

SOME REMARKS ON COMPLEX LIE GROUPS

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Abstract. First we show that any complex Lie group is complete Kähler. Moreover we obtain a plurisubharmonic exhaustion function on a complex Lie group as follows. Let \mathfrak{k} the real Lie algebra of a maximal compact real Lie subgroup K of a complex Lie group G . Put $q := \dim_{\mathbb{C}} \mathfrak{k} \cap \sqrt{-1}\mathfrak{k}$. Then we obtain that there exists a plurisubharmonic, strongly $(q + 1)$ -pseudoconvex in the sense of Andreotti-Grauert and K -invariant exhaustion function on G .

§1. Introduction

To get our aim of this paper, we may assume that every complex Lie group is always connected throughout this paper.

Let G be a complex Lie group of complex dimension n and \mathfrak{k} the real Lie algebra of a maximal compact real Lie subgroup K of G . Put $q := \dim_{\mathbb{C}} \mathfrak{k} \cap \sqrt{-1}\mathfrak{k}$. Since all maximal compact subgroups are conjugate each other ([2]), the integer q is independent of a choice of a maximal compact subgroup. Concerning the pseudoconvexity of complex Lie groups, we have the following theorem which was proved in [10] and, partially, in [3, 4].

THEOREM 1. *There exists a C^∞ plurisubharmonic function φ on G satisfying (1) and (2).*

- (1) *The Levi form of φ is positive semidefinite and has $n - q$ positive eigenvalues at every point of G , in other words, φ is plurisubharmonic and strongly $(q + 1)$ -pseudoconvex on G in the sense of [1].*
- (2) *φ is an exhaustion function on G , i.e., for any $c \in \mathbb{R}$.*

$$\{x \in G \mid \varphi(x) < c\} \subset\subset G.$$

On the Stein group $GL(n, \mathbb{C})$ there exists a natural strongly plurisubharmonic exhaustion function

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$$\Phi(a) := \text{trace}(a a^*) + \frac{1}{\det(a a^*)}$$

($a \in GL(n, \mathbb{C})$). This function Φ is invariant with respect to the left and right actions of the unitary subgroup $U(n) := \{a \mid a \in GL(n, \mathbb{C}), a^{-1} = a^*\}$ which is a maximal compact real Lie subgroup of $GL(n, \mathbb{C})$, i.e., for $a \in GL(n, \mathbb{C})$ and $x, y \in U(n)$,

$$\Phi(xay) = \Phi(a).$$

The purpose of this paper is to consider whether or not there exists a plurisubharmonic and strongly $(q+1)$ -pseudoconvex exhaustion function in the sense of [1] on G which is invariant on a given maximal compact real Lie subgroup K and to show, as it's application, the existence of complete Kähler metric for every complex Lie group.

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§2. Linear groups and abelian Lie groups

A closed complex Lie subgroup of the complex general linear group $GL(k, \mathbb{C})$ for some positive integer k is said to be a closed complex linear group.

LEMMA 1. *Let G be a closed complex linear group and K a maximal compact subgroup of G . Then there exists a C^∞ function*

$$\varphi : G \longrightarrow \mathbb{R}$$

such that

- (1) φ is strongly plurisubharmonic,
- (2) φ is an exhaustion function on G , i.e., for any $c \in \mathbb{R}$.

$$\{x \in G \mid \varphi(x) < c\} \subset\subset G,$$

- (3) φ is K -invariant, that is,

$$\varphi(x) = \varphi(yxz)$$

for any $x \in G, y, z \in K$.

Proof. Let G be a closed complex Lie subgroup of the complex general linear group $GL(k, \mathbb{C})$. There exists a maximal compact subgroup K_1 of $GL(k, \mathbb{C})$ such that

$$K \subset K_1.$$

Since all maximal compact subgroups are conjugate each other ([2]), one can find $a \in GL(k, \mathbb{C})$ so that

$$a K_1 a^{-1} = U(k),$$

where $U(k)$ is the unitary subgroup of $GL(k, \mathbb{C})$. Taking a function

$$\varphi(x) := \Phi(axa^{-1}),$$

where $\Phi(x) = \text{trace}(x x^*) + \frac{1}{\det(x x^*)}$, we get the assertion of this lemma.

In the case of complex abelian Lie groups we obtain a similar result as Lemma 1.

LEMMA 2. *Let G be a complex abelian Lie group of complex dimension n and K a maximal compact subgroup of G . Then there exists a K -invariant C^∞ function*

$$\varphi : G \longrightarrow \mathbb{R}$$

satisfying the same statements (1) and (2) in Theorem 1.

Proof. Let e be the unit element of the complex abelian Lie group G . Put

$$G^0 := \{x \mid f(x) = f(e) \text{ for every holomorphic function } f \text{ on } G\}$$

From the result of [7] G^0 is a complex Lie subgroup of G which is a toroidal group, that is, a connected complex Lie group without non-constant holomorphic functions and there exists a lattice Γ of \mathbb{C}^m such that $G^0 \cong \mathbb{C}^m / \Gamma$, where m is the complex dimension of G^0 . We may assume $\Gamma = \mathbb{Z}\{e_1, e_2, \dots, e_m, v_1, v_2, \dots, v_{q_0}\}$, where e_i is the i -th standard unit vector of \mathbb{C}^m and $e_1, e_2, \dots, e_m, v_1, v_2, \dots, v_{q_0}$ are \mathbb{R} -linearly independent, and $K^0 \cong \mathbb{R}\{e_1, e_2, \dots, e_m, v_1, v_2, \dots, v_{q_0}\} / \Gamma$. Let $v_i = (v_{i1}, v_{i2}, \dots, v_{im})$ and $v_i = \text{Re } v_i + \sqrt{-1} \text{Im } v_i$ ($\text{Re } v_i, \text{Im } v_i \in \mathbb{R}^m$). Since $e_1, e_2, \dots, e_m, v_1, v_2, \dots, v_{q_0}$ are \mathbb{R} -linearly independent, $\text{Im } v_1, \dots, \text{Im } v_{q_0}$ are \mathbb{R} -linearly independent. Without loss of generality we may assume the $q_0 \times q_0$ real matrix

$(v_{ij} ; 1 \leq i, j \leq q_0)$ is invertible. We put $v_{q_0+1} := \sqrt{-1}e_{q_0+1}, v_{q_0+2} := \sqrt{-1}e_{q_0+2}, \dots, v_m := \sqrt{-1}e_m$. Then $e_1, e_2, \dots, e_m, v_1, v_2, \dots, v_{q_0}, v_{q_0+1}, \dots, v_m$ are \mathbb{R} -linearly independent vectors of \mathbb{C}^m . For any $z = (z_1, z_2, \dots, z_m)$ there exists a unique vector $(t_1, t_2, \dots, t_{2m}) \in \mathbb{R}^{2m}$ such that

$$z = \sum_{i=1}^m t_i e_i + \sum_{i=1}^{q_0} t_{m+i} \sqrt{-1} \operatorname{Im} v_i + \sum_{i=q_0+1}^m t_{m+i} v_i.$$

We define the exhaustion function

$$\psi : \mathbb{C}^m / \Gamma \ni z + \Gamma \mapsto \sum_{i=q_0+1}^m t_{m+i}^2 \in \mathbb{R}$$

on \mathbb{C}^m / Γ . We denote by $A = (a_{ij})$ the inverse matrix of

$$\begin{pmatrix} \operatorname{Im} v_1 \\ \cdot \\ \cdot \\ \cdot \\ \operatorname{Im} v_{q_0} \\ e_{q_0+1} \\ \cdot \\ \cdot \\ \cdot \\ e_m \end{pmatrix}.$$

Let $x_i := \operatorname{Re} z_i$ and $y_i := \operatorname{Im} z_i$. Then we have $t_{m+i} = \sum_{k=1}^m y_k a_{ki}$ ($i \geq q_0 + 1$). The Levi form of ψ is given by

$$\left(\frac{\partial^2 \psi(z + \Gamma)}{\partial z_i \partial \bar{z}_j} \right) = \frac{1}{4} \left(\frac{\partial^2 \psi}{\partial y_i \partial y_j} \right) = \frac{1}{2} B B^t,$$

where B is the matrix

$$\begin{pmatrix} 0 & \dots & 0 & a_{1 \ q_0+1} & \dots & a_{1 \ m} \\ 0 & & 0 & a_{2 \ q_0+1} & \dots & a_{2 \ m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & a_{m \ q_0+1} & \dots & a_{m \ m} \end{pmatrix}.$$

Since B is a real (m, m) -matrix of rank $m - q_0$, $\frac{1}{2}B B^t$ is positive semi-definite with $m - q_0$ positive eigenvalues. By the definition of ψ we can see that $\psi(z + z^* + \Gamma) = \psi(z + \Gamma)$ for any $z \in \mathbb{C}^m$ and $z^* \in K^0$. This means ψ is K^0 -invariant. By the result of [8] $G \cong G^0 \times \mathbb{C}^{*p} \times \mathbb{C}^r$ for some non-negative integers p and r with $m + p + r = n$. Then we may assume $G = G^0 \times \mathbb{C}^{*p} \times \mathbb{C}^r$. We take the function

$$\varphi(z + \Gamma, \xi, \eta) := \psi(z + \Gamma) + \sum_{i=1}^p \left(\frac{1}{|\xi_i|^2} + |\xi_i|^2 \right) + \sum_{j=1}^r |\eta_j|^2$$

for $(z + \Gamma, \xi, \eta) \in G^0 \times \mathbb{C}^{*p} \times \mathbb{C}^r$, where $\xi = (\xi_1, \dots, \xi_p) \in \mathbb{C}^{*p}$ and $\eta = (\eta_1, \dots, \eta_r) \in \mathbb{C}^r$. Since G is abelian, G has the unique maximal compact subgroup $K = K^0 \times \{\xi; |\xi_i| = 1\} \times 0 \subset G^0 \times \mathbb{C}^{*p} \times \mathbb{C}^r$. It is easy to show that φ is K -invariant and the Levi form of φ has $n - q_0$ positive eigenvalues at every point of G . Let \mathfrak{k} be the Lie algebra of K . Then

$$\mathfrak{k} = \mathbb{R}\{e_1, e_2, \dots, e_m, v_1, v_2, \dots, v_{q_0}\} \oplus \mathbb{R}^q \oplus 0$$

that is a real Lie subalgebra of the Lie algebra $\mathbb{C}^m \times \mathbb{C}^p \times \mathbb{C}^r$ of G .

Hence $\mathfrak{k} \cap \sqrt{-1}\mathfrak{k} = \mathbb{R}\{e_1, e_2, \dots, e_m, v_1, v_2, \dots, v_{q_0}\} \cap \mathbb{R}\{\sqrt{-1}e_1, \sqrt{-1}e_2, \dots, \sqrt{-1}e_m, \sqrt{-1}v_1, \sqrt{-1}v_2, \dots, \sqrt{-1}v_{q_0}\} = \mathbb{C}\{\text{Im } v_1, \dots, \text{Im } v_{q_0}\}$. Then we have

$$q = \dim_{\mathbb{C}} \mathfrak{k} \cap \sqrt{-1}\mathfrak{k} = q_0.$$

§3. Complete Kähler metric on a complex Lie group

Prof. Hiroshi Yamaguchi proposed the following question.

QUESTION. *Is any complex Lie group a Kähler manifold?*

In this section we will give the affirmative answer to this question, that is, we will obtain the following theorem.

THEOREM 2. *Any complex Lie group is complete Kähler.*

Proof. Let \mathfrak{g} be the Lie algebra of G and $K_{\mathbb{C}}$ the complex Lie subgroup with the Lie subalgebra $\mathfrak{k}_{\mathbb{C}} := \mathfrak{k} + \sqrt{-1}\mathfrak{k}$. Then $K_{\mathbb{C}}$ is a closed complex Lie subgroup of G ([5]). Put $a := \dim_{\mathbb{C}} G/K_{\mathbb{C}}$. Then G is biholomorphic onto $K_{\mathbb{C}} \times \mathbb{C}^a$ ([5]). From this fact we may assume $G = K_{\mathbb{C}}$. By the result of [5] there exist closed connected complex Lie subgroups S and Z such that

- (1) S is semi-simple,
- (2) Z is the connected center of G ,
- (3)

$$\rho : Z \times S \ni (x, y) \mapsto xy \in G$$

is a finite covering homomorphism.

Z is a connected complex abelian Lie group and then is isomorphic onto \mathbb{C}^p/Γ for some discrete subgroup Γ of \mathbb{C}^p . Let (z_1, \dots, z_p) be the canonical coordinate system of \mathbb{C}^p . The complete Kähler metric

$$\sum_{i=1}^p dz_i d\bar{z}_i$$

induces a complete Kähler metric on Z that is Γ -invariant. Since $\text{Ker } \rho$ is a finite subgroup of $Z \times S$, there exist maximal compact subgroups K_Z and K_S of Z and S , respectively such that $\text{Ker } \rho \subset K_Z \times K_S$. Since S is semi-simple, S is isomorphic onto a complex linear group and further we may assume S is a closed complex Lie subgroup of $GL(k, \mathbb{C})$ for some positive integer k ([6]). By Lemma 1 there exists a C^∞ strongly plurisubharmonic exhaustion function

$$\varphi : S \longrightarrow \mathbb{R}$$

that is K_S -invariant. The form

$$\sum_{i=1}^p dz_i d\bar{z}_i + \sum_{j,\ell=1}^s \frac{\partial^2 \varphi}{\partial w_j \partial \bar{w}_\ell} dw_j d\bar{w}_\ell$$

is $\text{Ker } \rho$ -invariant and then induces a Kähler metric on $G \cong Z \times S/\text{Ker } \rho$, where $s := \dim_{\mathbb{C}} S$ and (w_1, \dots, w_s) is a local coordinate system of S . From the technique of Nakano (Proposition 1, [9]), we can find a strictly convex increasing C^∞ function

$$\chi : (0, \infty) \longrightarrow (0, \infty)$$

such that the Kähler metric

$$\sum_{i=1}^p dz_i d\bar{z}_i + \sum_{j,\ell=1}^s \frac{\partial^2 \chi(\varphi)}{\partial w_j \partial \bar{w}_\ell} dw_j d\bar{w}_\ell$$

is complete.

§4. Invariant plurisubharmonic exhaustion functions

LEMMA 3. Let K be a compact topological space with a positive measure m and $a_{ij}(x) : K \rightarrow \mathbb{C}$ be continuous functions for $1 \leq i, j \leq n$. If $n \times n$ matrix

$$A(x) := (a_{ij}(x))$$

is positive semidefinite Hermitian and has $n - q$ positive eigenvalues at any point x of K , then

$$B := \left(\int_K a_{ij}(x) dm \right)$$

is positive semidefinite and has at least $n - q$ positive eigen values.

Proof. We put $w := {}^t(w_1, \dots, w_n)$ for $(w_1, \dots, w_n) \in \mathbb{C}^n$. Since

$$(B w, w) := \sum_{i,j=1}^n w_j \int_K a_{ij}(x) dm \bar{w}_i = \int_K \sum_{i,j=1}^n w_j a_{ij}(x) \bar{w}_i dm \geq 0,$$

the matrix B is positive semidefinite Hermitian. Then there exists a unitary matrix U such that $B = {}^t\bar{U} \Lambda U$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \lambda_n \end{pmatrix}$$

and λ_i are non-negative eigenvalues of B . We get a positive semidefinite Hermitian matrix

$$\sqrt{B} := {}^t\bar{U} \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \sqrt{\lambda_n} \end{pmatrix} U.$$

If $(B w, w) = 0$, then $((\sqrt{B})^2 w, w) = (\sqrt{B} w, \sqrt{B} w) = 0$ and hence $\sqrt{B} w = 0$. $\sqrt{B} w = 0$ implies $B w = (\sqrt{B})(\sqrt{B}) w = 0$. This shows that

$$B w = 0$$

if and only if

$$(B w, w) = \sum_{i,j=1}^n w_j \int_K a_{ij}(x) dm \overline{w_i} = 0.$$

We consider the eigen space

$$E_0 := \{ w \mid B w = 0 \} = \{ w \mid (B w, w) = 0 \}$$

of the eigenvalue 0 of B . Since

$$(B w, w) = \int_K \sum_{i,j=1}^n w_i a_{ij}(x) \overline{w_j} d m = 0,$$

we have $\sum_{i,j=1}^n w_j a_{ij}(x) \overline{w_i} = 0$ for any $x \in K$. Then we obtain

$$E_0 \subset \cap_{x \in K} \{ w \mid A(x) w = 0 \} \subset \{ w \mid A(x_0) w = 0 \}$$

for any point $x_0 \in K$. From the assumption it follows that

$$\dim_C \{ w \mid A(x_0) w = 0 \} \leq q$$

and then $\dim_C E_0 \leq q$.

The following Theorem 3 implies Theorem 1. Moreover, using a different method from those given by [3], [4] and S. Takeuchi [10], we prove Theorem 3 here.

THEOREM 3. *Let G be a complex Lie group, K a maximal compact subgroup of G . Put $q := \dim_C \mathfrak{k} \cap \sqrt{-1}\mathfrak{k}$. Then there exists a C^∞ function*

$$\varphi : G \longrightarrow \mathbb{R}$$

such that

- (1) φ is plurisubharmonic and strongly $(q + 1)$ -pseudoconvex on G in the sense of [1] ,
- (2) φ is an exhaustion function on G , i.e., for any $c \in \mathbb{R}$.

$$\{ z \in G \mid \varphi(z) < c \} \subset\subset G,$$

(3) φ is K -invariant, that is,

$$\varphi(z) = \varphi(a z b)$$

for any $z \in G, a, b \in K$.

Proof. We obtain closed connected Lie subgroups K_C, S, Z, K_S, K_Z and a finite covering homomorphism

$$\rho : Z \times S \ni (x, y) \mapsto x y \in K_C$$

and G is biholomorphic onto $K_C \times \mathbb{C}^a$ as in the proof of Theorem 2. Put $n := \dim G, n_1 := \dim Z$ and $n_2 := \dim S$ ($n = n_1 + n_2 + a$). Since Z is abelian, we have isomorphisms

$$Z \cong G^0 \times \mathbb{C}^{*p} \times \mathbb{C}^r$$

and $G^0 \cong \mathbb{C}^m / \Gamma$ ($n_1 = m + p + r$), where we put $G^0 := \{x \mid f(x) = f(e) \text{ for every holomorphic function } f \text{ on } Z\}$ and $\Gamma = \mathbb{Z}\{e_1, e_2, \dots, e_m, v_1, v_2, \dots, v_{q_0}\}$, using the notations in the proof of Lemma 2. By Lemmas 1 and 2 we obtain a strongly plurisubharmonic exhaustion function $\varphi_S : S \rightarrow \mathbb{R}$ and a plurisubharmonic exhaustion function $\varphi_Z : Z \rightarrow \mathbb{R}$, where the Levi form of φ_Z has $n_1 - q_0$ positive eigen values at every point. Since φ_S and φ_Z are K_S - and K_Z -invariant, respectively and $\text{Ker } \rho \subset K_Z \times K_S$, they induce a plurisubharmonic exhaustion function

$$\varphi_{K_C} := \varphi_Z + \varphi_S : K_C \rightarrow \mathbb{R}.$$

The Levi form of φ_{K_C} has $n - a - q_0$ positive eigenvalues at every point. We take a strongly plurisubharmonic exhaustion function

$$\varphi_a : \mathbb{C}^a \ni (w_1, \dots, w_a) \mapsto \sum_{i=1}^a |w_i|^2 \in \mathbb{R}.$$

We put

$$\varphi_G := \varphi_{K_C} + \varphi_a : G \rightarrow \mathbb{R}.$$

Then φ_G is a plurisubharmonic exhaustion function whose Levi form has $n - q_0$ positive eigenvalues at every points. Since the maximal compact

subgroup K is a Lie group, we can obtain positive left (or right) invariant Haar measure μ_ℓ (or μ_r) on K , respectively. Finally we obtain a function

$$\varphi(z) := \int_{x \in K} \int_{y \in K} \varphi_G(y z x) d \mu_\ell(x) d \mu_r(y).$$

Since it's Levi form is

$$\left(\frac{\partial^2 \varphi(z)}{\partial z_i \partial \bar{z}_j} \right) = \left(\int_{x \in K} \int_{y \in K} \frac{\partial^2 \varphi_G(y z x)}{\partial z_i \partial \bar{z}_j} d \mu_\ell(x) d \mu_r(y) \right),$$

by Lemma 3 this is positive semi-definite and has at least $n - q_0$ positive eigenvalues at every point of G . Furthermore, for $a, b \in K$

$$\begin{aligned} \varphi(a z b) &= \int_{x \in K} \int_{y \in K} \varphi_G((y a) z (b x)) d \mu_\ell(x) d \mu_r(y) \\ &= \int_{x \in K} \int_{y \in K} \varphi_G(y z x) d \mu_\ell(x) d \mu_r(y) \\ &= \varphi(z). \end{aligned}$$

Let \mathfrak{k}_Z and \mathfrak{k}_S be the Lie algebras of K_Z and K_S , respectively. There exists $a \in G$ such that $K_S \subset aU(k)a^{-1}$. The isomorphism $\alpha : G \ni x \mapsto axa^{-1} \in G$ induces the isomorphism $\alpha : \mathfrak{g} \mapsto \mathfrak{g}$ of complex Lie algebras. Since the Lie algebra $\mathfrak{u} = \{x : \bar{t}x + x = 0\}$ of all skew-Hermitian matrices is the Lie algebra of the unitary group $U(k)$ and $\mathfrak{k}_S \cap \sqrt{-1}\mathfrak{k}_S \subset \alpha(\mathfrak{u} \cap \sqrt{-1}\mathfrak{u})$, we have $\mathfrak{u} \cap \sqrt{-1}\mathfrak{u} = 0$ and then

$$q = \dim_{\mathbb{C}} \mathfrak{k} \cap \sqrt{-1}\mathfrak{k} = \dim_{\mathbb{C}} \mathfrak{k}_Z \cap \sqrt{-1}\mathfrak{k}_Z = q_0.$$

REFERENCES

- [1] A. Andreotti and H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France, **99** (1962), 193–259.
- [2] K. Iwasawa, *On some types of topological groups*, Annals of Math., **50** (1949), 507–558.
- [3] H. Kazama, *On pseudoconvexity of complex abelian Lie groups*, J. Math. Soc. Japan, **25** (1973), 329–333.
- [4] H. Kazama, *On pseudoconvexity of complex Lie groups*, Mem. Fac. Sci. Kyushu Univ., **27** (1973), 241–247.
- [5] Y. Matsushima, *Espaces homogènes de Stein des groupes de Lie complexes*, Nagoya Math. J., **16** (1960), 205–218.

- [6] Y. Matsushima and A. Morimoto, *Sur certains espaces fibrés holomorphes sur une variété de Stein*, Bull. Soc. Math. France, **88** (1960), 137–155.
- [7] A. Morimoto, *Non-compact complex Lie groups without non-constant holomorphic functions*, Proc. Conf. on Complex Analysis, Minneapolis (1965), 256–272.
- [8] A. Morimoto, *On the classification of non-compact complex abelian Lie groups*, Trans. Amer. Math. Soc., **123** (1966), 200–228.
- [9] S. Nakano, *On the inverse of monoidal transformation*, Publ.RIMS, **6** (1970), 483–502.
- [10] S. Takeuchi, *On completeness of holomorphic principal bundles*, Nagoya Math. J., **57** (1974), 121–138.

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