{sa}}$. Let $n > \frac{p-1}{2}$ be an integer and let $m = n + \frac{3}{2}$. Then, for $D \in M_0$, $X \in T_D(M_0) = N_{\mathrm{sa}}$,*

$$X \longmapsto \frac{1}{\tilde{C}_m}\mathrm{Tr}\big(X(1+D^2)^{-m}\big)$$

is an exact 1-form on M_0. Moreover, if $\{D_t\}_{t\in[a,b]}$ is any piecewise-C^1 continuous path in M_0, then integrating this 1-form yields:

$$\mathrm{sf}(D_a, D_b) = \frac{1}{\tilde{C}_m}\int_a^b \mathrm{Tr}\Big(\frac{d}{dt}(D_t)(1+D_t^2)^{-m}\Big)\,dt + \beta_n\big(D_b(1+D_b^2)^{-\frac{1}{2}}\big) - \beta_n\big(D_a(1+D_a^2)^{-\frac{1}{2}}\big).$$

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PROOF. As usual, we let $F_t = D_t(1+D_t^2)^{-\frac{1}{2}}$. As observed in the proof of Theorem 2.14, the right-hand side of the equation equals $\mathrm{sf}(F_a, F_b)$ and so by the previous definition this is $\mathrm{sf}(D_a, D_b)$, as required. That the 1-form is exact is Theorem 2.14. ∎

THEOREM 2.17. *Let (N, D_0) be an odd p-summable unbounded Breuer-Fredholm module for the unital Banach $*$-algebra A, let n be an integer $n > \frac{p-1}{2}$ and let $m = n+\frac{3}{2}$. Let $P = \chi(D_0)$. Then, for each $u \in U(A)$ with $u(\mathrm{dom}\, D) \subseteq \mathrm{dom}\, D$ and $[D,u]$ bounded, PuP is a Breuer-Fredholm operator in PNP and if $\{D_t^u\}$ is any piecewise-C^1 continuous path in $M_0 = D_0 + N_{\mathrm{sa}}$ from D_0 to uD_0u^* (e.g., the linear path lies in M_0), then:*

$$\mathrm{ind}(PuP) = \mathrm{sf}(\{D_t^u\}) = \frac{1}{\tilde{C}_m}\int_0^1 \mathrm{Tr}\Big(\frac{d}{dt}(D_t^u)\big(1+(D_t^u)^2\big)^{-m}\Big)\, dt,$$

the integral of the exact 1-form, $\frac{1}{\tilde{C}_m}\mathrm{Tr}\big(X(1+D^2)^{-m}\big)$ along the path $\{D_t^u\}$.

PROOF. That PuP is a Breuer-Fredholm operator whose index is the spectral flow of the straight line path from F_0 to uF_0u^* is part of Theorem 1.9. That this is also $\mathrm{sf}(D_0, u\,D_0\,u^*)$ is just Definition 2.15. That the spectral flow is independent of path (in M_0) is in the previous theorem. The formula for the spectral flow is likewise part of the previous theorem, where it should be recalled that

$$uD_0u^* - D_0 = [u, D_0]u^*.$$ ∎

Appendix A. The Operator-Norm Continuity of Functions of Unbounded Self-Adjoint Operators. In this appendix, we prove sharp perturbation estimates (in the operator-norm) of the following sort: if D is an unbounded self-adjoint operator, A is a bounded self-adjoint operator, and f is an explicit, bounded continuous function on $\mathbf{R}$ then

$$\|f(D+A) - f(D)\| \le C_f\|A\|$$

where C_f is a constant depending only on f. While some of these results may be known, we have not found them in the standard references [DS, K, RS]. At a number of places in this paper, we need these sharp estimates, rather than just the usual continuity results of the form:

$$\|f(D+A_n) - f(D)\| \longrightarrow 0 \quad \text{as} \quad \|A_n\| \longrightarrow 0.$$

Many of these results can be generalized to relatively D-bounded symmetric operators, A [K, V.4]. While we do not use these results in this paper, we plan to use them in the future and they may be of independent interest to other workers in the field. For these reasons we include the more general D-bounded results and indicate the modifications needed to prove them.

We begin with some well-known facts that will help set the notation. Let D be an unbounded self-adjoint operator on the Hilbert space, H, and let

$$G_D = \{(\xi, D\xi) \mid \xi \in \mathrm{dom}\, D\}$$

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denote the graph of D, a closed subspace of $H \oplus H$. It is well-known [DS, Lemma XII.1.5] that the orthogonal complement to G_D is given by

$$G_D^{\perp} = \{(D\xi, -\xi) \mid \xi \in \operatorname{dom} D\}.$$

We denote the projection from $H \oplus H$ onto G_D by P_D and recall the result of B. Sz.-Nagy [DS, Exercise XII.9.36] that for $\xi \in H$,

$$P_D(\xi, 0) = \big((1 + D^2)^{-1}\xi, D(1 + D^2)^{-1}\xi\big).$$

From this, we easily deduce that the matrix of P_D relative to the decomposition $H \oplus H$ is

$$\left[\begin{array}{c|c} (1+D^2)^{-1} & D(1+D^2)^{-1} \\ \hline D(1+D^2)^{-1} & D^2(1+D^2)^{-1} \end{array}\right].$$

RELATIVELY BOUNDED OPERATORS. Let D_0 be a self-adjoint operator and let A be a symmetric operator with $\operatorname{dom} D_0 \subseteq \operatorname{dom} A$. We will say that A is D_0*-bounded* if there exists a positive constant C with

$$\|A\xi\| \le C(\|\xi\|^2 + \|D_0\xi\|^2)^{\frac{1}{2}}$$

for all $\xi \in \operatorname{dom} D_0$. We denote by $\|A\|_{D_0}$ the infimum of all such constants C, and note that if A is actually bounded then $\|A\|_{D_0} \le \|A\|$ and so A is also relatively D_0-bounded. We warn the reader that this number, $\|A\|_{D_0}$, is *not* the D_0-bound in the sense of [K, V.4.1]; however, the D_0-bound is $\le \|A\|_{D_0}$. Furthermore, by [K, V.4.1, Theorem 4.3], if $\|A\|_{D_0} < 1$ then $D_0 + A$ is self-adjoint. If A is actually bounded then $D_0 + A$ is also self-adjoint, independent of $\|A\|$. If A is a D_0-bounded symmetric operator with $\|A\|_{D_0} < 1$, and $D = D_0 + A$, then A is also D-bounded and one easily calculates that

$$\|A\|_D \le \frac{\sqrt{2}\|A\|_{D_0}}{1 - \|A\|_{D_0}}.$$

If we have $\|A\|_{D_0} \le .29$ ($< (1 - \frac{1}{\sqrt{2}})$ will do) we get $\|A\|_D \le 2\|A\|_{D_0}$. This lack of symmetry between $\|A\|_D$ and $\|A\|_{D_0}$ is the reason one often uses the distance between G_{D_0} and G_D to measure the distance between closed operators in perturbation theory [K, IV.2.4].

PROPOSITION 1. *Let D_0 be an unbounded self-adjoint operator and let A be a bounded self-adjoint operator. Then, $D = D_0 + A$ is also self-adjoint and $\|P_D - P_{D_0}\| \le \|A\|$. If A is a D_0-bounded symmetric operator and $\|A\|_{D_0} \le .29$, we get $\|P_D - P_{D_0}\| \le 2\|A\|_{D_0}$.*

PROOF. It is very easy to verify that D is also self-adjoint, and $\operatorname{dom} D = \operatorname{dom} D_0$. Let $\omega \in H \oplus H$, so that $P_{D_0}\omega = \zeta = (\xi, D_0\xi) \in G_{D_0}$. Let $\zeta' = (\xi, D\xi) \in G_D$ so then

$$\|\zeta' - \zeta\| = \|(0, A\xi)\| \le \|A\|\,\|\xi\| \le \|A\|\,\|\zeta\|.$$

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We can replace $\|A\|$ with $\|A\|_{D_0}$ if necessary. Since $P_D\zeta$ is the closest point to ζ in G_D we have

$$\|\zeta - P_D\zeta\| \leq \|\zeta - \zeta'\| \leq \|A\|\,\|\zeta\|.$$

That is,

$$\|(1-P_D)P_{D_0}\omega\| = \|\zeta - P_D\zeta\| \leq \|A\|\,\|\zeta\| = \|A\|\,\|P_{D_0}\omega\| \leq \|A\|\,\|\omega\|,$$

and hence,

$$\|(1-P_D)P_{D_0}\| \leq \|A\|, \quad (\text{or } \|A\|_{D_0}).$$

Since $D_0 = D - A$, reversing roles, we get by a similar calculation that

$$\|P_D(1-P_{D_0})\| = \|(1-P_{D_0})P_D\| \leq \|-A\| = \|A\| \quad (\text{or } \|A\|_D \leq 2\|A\|_{D_0}).$$

Finally,

$$\begin{aligned}\|P_D - P_{D_0}\| &= \|P_D(1-P_{D_0}) - (1-P_D)P_{D_0}\|\\ &= \max\{\|P_D(1-P_{D_0})\|, \|(1-P_D)P_{D_0}\|\} \leq \|A\|, \quad (\text{or } 2\|A\|_{D_0}),\end{aligned}$$

since these two operators have orthogonal initial spaces and orthogonal ranges. ∎

COROLLARY 2. *Let D_0 be an unbounded self-adjoint operator and let A be a bounded self-adjoint operator. Letting $D = D_0 + A$, we have:*

(1) $\|(1+D^2)^{-1} - (1+D_0^2)^{-1}\| \leq \|A\|$,

(2) $\|D(1+D^2)^{-1} - D_0(1+D_0^2)^{-1}\| \leq \|A\|$, *and*

(3) $\|D^2(1+D^2)^{-1} - D_0^2(1+D_0^2)^{-1}\| \leq \|A\|$.

If A is a D_0-bounded symmetric operator and $\|A\|_{D_0} \leq .29$, we can replace $\|A\|$ by $2\|A\|_{D_0}$ in each case.

PROOF. These all follow directly from Proposition 1 and the matrix form of P_D given above. ∎

These are careful versions of results of Sz.-Nagy [DS, Exercise XII.9.37].

REMARK 3. Each of the functions, $f_1(x) = (1+x^2)^{-1}$, $f_2(x) = x(1+x^2)^{-1}$ and $f_3(x) = x^2(1+x^2)^{-1}$ used above satisfy $f_i(\infty) = f_i(-\infty)$. In order to get an estimate for a function with different limits at $+\infty$ and $-\infty$, we must work harder. We are particularly interested in $f(x) = x(1+x^2)^{-\frac{1}{2}}$.

To begin, we recall [Ped, p. 8] that for any bounded positive operator B, and any r, $0 < r < 1$ that

$$B^r = \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}(1+\lambda B)^{-1}B\,d\lambda,$$

where the integrand is a norm-continuous function of λ and the finite Riemann Sums converge in norm. Here,

$$\frac{\pi}{\sin(r\pi)} = \int_0^\infty u^{-r}(1+u)^{-1}\,du.$$

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This integral formula for B^r can also be proved by the method of Lemma 2.10. [See also [K, V.3.50]]. Letting $B = (1 + D^2)^{-1}$ we calculate:

$$\begin{aligned} B(1 + \lambda B)^{-1} &= (1 + D^2)^{-1}\big(1 + \lambda(1 + D^2)^{-1}\big)^{-1} \\ &= \big[(1 + D^2)\big(1 + \lambda(1 + D^2)^{-1}\big)\big]^{-1} \\ &= (1 + D^2 + \lambda)^{-1} \end{aligned}$$

so

$$(1 + D^2)^{-r} = \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}(1 + D^2 + \lambda)^{-1}\, d\lambda$$

where the integrand is a norm-continuous function of λ and the integral converges in operator norm. We would like to apply the operator, D, to both sides of this equation and pass D through the integral. However, the integrand, $\lambda^{-r}D(1 + D^2 + \lambda)^{-1}$ is *not* absolutely integrable in operator norm for $r \leq \frac{1}{2}$. Fortunately, for $\xi \in \operatorname{dom} D$, the integrand $\lambda^{-r}D(1 + D^2 + \lambda)^{-1}\xi$ is integrable in H!

LEMMA 4. *If D is a self-adjoint operator, then for all $\xi \in \operatorname{dom} D$ and $0 < r < 1$*

$$D(1 + D^2)^{-r}\xi = \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}D(1 + D^2 + \lambda)^{-1}\xi\, d\lambda$$

where the integrand on the right converges in H.

PROOF. Since $\xi \in \operatorname{dom} D$

$$\begin{aligned} D(1 + D^2)^{-r}\xi &= (1 + D^2)^{-r}(D\xi) \\ &= \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}(1 + D^2 + \lambda)^{-1}\, d\lambda(D\xi) \\ &= \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}(1 + D^2 + \lambda)^{-1}D\xi\, d\lambda \\ &= \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}D(1 + D^2 + \lambda)^{-1}\xi\, d\lambda. \end{aligned}$$

■

REMARK 5. By the Spectral Theorem for self-adjoint operators one can easily prove the following estimates:

(1) $\|(1 + D^2 + \lambda)^{-1}\| \leq \frac{1}{1+\lambda}$

(2) $\|(1 + D^2 + \lambda)^{-1} - (1 + D^2 + \gamma)^{-1}\| \leq \frac{1}{1+\lambda}|\lambda - \gamma|\frac{1}{1+\gamma}$

(3) $\|D(1 + D^2 + \lambda)^{-1}\| \leq \frac{1}{2\sqrt{1+\lambda}}$

(4) $\|D(1 + D^2 + \lambda)^{-1} - D(1 + D^2 + \gamma)^{-1}\| \leq \frac{1}{2\sqrt{1+\lambda}}|\lambda - \gamma|\frac{1}{1+\gamma}$

for all $\lambda, \gamma \geq 0$.

From (1) and (2) one concludes that the function $\lambda \longmapsto \lambda^{-r}(1+D^2+\lambda)^{-1}$ is norm continuous, and provided $0 < r < 1$, absolutely integrable. From (3) and (4) one concludes that the function $\lambda \longmapsto \lambda^{-r}D(1+D^2+\lambda)^{-1}$ is norm continuous, and absolutely integrable for $r > \frac{1}{2}$; but, for $0 < r \leq \frac{1}{2}$ this function is generally *not* absolutely integrable. If D is "multiplication by x" on $L^2(\mathbf{R})$ or "multiplication by n" on $\ell^2(\mathbf{Z})$, then these integrals do *not* converge in norm. For $0 < r < \frac{1}{2}$, the operator, $D(1+D^2)^{-r}$ is not even bounded!

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LEMMA 6. *Let D_0 be an unbounded self-adjoint operator, let A be a bounded self-adjoint operator, and let $D = D_0 + A$. Then for all $\lambda \geq 0$,*

(1) $\|(1+D^2+\lambda)^{-1} - (1+D_0^2+\lambda)^{-1}\| \leq (\frac{1}{1+\lambda})^{\frac{3}{2}}\|A\|$ *and*

(2) $\|D(1+D^2+\lambda)^{-1} - D_0(1+D_0^2+\lambda)^{-1}\| \leq \frac{1}{1+\lambda}\|A\|$.

If A is only D_0-bounded and $\|A\|_{D_0} \leq .29$ then we get:

(3) $\|(1+D^2+\lambda)^{-1} - (1+D_0^2+\lambda)^{-1}\| \leq \frac{1}{1+\lambda}2\|A\|_{D_0}$ *and*

(4) $\|D(1+D^2+\lambda)^{-1} - D_0(1+D_0^2+\lambda)^{-1}\| \leq (\frac{1}{1+\lambda})^{\frac{1}{2}}2\|A\|_{D_0}$.

PROOF.

$$(1+D^2+\lambda)^{-1} = \frac{1}{1+\lambda}\left(1 + \left[\left(\frac{1}{1+\lambda}\right)^{\frac{1}{2}} D_0 + \left(\frac{1}{1+\lambda}\right)^{\frac{1}{2}} A\right]^2\right)^{-1}.$$

We apply Corollary 2 part (1) and observe that

$$\left\|\left(\frac{1}{1+\lambda}\right)^{\frac{1}{2}} A\right\| = \left(\frac{1}{1+\lambda}\right)^{\frac{1}{2}} \|A\|$$

to get (1) above, and that

$$\left\|\left(\frac{1}{1+\lambda}\right)^{\frac{1}{2}} A\right\|_{(\frac{1}{1+\lambda})^{\frac{1}{2}} D_0} \leq \|A\|_{D_0}$$

to get (3).

To see (2) and (4) we apply Corollary 2 part (2) to:

$$D(1+D^2+\lambda)^{-1} = \left(\frac{1}{1+\lambda}\right)^{\frac{1}{2}}\left[\left(\frac{1}{1+\lambda}\right)^{\frac{1}{2}} D\left(1 + \left[\left(\frac{1}{1+\lambda}\right)^{\frac{1}{2}} D\right]^2\right)^{-1}\right]. \qquad \blacksquare$$

PROPOSITION 7. *Let D_0 be an unbounded self-adjoint operator, let A be a bounded self-adjoint operator, and let $D = D_0 + A$. Then for $0 \leq r \leq 1$ we get*

$$\|(1+D^2)^{-r} - (1+D_0^2)^{-r}\| \leq \|A\|.$$

If A is only D_0-bounded and $\|A\|_{D_0} \leq .29$ we get the same estimate with $\|A\|$ replaced by $2\|A\|_{D_0}$.

PROOF. By Lemma 6 part (1) we get for $0 < r < 1$

$$\begin{aligned}
\|(1+D^2)^{-r} - (1+D_0^2)^{-r}\| &\leq \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}\|(1+D^2+\lambda)^{-1} - (1+D_0^2+\lambda)^{-1}\|\,d\lambda \\
&\leq \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}\left(\frac{1}{1+\lambda}\right)^{\frac{3}{2}}\|A\|\,d\lambda \\
&\leq \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}\left(\frac{1}{1+\lambda}\right)d\lambda\|A\| = \|A\|.
\end{aligned}$$

If A is only D_0-bounded we appeal to Lemma 6 part (3) to get the final estimate. Of course, we could do a little better in the bounded case by evaluating

$$\int_0^\infty \lambda^{-r}\left(\frac{1}{1+\lambda}\right)^{\frac{3}{2}} d\lambda$$

exactly. For example, if $r = \frac{1}{2}$ we get the estimate $\frac{2}{\pi}\|A\|$. $\blacksquare$

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THEOREM 8. *Let D_0 be an unbounded self-adjoint operator, let A be a bounded self-adjoint operator and let $D = D_0 + A$. Then, for $\frac{1}{2} \leq r \leq 1$ we have:*

$$\|D(1+D^2)^{-r} - D_0(1+D_0^2)^{-r}\| \leq \|A\|.$$

PROOF. The case $r = 1$ has already been done. For $r < 1$, we use Lemma 4: let $\xi \in \operatorname{dom} D_0 = \operatorname{dom} D$ so that for $0 < r < 1$

$$D(1+D^2)^{-r}\xi - D_0(1+D_0^2)^{-r}\xi = \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}[D(1+D^2+\lambda)^{-1} - D_0(1+D_0^2+\lambda)^{-1}]\xi\, d\lambda$$

and hence by Lemma 6 part (2):

$$\|[D(1+D^2)^{-r} - D_0(1+D_0^2)^{-r}]\xi\| \leq \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}\frac{1}{1+\lambda}\|A\|\,\|\xi\|\, d\lambda = \|A\|\,\|\xi\|.$$

Since both operators are bounded for $r \geq \frac{1}{2}$ and $\operatorname{dom} D_0$ is dense in H, the result follows. For $r = \frac{1}{2}$ see Theorem 4.7 of [BF] for a slightly weaker version of this result. ∎

REMARK 9. Even for $0 < r < \frac{1}{2}$ where the two operators $D(1+D^2)^{-r}$ and $D_0(1+D_0^2)^{-r}$ are both unbounded, the above proof shows that their difference is bounded by $\|A\|$ on $\operatorname{dom} D_0$!

If A is only D_0-bounded with $\|A\|_{D_0} \leq .29$ and $r > \frac{1}{2}$ then the same proof using Lemma 6 part (4) yields:

$$\|D(1+D^2)^{-r} - D_0(1+D_0^2)^{-r}\| \leq C_r 2\|A\|_{D_0}$$

where

$$C_r = \frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{-r}(1+\lambda)^{-\frac{1}{2}}\, d\lambda < \infty.$$

If $r = \frac{1}{2}$ and A is D_0-bounded then we don't know if such a result holds. However, we can show a weaker result.

If $\xi \in \operatorname{dom} D_0$, we let

$$\|\xi\|_{D_0} = (\|\xi\|^2 + \|D_0\xi\|^2)^{\frac{1}{2}}.$$

THEOREM 10. *Let D_0 be an unbounded self-adjoint operator, and let A be a D_0-bounded symmetric operator such that $\|A\|_{D_0} \leq .29$. Then*

$$\|D(1+D^2)^{-\frac{1}{2}} - D_0(1+D_0^2)^{-\frac{1}{2}}\|_{D_0} \leq 3\|A\|_{D_0}.$$

PROOF. For $\xi \in \operatorname{dom} D = \operatorname{dom} D_0$ we have:

$$\begin{aligned}
\|D(1+D^2)^{-\frac{1}{2}}\xi - &D_0(1+D_0^2)^{-\frac{1}{2}}\xi\| \\
&= \|(1+D^2)^{-\frac{1}{2}}D\xi - (1+D_0^2)^{-\frac{1}{2}}D_0\xi\| \\
&\leq \|(1+D^2)^{-\frac{1}{2}}A\xi\| + \|(1+D^2)^{-\frac{1}{2}} - (1+D_0^2)^{-\frac{1}{2}}\|\,\|D_0\xi\| \\
&\leq \|A\xi\| + 2\|A\|_{D_0}\|D_0\xi\| \quad \text{by Proposition 7} \\
&\leq \|A\|_{D_0}\|\xi\|_{D_0} + 2\|A\|_{D_0}\|D_0\xi\| \leq 3\|A\|_{D_0}\|\xi\|_{D_0}.
\end{aligned}$$

∎

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FOURIER TRANSFORM METHODS.

LEMMA 11. *If D is an unbounded self-adjoint operator then*

$$e^{itD} = \text{strong} \lim_{n\to\infty} \Big(1 - \frac{itD}{n}\Big)^{-n}$$

for all $t \in \mathbf{R}$, where the left-hand side of the equation is defined by the functional calculus for unbounded self-adjoint operators (i.e., the spectral theorem).

PROOF. Without loss of generality $t = 1$. In this case, the functions $(1 - \frac{ix}{n})^{-n}$ and e^{ix} are all bounded by 1 on $\mathbf{R}$, as are their derivatives. Since

$$\lim_{n\to\infty} \Big(1 - \frac{ix}{n}\Big)^{-n} = e^{ix}$$

pointwise on $\mathbf{R}$, the convergence is thus uniform on compact subsets. Let E_N be the spectral projection for D corresponding to $[-N, N]$. Then,

$$E_N(e^{iD})E_N = \|\cdot\| - \lim_{n\to\infty} E_N\Big(1 - \frac{iD}{n}\Big)^{-n} E_N.$$

So, for vectors ξ in the dense set $\bigcup_{N=1}^{\infty} E_N(H)$, we see that

$$\lim_{n\to\infty} \Big(1 - \frac{iD}{n}\Big)^{-n} \xi = e^{iD}\xi.$$

As all the operators are bounded by 1, this is sufficient to conclude that

$$e^{iD} = \text{strong} \lim_{n\to\infty} \Big(1 - \frac{iD}{n}\Big)^{-n}. \qquad \blacksquare$$

PROPOSITION 12. *Let D_0 be an unbounded self-adjoint operator; let A be a bounded self-adjoint operator, and let t be a real number. Then*

$$\|e^{itD} - e^{itD_0}\| \leq |t|\,\|A\|,$$

where $D = D_0 + A$.

PROOF.

$$\begin{aligned}
\Big(1 - i\frac{t}{n}D\Big)^{-1} &- \Big(1 - i\frac{t}{n}D_0\Big)^{-1} \\
&= \Big(1 - i\frac{t}{n}D)\Big)^{-1}\Big[\Big(1 - i\frac{t}{n}D_0\Big) - \Big(1 - i\frac{t}{n}D\Big)\Big]\Big(1 - i\frac{t}{n}D_0\Big)^{-1} \\
&= \Big(1 - i\frac{t}{n}D\Big)^{-1}\Big[i\frac{t}{n}A\Big]\Big(1 - i\frac{t}{n}D_0\Big)^{-1}.
\end{aligned}$$

So,

$$\Big\|\Big(1 - i\frac{t}{n}D\Big)^{-1} - \Big(1 - i\frac{t}{n}D_0\Big)^{-1}\Big\| \leq \frac{|t|}{n}\|A\|.$$

(Since $\operatorname{dom} D = \operatorname{dom} D_0 = \operatorname{range}(1 - i\frac{t}{n}D_0)^{-1} = \operatorname{range}(1 - i\frac{t}{n}D)^{-1}$, the first equality holds for all vectors in H.)

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We now apply the identity

$$X^n - Y^n = \sum_{k=0}^{n-1} X^k (X - Y) Y^{n-k-1}$$

to

$$\Big(1 - i\frac{t}{n}D\Big)^{-n} - \Big(1 - i\frac{t}{n}D_0\Big)^{-n}$$

to obtain:

$$\begin{aligned}\Big\|\Big(1 - i\frac{t}{n}D\Big)^{-n} - \Big(1 - i\frac{t}{n}D_0\Big)^{-n}\Big\| &\leq \sum_{k=0}^{n-1}\Big\|\Big(1 - i\frac{t}{n}D\Big)^{-k}\Big\|\frac{|t|}{n}\|A\|\,\Big\|\Big(1 - i\frac{t}{n}D_0\Big)^{-(n-k-1)}\Big\| \\ &\leq |t|\,\|A\|.\end{aligned}$$

Taking strong operator limits gives us

$$\|e^{itD} - e^{itD_0}\| \leq |t|\,\|A\|. \qquad \blacksquare$$

REMARK 13. If we only assume that A is D_0-bounded, then the map $A \longmapsto e^{it(D_0+A)}$ is *not* continuous in general. For example, if D_0 is "multiplication by x" on $L^2(\mathbf{R})$ and $A_n = \frac{1}{n}D_0$ then $\|A_n\|_{D_0} \leq \frac{1}{n}$, but

$$\begin{aligned}\|e^{it(D_0+A_n)} - e^{itD_0}\| &= \Big\|e^{itD_0}\Big(e^{i\frac{t}{n}D_0} - 1\Big)\Big\| \\ &= \|e^{i\frac{t}{n}D_0} - 1\| = 2 \quad \text{for } t \neq 0.\end{aligned}$$

However, we do have the following:

PROPOSITION 14. *Let D_0 be an unbounded self-adjoint operator, let A be a D_0-bounded symmetric operator such that $D = D_0 + A$ is self-adjoint. For each real number t we have*

$$\|e^{itD} - e^{itD_0}\|_{D_0} \leq |t|\,\|A\|_{D_0}.$$

PROOF. Let $\xi \in \operatorname{dom} D_0$. Following the ideas of Proposition 12,

$$\begin{aligned}&\Big\|\Big(1 - i\frac{t}{n}D\Big)^{-n}\xi - \Big(1 - \frac{it}{n}D_0\Big)^{-n}\xi\Big\| \\ &\quad\leq \sum_{k=0}^{n-1}\Big\|\Big(1 - \frac{it}{n}D\Big)^{-k}\Big\|\Big\|\frac{it}{n}A\Big(1 - \frac{it}{n}D_0\Big)^{-(n-k-1)}\xi\Big\| \\ &\quad\leq \sum_{k=0}^{n-1}\frac{|t|}{n}\|A\|_{D_0}\bigg(\Big\|\Big(1 - \frac{it}{n}D_0\Big)^{-(n-k-1)}\xi\Big\|^2 + \Big\|D_0\Big(1 - \frac{it}{n}D_0\Big)^{-(n-k-1)}\xi\Big\|^2\bigg)^{\frac{1}{2}} \\ &\quad\leq \sum_{k=0}^{n-1}\frac{|t|}{n}\|A\|_{D_0}(\|\xi\|^2 + \|D_0\xi\|^2)^{\frac{1}{2}} = |t|\,\|A\|_{D_0}\|\xi\|_{D_0}.\end{aligned}$$

By Lemma 11, we obtain for $\xi \in \operatorname{dom} D_0$

$$\|e^{itD}\xi - e^{itD_0}\xi\| \leq |t|\,\|A\|_{D_0}\|\xi\|_{D_0}$$

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and the result follows. ∎

15. FOURIER TRANSFORM. Let $f \in L^1(\mathbf{R})$, then for $t \in \mathbf{R}$,

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixt} f(x)\, dx$$

is the Fourier transform of f, a continuous function vanishing at $\pm\infty$. We wish to apply this to an unbounded self-adjoint operator D.

LEMMA 16. *Let D be an unbounded self-adjoint operator and let $f \in L^1(\mathbf{R})$. Then*

$$\hat{f}(D) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixD} f(x)\, dx,$$

where the integral on the right converges in the strong-operator topology.

PROOF. A careful application of the spectral theorem. ∎

THEOREM 17. *Let D_0 be an unbounded self-adjoint operator and suppose $f(x)$ and $xf(x)$ are in $L^1(\mathbf{R})$. Let $D = D_0 + A$ where A is bounded and self-adjoint. Then*

$$\|\hat{f}(D) - \hat{f}(D_0)\| \leq \frac{\|A\|}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |xf(x)|\, dx.$$

PROOF. For $\xi \in H$ we have

$$\begin{aligned}
\big\|\big(\hat{f}(D) - \hat{f}(D_0)\big)\xi\big\| &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \|(e^{-ixD} - e^{-ixD_0})\xi\|\, |f(x)|\, dx \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|\, \|A\|\, \|\xi\| \cdot |f(x)|\, dx \\
&= \left(\frac{\|A\|}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |xf(x)|\, dx\right) \|\xi\|.
\end{aligned}$$ ∎

COROLLARY 18. $\|e^{-tD^2} - e^{-tD_0^2}\| \leq 2\sqrt{\frac{t}{\pi}}\|A\|$, *for* $t > 0$.

PROOF. For $t = \frac{1}{2}$ the function $f(x) = e^{-\frac{1}{2}x^2}$ satisfies $\hat{f} = f$ and so we get

$$\|e^{-\frac{1}{2}D^2} - e^{-\frac{1}{2}D_0^2}\| \leq \frac{\|A\|}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x| e^{-\frac{1}{2}x^2}\, dx = \sqrt{\frac{2}{\pi}}\|A\|.$$

For arbitrary t, we get $e^{-tD^2} = e^{-\frac{1}{2}(\sqrt{2t}D)^2}$ so we replace $\|A\|$ by $\sqrt{2t}\|A\|$ in the calculation for $t = \frac{1}{2}$. ∎

REMARK 19. In the case that A is only D_0-bounded and $D = D_0 + A$ is self-adjoint, the same proofs show that

$$\|\hat{f}(D) - \hat{f}(D_0)\|_{D_0} \leq \frac{\|A\|_{D_0}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |xf(x)|\, dx$$

under the hypotheses of Theorem 17 and, therefore

$$\|e^{-tD^2} - e^{-tD_0^2}\|_{D_0} \leq 2\sqrt{\frac{t}{\pi}}\|A\|_{D_0}.$$

REMARK 20. Let $\mathcal{A}$ be the C^*-subalgebra of $C_b(\mathbf{R})$ generated by the almost periodic functions and the functions having limits at $+\infty$ and $-\infty$. Let D_0 be an unbounded self-adjoint operator on H. Then for each fixed $f \in \mathcal{A}$ the mapping $A \longmapsto f(D_0+A)\colon \mathcal{B}(H)_{\mathrm{sa}} \longrightarrow \mathcal{B}(H)$ is continuous: to see this we first observe that the set of $f \in C_b(\mathbf{R})$ for which this holds is a C^*-algebra. Once this is established, Corollary 2, Theorem 8 and Proposition 12 together with the Stone-Weierstrass Theorem complete the proof.

By a similar argument, if $\{A_n\}$ is a sequence of D_0-bounded symmetric operators and $\|A_n\|_{D_0} \longrightarrow 0$, then

$$\|f(D_0 + A_n) - f(D_0)\|_{D_0} \longrightarrow 0 \quad \text{for all } f \in \mathcal{A}.$$

For f in the unitization of $C_0(\mathbf{R})$ we can prove the stronger result:

$$\|f(D_0 + A_n) - f(D_0)\| \longrightarrow 0$$

by this argument and an appeal to Corollary 2.

REMARK 21. If D_0 is an unbounded self-adjoint operator, then there exists a bounded continuous (even smooth) function f and a sequence $\{A_n\}$ of bounded self-adjoint operators with $\|A_n\| \longrightarrow 0$ and $\|f(D_0 + A_n) - f(D_0)\| \geq 1$ for all n.

We outline the proof for D_0 = "multiplication by n" on $\ell^2(\mathbf{Z})$: the general case is similar but messier. Let f be a continuous function on $\mathbf{R}$ with range in $[0,1]$ which is 0 at all integer points but so that $f(n+\frac{1}{n}) = 1$ for all positive integers $n \geq 2$. For $n \geq 2$ let $A_n = \frac{1}{n}P_n$ where P_n is the projection on the nth basis vector. Then $\|A_n\| = \frac{1}{n} \longrightarrow 0$ but $\|f(D_0 + A_n) - f(D_0)\| \geq 1$.

NOTE. Remarks 20 and 21 are carefully stated versions of results of Rellich [DS, Exercise XII.9.38]. The example given above shows that the result stated in [DS] is not quite correct. See also Remark 13.

Appendix B. The Trace-Norm Continuity of Certain Functions of Unbounded Self-Adjoint Operators. In this appendix, we prove continuity results of the following sort. We assume that D is an unbounded self-adjoint operator affiliated with the semifinite factor, N, and that $(1+D^2)^{-1}$ is *n-summable* $\Big(\mathrm{Tr}\big((1+D^2)^{-n}\big) < +\infty\Big)$. We then show (with explicit bounds) that the map $N_{\mathrm{sa}} \longrightarrow \mathcal{L}^1(N)$ given by $A \longmapsto [1+(D+A^2)]^{-n}$ is well-defined and continuous. We also prove D-bounded versions of these results.

The following lemma is certainly known, although we have not found a proof in the literature. We outline its proof for completeness.

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LEMMA 1. *If A and B are (unbounded) self-adjoint operators with* $\operatorname{dom} A = \operatorname{dom} B$ *and* $0 < c1 \leq A \leq B$ *on their common domain, then* $0 \leq B^{-1} \leq A^{-1} \leq \frac{1}{c}1$ *on all of H.*

PROOF. For $\xi \in \operatorname{dom} B$, $\theta(B^{\frac{1}{2}}\xi) = A^{\frac{1}{2}}\xi$ is well-defined and $\|\theta\| \leq 1$. Since the closure of $B^{\frac{1}{2}}|_{\operatorname{dom} B}$ is $B^{\frac{1}{2}}$ one checks that $\theta(B^{\frac{1}{2}}\xi) = A^{\frac{1}{2}}\xi$ makes sense for all $\xi \in \operatorname{dom} B^{\frac{1}{2}} \subseteq \operatorname{dom} A^{\frac{1}{2}}$ and so $\theta B^{\frac{1}{2}} = A^{\frac{1}{2}}$. Since $B^{\frac{1}{2}} \geq c^{\frac{1}{2}}1$, $\operatorname{range} B^{\frac{1}{2}} = H$ and θ is everywhere defined and $1:1$ and $\operatorname{range}\theta \supseteq \operatorname{range} A^{\frac{1}{2}} = H$. So, θ^{-1} is bounded. Thus, $B^{\frac{1}{2}} = \theta^{-1}A^{\frac{1}{2}}$ and so $B^{-\frac{1}{2}} = A^{-\frac{1}{2}}\theta$ or $B^{-\frac{1}{2}} = \theta^* A^{-\frac{1}{2}}$ and $\|\theta^*\| \leq 1$. This implies $B^{-1} \leq A^{-1}$. ∎

2. GENERALIZED SINGULAR VALUES. In order to prove our results in full generality we need the concept of *generalized singular values* due to Fack and Kosaki, [FK]. We let N be a fixed von Neumann algebra with faithful, normal, semifinite trace, Tr. If $A \in N$ we define for each $t > 0$, the *t-th singular value of A*, $\mu_t(A)$, by

$$\mu_t(A) = \inf\{\|AE\| \mid E \text{ is a projection in } N \text{ with } \operatorname{Tr}(1-E) \leq t\}.$$

LEMMA 3. *If N is a von Neumann algebra with faithful, normal, semifinite trace,* Tr*; $0 \leq A \leq B$ are self-adjoint operators in N; and g is a continuous, increasing function on $\mathbf{R}^+$ with $g(0) = 0$ then*

$$\operatorname{Tr}\big(g(A)\big) \leq \operatorname{Tr}\big(g(B)\big).$$

PROOF. This follows immediately from Lemma 2.5 and Proposition 2.7 of [FK]. ∎

COROLLARY 4. *With N as above and* $0 \leq A \leq B$ *in N and* $k > 0$ *then*

$$\operatorname{Tr}(A^k) \leq \operatorname{Tr}(B^k).$$

LEMMA 5. *Let N be a von Neumann algebra with faithful, normal, semifinite trace,* Tr. *If S, T in N are self-adjoint with S positive and $-S \leq T \leq S$, then for any continuous, increasing function g on $\mathbf{R}^+$ with $g(0) = 0$ we have*

$$\operatorname{Tr}\big(g(|T|)\big) \leq 2\operatorname{Tr}\big(g(S)\big).$$

PROOF. Let P_+ be the spectral projection for T corresponding to $\mathbf{R}^+$. Let $T_+ = P_+TP_+$ so that $T_+ \leq P_+SP_+$ and so by Lemma 2.5 of [FK], for each $t > 0$

$$\begin{aligned}\mu_t\big(g(T_+)\big) &= g\big(\mu_t(T_+)\big) \leq g\big(\mu_t(P_+SP_+)\big) \leq g\big(\mu_t(S)\big)\\ &= \mu_t\big(g(S)\big)\end{aligned}$$

and so by Proposition 2.7 of [FK]

$$\operatorname{Tr}\big(g(T_+)\big) \leq \operatorname{Tr}\big(g(S)\big).$$

Similarly,

$$\mathrm{Tr}\big(g(T_-)\big) \leq \mathrm{Tr}\big(g(S)\big)$$

where

$$T_- := (1 - P_+)(-T)(1 - P_+).$$

Since $|T| = T_+ + T_-$ is a direct sum, $g(|T|) = g(T_+) + g(T_-)$ and hence

$$\mathrm{Tr}\big(g(|T|)\big) \leq 2\,\mathrm{Tr}\big(g(S)\big). \qquad \blacksquare$$

LEMMA 6. *If D_0 is an unbounded self-adjoint operator, A is a bounded self-adjoint operator, and $D = D_0 + A$ then*

$$(1 + D^2)^{-1} \leq f(\|A\|)(1 + D_0^2)^{-1}$$

where $f(a) = 1 + \frac{1}{2}a^2 + \frac{1}{2}a\sqrt{a^2 + 4}$.

PROOF. We first assume that $A(\mathrm{dom}\, D_0) \subseteq \mathrm{dom}\, D_0$ so that $\mathrm{dom}\, D^2 = \mathrm{dom}\, D_0^2$. We seek a positive constant C so that:

$$1 + D_0^2 \leq C(1 + D^2) \quad \text{on } \mathrm{dom}\, D^2 = \mathrm{dom}\, D_0^2.$$

That is, for all vectors ξ of norm 1 in $\mathrm{dom}\, D_0^2$ we want:

$$\langle \xi, \xi \rangle + \langle D_0\xi, D_0\xi \rangle \leq C\big[\langle \xi, \xi \rangle + \langle D_0\xi, D_0\xi \rangle + \langle A\xi, A\xi \rangle + \langle D_0\xi, A\xi \rangle + \langle A\xi, D_0\xi \rangle\big]$$

or,

$$1 + \|D_0\xi\|^2 \leq C\big[1 + \|D_0\xi\|^2 + \|A\xi\|^2 + 2\,\mathrm{Re}\langle D_0\xi, A\xi \rangle\big]$$

which would follow from:

$$1 + \|D_0\xi\|^2 \leq C\big[1 + \|D_0\xi\|^2 + \|A\xi\|^2 - 2\|D_0\xi\|\,\|A\xi\|\big].$$

Letting $x = \|D_0\xi\|$ and $a = \|A\xi\|$ one easily calculates the maximum value of $\frac{1+x^2}{1+(x-a)^2}$ to be $f(a) = 1 + \frac{1}{2}a^2 + \frac{1}{2}a\sqrt{a^2 + 4}$. Since $a = \|A\xi\| \leq \|A\|$ and f is clearly increasing, we get

$$(1 + x^2) \leq f(\|A\|)\big(1 + (x - a)^2\big)$$

and so $(1 + D_0^2) \leq f(\|A\|)(1 + D^2)$ on $\mathrm{dom}(1 + D_0^2) = \mathrm{dom}(1 + D^2)$. By Lemma 1,

$$(1 + D^2)^{-1} \leq f(\|A\|)(1 + D_0^2)^{-1}.$$

To rid ourselves of the restrictive hypothesis that $A(\mathrm{dom}\, D_0) \subseteq \mathrm{dom}\, D_0$, we let E_k be the spectral projection of D_0 for the interval $[-k, k]$ and let $A_k = E_k A E_k$. Then A_k is self-adjoint, $\|A_k\| \leq \|A\|$, $A_k(\mathrm{dom}\, D_0) \subseteq \mathrm{dom}\, D_0$ and $A\xi = \lim_{k\to\infty} A_k\xi$ for all $\xi \in H$. If $D_k := (D_0 + A_k)$ then

$$(1 + D_k^2)^{-1} \leq f(\|A\|)(1 + D_0^2)^{-1}.$$

By a result of Sz.-Nagy [DS, Exercise XII.9.37],

$$(1 + D_k^2)^{-1}\xi \longrightarrow (1 + D^2)^{-1}\xi \quad \text{for all } \xi \in H$$

and so the inequality holds for all A. $\blacksquare$

LEMMA 7. *If D_0 is an unbounded self-adjoint operator, A is a symmetric D_0-bounded operator with $\|A\|_{D_0} < 1$, and $D = D_0 + A$ then*

$$(1 + D^2)^{-1} \le h(\|A\|_{D_0})(1 + D_0^2)^{-1}$$

where $h(a) = (1 - a)^{-2}$.

PROOF. We follow the proof of the previous lemma, with $\xi \in \operatorname{dom} D_0^2$, $\|\xi\|_{D_0} = 1$. So we want $C > 0$ satisfying:

$$1 \le C[1 + \|A\xi\|^2 - 2\|D_0\xi\| \, \|A\xi\|].$$

Since $\|D_0\xi\| < \|\xi\|_{D_0} = 1$ this will be satisfied if

$$1 \le C[1 + \|A\xi\|^2 - 2\|A\xi\|]$$

or

$$(1 - \|A\xi\|)^{-2} \le C.$$

Choosing $C = (1 - \|A\|_{D_0})^{-2}$ works, since $(1 - a)^{-2}$ is increasing for $a < 1$.

We rid ourselves of the restriction $A(\operatorname{dom} D_0) \subseteq \operatorname{dom} D_0$ in the same way: each $A_k = E_k A E_k$ is self-adjoint (bounded, in fact!), and it is easy to see $A_k\xi \longrightarrow A\xi$ for all $\xi \in \bigcup_{k=1}^{\infty} E_k(H)$ which is a core for $(D_0 + A)$. Hence, by the same result of Sz.-Nagy (using its full power) we obtain

$$(1 + D^2)^{-1} \le (1 - \|A\|_{D_0})^{-2}(1 + D_0^2)^{-1}. \qquad \blacksquare$$

COROLLARY 8. *Let N be as above, let D_0 be an unbounded self-adjoint operator affiliated with N, let $A \in N_{\mathrm{sa}}$ and let $D = D_0 + A$.*

(1) If $\operatorname{Tr}\big((1 + D_0^2)^{-n}\big) < +\infty$ for some positive n, then so is $\operatorname{Tr}\big((1 + D^2)^{-n}\big)$ and

$$\operatorname{Tr}\big((1 + D^2)^{-n}\big) \le f(\|A\|)^n \operatorname{Tr}\big((1 + D_0^2)^{-n}\big).$$

(2) If $\operatorname{Tr}(e^{-tD_0^2}) < +\infty$ for $t > 0$ then

$$\operatorname{Tr}(e^{-tD^2}) \le \exp\left(1 - \frac{1}{f(t^{\frac{1}{2}}\|A\|)}\right) \operatorname{Tr}\left(\exp\left(-\frac{t}{f(t^{\frac{1}{2}}\|A\|)} D_0^2\right)\right).$$

If we only assume that A is D_0-bounded and symmetric then we must also assume that $\|A\|_{D_0} < 1$ and $D = D_0 + A$ is affiliated with N. In this case we get:

(3) If $\operatorname{Tr}\big((1 + D_0^2)^{-n}\big) < +\infty$ for some positive n, then

$$\operatorname{Tr}\big((1 + D^2)^{-n}\big) \le \big(h(\|A\|_{D_0})\big)^n \operatorname{Tr}\big((1 + D_0^2)^{-n}\big).$$

(4) If $\operatorname{Tr}(e^{-tD_0^2}) < \infty$ for $t > 0$ then if $t \le 1$

$$\operatorname{Tr}(e^{-tD^2}) \le e^{\|A\|_{D_0}(2 - \|A\|_{D_0})} \operatorname{Tr}(e^{-(1 - \|A\|_{D_0})^2 t D_0^2})$$

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while for $t \geq 1$ we get

$$\mathrm{Tr}(e^{-tD^2}) \leq \mathrm{Tr}(e^{-D^2}) \leq e^{\|A\|_{D_0}(2-\|A\|_{D_0})}\,\mathrm{Tr}(e^{-(1-\|A\|_{D_0})^2 D_0^2})$$

by the previous line.

PROOF. (1) and (3) follow from Lemmas 6 and 7 and Corollary 4.

To see (2) and (4), we let

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-(\frac{1}{x}-1)} & \text{if } x > 0 \end{cases}$$

so that g is a continuous increasing function on $\mathbf{R}^+$ with $g(0) = 0$. Then, $e^{-tD^2} = g\big((1+tD^2)^{-1}\big)$ and applying Lemmas 6 and 7 yields:

$$(1+tD^2)^{-1} \leq f(t^{\frac{1}{2}}\|A\|)(1+tD_0^2)^{-1} \quad \text{in case (2);}$$

while

$$\begin{aligned}(1+tD^2)^{-1} &\leq h(\|t^{\frac{1}{2}}A\|_{t^{\frac{1}{2}}D_0})(1+tD_0^2)^{-1} \\ &\leq h(\|A\|_{D_0})(1+tD_0^2)^{-1} \quad \text{in case (4)}\end{aligned}$$

since for $s \leq 1$,

$$\|sA\|_{sD_0} \leq \|A\|_{D_0}.$$

A straightforward application of Lemma 3 and the fact that

$$g(cx) = e^{(1-\frac{1}{c})}[g(x)]^{\frac{1}{c}}$$

gives us (2) and (4). ∎

REMARK 9. (1) In the following propositions we will use the easily derived estimates:

$$\begin{cases} 1 \leq f(a) \leq 1+2a & \text{if } 0 \leq a \leq 1.5 \text{ and} \\ 1 \leq h(a) \leq 1+4a & \text{if } 0 \leq a \leq .35. \end{cases}$$

(2) We will also use the following estimates (with $x = f(a)$ or $h(a)$):

(i) $x^r - 1 \leq (x-1)^r$ for $x \geq 1$ and $0 < r \leq 1$

(ii) $(\frac{1}{x})^r - 1 \geq -(x-1)^r$ for $x \geq 1$ and $0 < r \leq 1$.

Inequality (ii) follows easily from (i) and (i) is proved using the usual calculus techniques after letting $(x-1)^r = b$ and $s = \frac{1}{r}$, to convert (i) into:

$$b^s + 1 \leq (b+1)^s \quad \text{for } b \geq 0,\ s \geq 1.$$

(3) In the following propositions we could also use the same techniques to get operator norm estimates of $\|(1+D^2)^{-r} - (1+D_0^2)^{-r}\|$. However, they would not be nearly as sharp as the estimates obtained in Appendix A.

(4) In the following propositions, if we were only interested in positive integers n, we could restrict to the case $r = 1$ and things would simplify somewhat. However, we need the greater generality.

PROPOSITION 10. *If D_0 is an unbounded self-adjoint operator affiliated with N and if* $\mathrm{Tr}\big((1+D_0^2)^{-n}\big) < \infty$ *for some $n \geq 1$ (not necessarily an integer), then for all A in N_{sa},* $\mathrm{Tr}\Big(\big(1+(D_0+A)^2\big)^{-n}\Big) < \infty$ *and if $0 < r \leq 1$ and $\|A\| \leq 1$ then*

$$\Big\|(1+D_0^2)^{-r} - \big(1+(D_0+A^2)\big)^{-r}\Big\|_{\frac{n}{r}} \leq 2^{\frac{r}{n}}(2\|A\|)^r\|(1+D_0^2)^{-1}\|_n^r.$$

If we only assume that A is a D_0-bounded symmetric operator with $D = D_0 + A$ also affiliated with N, then the inequality becomes:

$$\|(1+D_0^2)^{-r} - (1+D^2)^{-r}\|_{\frac{n}{r}} \leq 2^{\frac{r}{n}}(8\|A\|_{D_0})^r\|(1+D_0^2)^{-1}\|_n^r,$$

provided $\|A\|_{D_0} \leq .29$.

PROOF. By Lemma 6 we have

$$(1+D^2)^{-1} \leq f(\|A\|)(1+D_0^2)^{-1} \quad \text{and}$$
$$(1+D_0^2)^{-1} \leq f(\|-A\|)(1+D^2)^{-1}.$$

[Or, by Lemma 7, $\quad (1+D^2)^{-1} \leq h(\|A\|_{D_0})(1+D_0^2)^{-1} \quad$ and

$$\begin{aligned}(1+D_0^2)^{-1} &\leq h(\|-A\|_D)(1+D^2)^{-1}\\ &\leq h(2\|A\|_{D_0})(1+D^2)^{-1} \quad \text{if } A \text{ is } D_0\text{-bounded.}\end{aligned}$$]

Thus,

$$\frac{1}{f(\|A\|)}(1+D_0^2)^{-1} \leq (1+D^2)^{-1} \leq f(\|A\|)(1+D_0^2)^{-1}.$$

Hence, by operator monotonicity:

$$\left(\frac{1}{f(\|A\|)}\right)^r(1+D_0^2)^{-r} \leq (1+D^2)^{-r} \leq \big(f(\|A\|)\big)^r(1+D_0^2)^{-r}.$$

And so,

$$\left[\left(\frac{1}{f(\|a\|)}\right)^r - 1\right](1+D_0^2)^{-r} \leq (1+D^2)^{-r} - (1+D_0^2)^{-r} \leq \left[\Big(f(\|A\|)\Big)^r - 1\right](1+D_0^2)^{-r},$$

which, by the previous remarks, yields

$$-[f(\|A\|)-1]^r(1+D_0^2)^{-r} \leq (1+D^2)^{-r} - (1+D_0^2)^{-r} \leq [f(\|A\|)-1]^r(1+D_0^2)^{-r}.$$

So, if $\|A\| \leq 1.5$ we get:

$$-(2\|A\|)^r(1+D_0^2)^{-r} \leq (1+D^2)^{-r} - (1+D_0^2)^{-r} \leq (2\|A\|)^r(1+D_0^2)^{-r}.$$

(If A is only D_0-bounded and $\|A\|_{D_0} \leq .29$ we get:

$$\begin{aligned}-(8\|A\|_{D_0})^r(1+D_0^2)^{-r} &\leq (1+D^2)^{-r} - (1+D_0^2)^{-r}\\ &\leq (4\|A\|_{D_0})^r(1+D_0^2)^{-r}\\ &\leq (8\|A\|_{D_0})^r(1+D_0^2)^{-r}.\,)\end{aligned}$$

An application of Lemma 5 with $g(x) = x^{\frac{n}{r}}$ yields the result. ∎

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PROPOSITION 11. *With the hypotheses as in Proposition 10 and* $r = \frac{n}{\lceil n\rceil+1}$, *we get:*

$$\|(1+D_0^2)^{-n} - (1+D^2)^{-n}\|_1 \le 2^{\frac{r}{n}}(2\|A\|)^r[f(\|A\|)]^n(n+1)\|(1+D_0^2)^{-n}\|_1.$$

If A is only D_0*-bounded with* $\|A\|_{D_0} \le .29$ *we get:*

$$\|(1+D_0^2)^{-n} - (1+D^2)^{-n}\|_1 \le 2^{\frac{r}{n}}(8\|A\|_{D_0})^r[h(\|A\|_{D_0})]^n(n+1)\|(1+D_0^2)^{-n}\|_1.$$

PROOF. Let $k = \lceil n\rceil$, the greatest integer in n, and let $r = \frac{n}{k+1} < 1$. So,

$$\begin{aligned}(1+D^2)^{-n} - (1+D_0^2)^{-n} &= [(1+D^2)^{-r}]^{k+1} - [(1+D_0^2)^{-r}]^{k+1}\\ &= \sum_{j=0}^{k}(1+D^2)^{-rj}\big((1+D^2)^{-r} - (1+D_0^2)^{-r}\big)(1+D_0^2)^{-r(k-j)}.\end{aligned}$$

Applying the Hölder inequality [D] we get:

$$\begin{aligned}&\|(1+D^2)^{-n} - (1+D_0^2)^{-n}\|_1\\ &\quad\le \sum_{j=0}^{k}\|(1+D^2)^{-rj}\|_{\frac{n}{rj}}\|(1+D^2)^{-r} - (1+D_0^2)^{-r}\|_{\frac{n}{r}}\|(1+D_0^2)^{-r(k-j)}\|_{\frac{n}{r(k-j)}}\\ &\quad\le \sum_{j=0}^{k}\|(1+D^2)^{-1}\|_n^{rj}2^{\frac{r}{n}}(2\|A\|^r)\|(1+D_0^2)^{-1}\|_n^{r}\|(1+D_0^2)^{-1}\|_n^{r(k-j)}\\ &\quad\le \sum_{j=0}^{k}f(\|A\|)^{rj}\|(1+D_0^2)^{-1}\|_n^{rj}2^{\frac{r}{n}}(2\|A\|^r)\|(1+D_0^2)^{-1}\|_n^{r(k-j+1)}\\ &\quad= 2^{\frac{r}{n}}(2\|A\|)^r\|(1+D_0^2)^{-1}\|_n^{n}\sum_{j=0}^{k}f(\|A\|)^{rj}\\ &\quad\le 2^{\frac{r}{n}}(2\|A\|)^r\|(1+D_0^2)^{-n}\|_1(k+1)f(\|A\|)^{rk}\\ &\quad\le 2^{\frac{r}{n}}(2\|A\|)^r\|(1+D_0^2)^{-n}\|_1(n+1)\big(f(\|A\|)\big)^n.\end{aligned}$$

The proof of the D_0-bounded version is similar. ∎

At this point we are in a position to prove the trace-class continuity of the map $A \longmapsto e^{-(D_0+A)^2}\colon N_{sa} \longrightarrow L^1(N)$ assuming that D_0 is θ*-summable* (*i.e.*, $\mathrm{Tr}(e^{-tD_0^2}) < \infty$ for all $t > 0$). We can also prove a D_0-bounded version of this result. However, in order to keep this paper to a reasonable length we leave these results to the sequel on θ-summable Fredholm modules and spectral flow where they will be directly useful.

REMARK 12. Motivated by our work, F. A. Sukochev has generalized and improved some of our estimates by other methods [Su].

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Appendix C. Examples. To illustrate the theory we present some nontrivial examples.

EXAMPLE I. Let $A = C(\mathbf{T})$, the C^*-algebra of continuous functions on the unit circle, $\mathbf{T}$, and let $N = B\big(L^2(\mathbf{T})\big)$, the (type I) von Neumann factor of all bounded operators on the Hilbert space, $L^2(\mathbf{T})$. We represent A as multiplication operators on $L^2(\mathbf{T})$ so that $f \longmapsto M_f : A \longrightarrow N$ is faithful. We let $D = \frac{1}{i}\frac{d}{dt}$ be the unique self-adjoint unbounded operator on $L^2(\mathbf{T})$ which is diagonal relative to the orthonormal basis,

$$\left\{\frac{1}{\sqrt{2\pi}}e^{int}\right\}_{n\in\mathbf{Z}}.$$

If f is a continuously differentiable function in A, then one easily calculates that

$$[D, M_f] = \frac{1}{i}M_{f'}$$

so that axiom 2 of Definition 2.1 holds. Using the orthonormal basis which diagonalizes D, we easily calculate that

$$\mathrm{Tr}\Big((1+D^2)^{-\frac{p}{2}}\Big) = \sum_{n\in\mathbf{Z}}\Big(\frac{1}{1+n^2}\Big)^{\frac{p}{2}},$$

which is finite for any $p > 1$. Thus, (N, D) is a p-summable Fredholm module for $A = C(\mathbf{T})$ for any $p > 1$. We take $p = 2$ in the following calculations to be definite.

We let $u \in U(A)$ be the function $u(t) = e^{-it}$, so that (suppressing the representation M) we have

$$uDu^* = D + u[D, u^*] = D + u\frac{1}{i}(u^*)' = D + 1.$$

Thus, the straight line path from D to uDu^* is $D_t^u = D + t1$ for $t \in [0, 1]$. As t increases from 0 to 1, the *eigenvectors* of the operators D_t^u remain the same, but the *eigenvalues* each increase by 1. Exactly one eigenvalue changes from negative to nonnegative so that $\mathrm{sf}(\{D_t^u\}) = +1$. In this setting, $P = \chi(D)$ is the projection of $L^2(\mathbf{T})$ onto $H^2(\mathbf{T})$ and $T_u = PuP$ is the classical Toeplitz operator corresponding to the backward shift so that $\mathrm{ind}(T_u) = +1$ also.

Letting $k = \frac{p}{2} = \frac{2}{2} = 1$ we have

$$\tilde{C}_k := \int_{-\infty}^{\infty}(1+x^2)^{-k}\,dx = \int_{-\infty}^{\infty}(1+x^2)^{-1}\,dx = \pi.$$

Thus,

$$\begin{aligned}
&\frac{1}{\tilde{C}_k}\int_0^1 \mathrm{Tr}\Big(\frac{d}{dt}(D_t^u)\big(1+(D_t^u)^2\big)^{-k}\Big)\,dt\\
&\quad = \frac{1}{\pi}\int_0^1 \mathrm{Tr}\Big(1\cdot\big(1+(D+t1)^2\big)^{-1}\Big)\,dt = \frac{1}{\pi}\int_0^1\Big(\sum_{n\in\mathbf{Z}}\frac{1}{1+(n+t)^2}\Big)\,dt\\
&\quad = \frac{1}{\pi}\sum_{n\in\mathbf{Z}}\int_0^1\frac{1}{1+(n+t)^2}\,dt = \frac{1}{\pi}\sum_{n\in\mathbf{Z}}\int_n^{n+1}\frac{1}{1+u^2}\,du = +1.
\end{aligned}$$

Thus, we have verified that, in this case,

$$\begin{aligned}\operatorname{ind}(PuP) &= \operatorname{sf}(\{D_t^u\}) \\ &= \frac{1}{\pi}\int_0^1 \operatorname{Tr}\Big(\frac{d}{dt}(D_t^u)\big(1+(D_t^u)^2\big)^{-1}\Big)\,dt = 1.\end{aligned}$$

Theorem 2.17 (with $p = 1+\epsilon$) would require us to use the exponent $m = 1+\frac{3}{2} = 2.5$ and the constant,

$$\tilde{C}_m = \int_{-\infty}^{\infty} (1+x^2)^{-2.5}\,dx = \frac{4}{3}.$$

We fully believe that this exponent m is only an artifact of the proof, and furthermore that in general one need only use the minimal exponent for which the integral formula converges, namely $k = \frac{p}{2}$.

EXAMPLE II. Let $A = C(\mathbf{T}^2)$, the C^*-algebra of continuous functions on the torus, and let A act as multiplication operators on $H = L^2(\mathbf{T}^2)$. We let $\alpha\colon \mathbf{R} \longrightarrow \operatorname{Aut}(A)$ be the Kronecker flow on A determined by the irrational number, θ. That is, for $s \in \mathbf{R}$, $f \in A$, and $(z_1, z_2) \in \mathbf{T}^2$ we have:

$$(\alpha_s f)(z_1, z_2) = f(e^{-2\pi i s} z_1, e^{-2\pi i\theta s} z_2).$$

Now, the C^*-crossed product $A \rtimes_\alpha \mathbf{R}$ acts on $L^2(\mathbf{R}, H)$ as follows: for

$$s, t \in \mathbf{R}, \quad \xi \in L^2(\mathbf{R}, H) \quad \text{and} \quad f \in A$$

we define

$$\begin{aligned}\big(\pi(f)\xi\big)(s) &= \alpha_s^{-1}(f)\cdot\xi(s) \quad \text{and} \\ \big(\lambda(t)\xi\big)(s) &= \xi(s-t).\end{aligned}$$

Thus, $\pi \times \lambda$ is a faithful representation of $A \rtimes_\alpha \mathbf{R}$ on $L^2(\mathbf{R}, H)$. It is well-known that $N = \big(\pi \times \lambda(A \rtimes_\alpha \mathbf{R})\big)''$ is a II_∞ factor, [CMX], and so we have $\pi\colon A \longrightarrow N$. We let

$$D = \frac{1}{2\pi i}\frac{d}{ds},$$

the usual generator of the one-parameter unitary group $\lambda\colon \mathbf{R} \longrightarrow N$, so that D is affiliated with N. Now, if δ is the densely defined (unbounded) derivation on A generating the representation $\alpha\colon \mathbf{R} \longrightarrow \operatorname{Aut}(A)$ and $f \in A$ is a smooth element for δ then

$$\pi\big(\delta(f)\big) = 2\pi i[D, \pi(f)]$$

by [L] so that axiom (2) of Definition 2.1 holds.

Now, D is really

$$\frac{1}{2\pi i}\frac{d}{ds} \otimes 1$$

and $\lambda(t)$ is really

$$\lambda(t) \otimes 1$$

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on $L^2(\mathbf{R})\otimes H$. Since the trace on A (given by integration) is finite, we have that the trace on N restricts to the usual trace on $\lambda(\mathbf{R})''\otimes 1$: that is, $x\otimes 1$ in $\lambda(\mathbf{R})''\otimes 1$ is trace class if and only if $x=\lambda(\hat{g})$ for some $g\in L^1(\mathbf{R})\cap L^\infty(\mathbf{R})$ and then

$$\mathrm{Tr}\Big(\lambda(\hat{g})\otimes 1\Big)=\int_{-\infty}^{\infty} g(r)\,dr.$$

In this Fourier Transform picture, D becomes multiplication by the independent variable, r. Hence, $(1+D^2)^{-1}$ becomes multiplication by the function $g(r)=(1+r^2)^{-1}$ which is in $L^1(\mathbf{R})\cap L^\infty(\mathbf{R})$. That is,

$$(1+D^2)^{-1}=\lambda(\hat{g})\otimes 1.$$

Therefore if $p>1$, then

$$\mathrm{Tr}\Big((1+D^2)^{-\frac{p}{2}}\Big)=\int_{-\infty}^{\infty}(1+r^2)^{-\frac{p}{2}}\,dr<+\infty.$$

That is, (N,D) is a p-summable (type II_∞) unbounded Breuer-Fredholm module for A for any $p>1$. In particular, (N,D) is 2-summable and

$$\mathrm{Tr}\Big((1+D^2)^{-1}\Big)=\int_{-\infty}^{\infty}(1+r^2)^{-1}\,dr=\pi.$$

We let $u\in U(A)$ be the function $u(z_1,z_2)=z_2$ which is easily seen to be a smooth element for the derivation δ and that $\delta(u^*)=2\pi i\theta u^*$. This implies that

$$[D,\pi(u^*)]=\frac{1}{2\pi i}\pi\Big(\delta(u^*)\Big)=\theta\pi(u^*).$$

Now, suppressing the representation π we get:

$$uDu^*=D+u[D,u^*]=D+u(\theta u^*)=D+\theta 1.$$

Thus, the straight line path from D to uDu^* is $D_t^u=D+t\theta 1$ for $t\in[0,1]$. As t increases from 0 to 1, the spectral *subspaces* of the operators D_t^u remain the same, but the spectral *values* each increase by θ. The spectral subspace of D corresponding to the interval $[-\theta,0)$, $E=E_{[-\theta,0)}$, is exactly the subspace where the spectral values change from negative to nonnegative. By a calculation very similar to Example 2.6 of [P2], the spectral flow of the path $\{D_t^u\}$ is exactly $\mathrm{Tr}(E)$ and since $E=\lambda(\hat{g})\otimes 1$ where $g=\chi_{[-\theta,0)}$ we have

$$\mathrm{sf}(\{D_t^u\})=\mathrm{Tr}(E)=\int_{-\infty}^{\infty}\chi_{[-\theta,0)}(r)\,dr=\theta.$$

It is also easy to show directly that θ is the Breuer index of the "Toeplitz" operator $T_u:=PuP$ (where $P=\chi_{\mathbf{R}^+}(D)$) computed in the II_∞ factor, PNP. Finally,

$$\begin{aligned}\frac{1}{\pi}\int_0^1\mathrm{Tr}\Big(\frac{d}{dt}(D_t^u)\big(1+(D_t^u)^2\big)^{-1}\Big)\,dt &= \frac{1}{\pi}\int_0^1\mathrm{Tr}\Big(\theta\big(1+(D+t\theta)^2\big)^{-1}\Big)\,dt\\ &=\frac{\theta}{\pi}\int_0^1\Big(\int_{-\infty}^{\infty}\frac{1}{1+(r+t\theta)^2}\,dr\Big)\,dt\quad.\\ &=\frac{\theta}{\pi}\int_0^1\Big(\int_{-\infty}^{\infty}\frac{1}{1+u^2}\,du\Big)\,dt=\theta\end{aligned}$$

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Hence, we have verified in the example that:

$$\begin{aligned}\operatorname{ind}(PuP) &= \operatorname{sf}(\{D_t^u\}) \\ &= \frac{1}{\pi}\int_0^1 \operatorname{Tr}\Big(\frac{d}{dt}(D_t^u)\big(1+(D_t^u)^2\big)^{-1}\Big)\,dt = \theta.\end{aligned}$$

As mentioned in Example I, Theorem 2.17 would require us to use the exponent $m = 2.5$ (or 3.5 or $4.5\cdots$) to get:

$$\operatorname{ind}(PuP) = \operatorname{sf}(\{D_t^u\}) = \frac{3}{4}\int_0^1 \operatorname{Tr}\Big(\frac{d}{dt}(D_t^u)\big(1+(D_t^u)^2\big)^{-2.5}\Big)\,dt = \theta.$$

These examples serve to illustrate our conjecture that one need only use the minimal exponent necessary (namely $\frac{p}{2}$) in the integral formula for the spectral flow.

For more general examples of this type (*i.e.*, given by action of $\mathbf{R}$) see [L] and [PR] where the index formula

$$\operatorname{ind}(PuP) = \frac{-1}{2\pi i}\tau\big(\delta(u)u^*\big)$$

is proved (here τ is an $\mathbf{R}$-invariant trace on the C^*-algebra, A).

Higher dimensional examples (larger p) can be constructed from Dirac operators on Riemannian manifolds, but we do not present the construction here.

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