BULL. AUSTRAL. MATH. SOC. 16A76
VOL. 35 (1987) 73-80.

# DIVISION DISTRIBUTIVELY GENERATED NEAR-RINGS II

V. THARMARATNAM

The definition of division d.g. near-rings given by the author in 1976 has been found to be too restrictive. In this paper we generalise the definition of division d.g. near-rings and extend the results obtained earlier for division d.g. near-rings to the new wider class of near-rings.

## 1. Introduction.

In [6] we defined a non-zero topological distributively generated (d.g.) near-ring $R$ with identity to be a division topological d.g. near-ring if every continuous non-zero right $R$-endomorphism of the topological d.g. right $R$-group $R^+$ is an $R$-automorphism; that is, every continuous non-zero d.g. right $R$-endomorphism of the topological d.g. right $(R,T_0(R))$-group $(R^+,T_0(R))$ is a d.g. right $R$-automorphism. In this paper we generalise the above definition, which is in a sense too restrictive, and extend the results obtained in [6] to this wider class of near-rings. We now define a non-zero topological d.g. near-ring $R$ with the identity to be a division topological d.g. near-ring if $R^+$ is generated topologically by a distributive semigroup $S$ such that every continuous non-zero d.g. right $R$-endomorphism of the topological d.g.

Received 26 February 1986.





https://doi.org/10.1017/S0004972700013058 Published online by Cambridge University Press

right $(R,S)$-group $(R^+,S)$ is a d.g. right $R$-automorphism. Clearly division topological d.g. near-rings as defined in [6] continue to be division topological d.g. near-rings under this new definition.

## 2. Preliminaries.

Throughout this paper we will assume (i) that the term near-ring refers to a non-zero right near-ring with identity; (ii) the basic definitions and notation given in [4], [5] and [6]; and (iii) that $R$ is a topological d.g. near-ring with identity $1$, $S$ is a distributive semi-group generating $R^+$ topologically, $0$ and $1$ are in $S$, $T_0(R)$ is the set of distributive elements in $R$, $T(R) = T_0(R)\backslash\{0\}$, and $S^* = S\backslash\{0\}$.

DEFINITION 2.1. $(\Omega,\Lambda)$ is said to be a *topological d.g. right $(R,S)$-group* if

(i) $\Omega$ is a topological right $R$-group;

(ii) $\Lambda$ is a subset of $\Lambda_0$ (the set of all distributive elements in $\Omega$), $0 \in \Lambda$ and $\Lambda S \subseteq \Lambda$;

(iii) $\Lambda$ generates $\Omega$ topologically.

DEFINITION 2.2. $\phi$ is said to be a *continuous d.g. right $R$-homomorphism* of the topological d.g. right $(R,S)$-group $(\Omega,\Lambda)$ into the topological d.g. right $(R,S)$-group $(\Omega^1,\Lambda^1)$ if $\phi$ is a continuous right $R$-homomorphism from $\Omega$ into $\Omega^1$ such that $\phi(\Lambda) \subseteq \Lambda^1$. If in addition $\phi(\Omega) = \Omega^1$ and $\phi(\Lambda) = \Lambda^1$ then $\phi$ is said to be a *continuous d.g. right $R$-epimorphism*.

DEFINITION 2.3. $\Delta$ is said to be a *d.g. right $R$-subgroup* of the topological d.g. right $(R,S)$-group $(\Omega,\Lambda)$ if

(i) $\Delta$ is a subgroup of $\Omega$ and $\Delta R \subseteq \Delta$;

(ii) $\Lambda \cap \Delta$ generates $\Delta$ topologically.

DEFINITION 2.4. A subgroup $\rho$ of $R^+$ is said to be

(i) a *right $R$-module if* $\rho R \subseteq \rho$;

(ii) a *d.g. right $(R,S)$-module* if $\rho R \subseteq \rho$ and $\rho \cap S$ generates $\rho$ topologically;

(iii) a *right ideal* of $R$ if $\rho$ is a normal subgroup of $R^+$ and $\rho R \subseteq \rho$;

https://doi.org/10.1017/S0004972700013058 Published online by Cambridge University Press

(iv) a *d.g. right $(R,S)$-ideal* if $\rho$ is a d.g. right $(R,S)$-module and a right ideal of $R$.

## 3. Division topological d.g. near-rings.

DEFINITION 3.1. A topological d.g. near-ring $R$ is said to be a *division topological d.g. near-ring* if $R$ has a distributive semigroup $S$ generating $R^+$ topologically such that every continuous non-zero d.g. right $R$-endomorphism of the topological d.g. right $(R,S)$-group $(R^+,S)$ is a d.g. right $R$-automorphism.

If we wish to specify the distributive semigroup $S$ then we shall speak of the division topological d.g. near-ring $(R,S)$.

PROPOSITION 3.1. *A topological d.g. near-ring $R$ is a division topological d.g. near-ring if and only if*

*(i) $R$ has no non-trivial closed right ideals;*

*(ii) $S^*$ forms a multiplicative group for some distributive semigroup $S$ generating $R^+$ topologically.*

Proof. Suppose $(R,S)$ is a division topological d.g. near-ring. Since the kernel of a continuous non-zero d.g. right $R$-endomorphism of the topological d.g. right $(R,S)$-group $(R^+,S)$ is zero, $R$ has no non-trivial closed right ideals. Now let $s \in S^*$ and define $\psi : R \longrightarrow R$ by $\psi(x) = sx$. Then $\psi$ is a continuous non-zero d.g. right $R$-endomorphism of the topological d.g. right $(R,S)$-group $(R^+,S)$ and so is a d.g. right $R$-automorphism. Hence $\psi(S) = S$ and $\psi(S^*) = S^*$. Thus there exists $t \in S^*$ such that $st = 1$. Consequently $S^*$ forms a multiplicative group.

Conversely, suppose $R$ is a topological d.g. near-ring satisfying (i) and (ii) and $\psi$ is a continuous non-zero d.g. right $R$-endomorphism of the topological d.g. right $(R,S)$-group $(R^+,S)$. Then by (i) we have $\ker \psi = 0$ and by Proposition 1.3 of [5], $(\psi(R), \psi(S))$ is a non-zero topological d.g. right $(R,S)$-group. Now $\psi(S) \subseteq S$ and so $\psi(R)$ is a non-zero d.g. right $(R,S)$-module. Hence $\psi(R) \cap S^* \neq \emptyset$ and by (ii) we have $\psi(R) = R$. Also if $\psi(1) = t$ then $t \in S$ and given $s \in S$ we have $\psi(t^{-1}s) = \psi(1)t^{-1}s = tt^{-1}s = s$ and so $\psi(S) = S$. Thus $\psi$ is a

https://doi.org/10.1017/S0004972700013058 Published online by Cambridge University Press

d.g. right $R$-automorphism.

EXAMPLES. 1. Every division ring is a division topological d.g. near-ring. We note that a division topological d.g. near-ring which is a ring is a division ring.

2. The near-ring $R$ generated by the inner automorphisms of a finite non-abelain simple group $\Omega$ is a division discrete d.g. near-ring.

3. Let $\Omega$ be a non-abelian simple group, $S^*$ a subgroup of Aut$(\Omega)$ such that $S^*$ contains all inner automorphisms of $\Omega$, $S = S^* \cup 0$, $R$ the near-ring generated by $S$ and $\overline{R}$ the completion of $R$ under the finite topology induced by $\Omega$. Then as in Example 2 of [6], $\overline{R}$ has no non-trivial closed right ideals and so $(\overline{R},S)$ is a division topological d.g. near-ring. However we observe that by Example 2 of [6], $\overline{R}$ is not necessarily a division topological d.g. near-ring in the sense of the definition given in [6].

LEMMA 3.1. *Let* $(R,S)$ *be a topological d.g. near-ring,* $t \in S^*$ *and* $t$ *not a left divisor of zero. If* $z \in R$ *such that* $tz = 1$ *then* $z \in T(R)$.

Proof. Clearly $z \neq 0$. Now for all $x,y \in R$ we have
$t\{z(x + y) - zy - zx\} = tz(x + y) - tzy - tzx = (x + y) - y - x = 0$
and since $t$ is not a left divisor of zero we have $z(x + y) = zx + zy$. Thus $z \in T(R)$.

LEMMA 3.2. *Let* $(R,S)$ *be a topological d.g. near-ring such that for each* $t \in S^*$, $t^{-1}$ *exists and* $t^{-1} \in T(R)$. *If* $S_1^*$ *is the semigroup of* $T(R)$ *generated by* $S^* \cup S^{*-1}$ *then* $S_1^*$ *is a multiplicative group.*

Proof. If $t \in S_1^*$ we have $t = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots\cdots s_n^{\varepsilon_n}$ where $s_i \in S^*$ and $\varepsilon_i = \pm 1$ and so $t_1 = s_n^{-\varepsilon_n} \cdots\cdots s_2^{-\varepsilon_2} s_1^{-\varepsilon_1} \in S_1^*$ and $tt_1 = 1$.

Now by Proposition 3.1, Lemma 3.1 and Lemma 3.2 we have

PROPOSITION 3.2. *$R$ is a division topological d.g. near-ring if and only if*

*(i)* $R$ *has no non-trivial closed right ideals;*

https://doi.org/10.1017/S0004972700013058 Published online by Cambridge University Press

*(ii)* $R$ *has no non-trivial d.g. right* $(R,S)$*-modules for some distributive semigroup* $S$ *generating* $R^+$ *topologically.*

PROPOSITION 3.3. *If* $R$ *is a division topological d.g. near-ring then*

*(i)* $R$ *has no non-zero distributive left divisors of zero;*

*(ii)* $R$ *has no non-zero nilpotent distributive elements;*

*(iii) The only distributive idempotents in* $R$ *are* $0$ *and* $1$ .

Proof. The results follow from the fact that $R$ has no non-trivial closed right ideals.

Note that by Example 2 of [6], a division topological d.g. near-ring can have non-zero distributive right divisors of zero.

PROPOSITION 3.4. *If* $(R,S)$ *is a finite d.g. near-ring having every right ideal as a d.g. right* $(R,S)$*-module and if* $S^*$ *contains no left divisors of zero then* $R$ *is a division d.g. near-ring.*

Proof. Let $s \in S^*$ . Then since $s$ is not a left divisor of zero and $R$ is finite we have $sR = R$ . Thus there exists $z \in R$ such that $sz = 1$ and by Lemma 3.1 we have $z \in T(R)$ . Now define $S_1^*$ as in Lemma 3.1 and let $S_1 = S^* \cup 0$ . Then $S_1^*$ is a multiplicative group, $S_1$ generates $R^+$ and $S_1 \supseteq S$ . Hence every right ideal is a d.g. right $(R,S_1)$-module and consequently $R$ has no non-trivial right ideals. Thus $(R,S_1)$ is a division d.g. near-ring.

We observe that by Example 3 of [6], the condition that every right ideal is a d.g. right $(R,S)$-module is essential in the above Proposition.

COROLLARY. *If* $R$ *is a finite d.g. near-ring having no non-zero distributive left divisors of zero then* $T(R)$ *forms a multiplicative group.*

Now as in [6] we have the following results.

PROPOSITION 3.5. *Let* $R$ *be a finite d.g. near-ring having no non-trivial right ideals. Then* $R$ *is a division d.g. near-ring.*

COROLLARY 1. *If* $R$ *is a finite division d.g. near-ring which is not a ring then there exists a finite non-abelian simple group* $\Omega$ *such that* $R$ *is near-ring isomorphic to the near-ring* $R_0(\Omega)$ *of all maps of* $\Omega$ *into itself which take* $0_\Omega$ *onto itself.*

https://doi.org/10.1017/S0004972700013058 Published online by Cambridge University Press

COROLLARY 2. *If* $(R,S)$ *is a finite division d.g. near-ring then* $(R,T_0(R))$ *is a division d.g. near-ring.*

By Example 2 of [6], we see that Corollary 2 is not true when $R$ is infinite.

## 4. Right primitive topological d.g. near-rings.

DEFINITION 4.1. A non-zero topological d.g. right $(R,S)$-group $(\Omega,\Lambda)$ is said to be irreducible if

(i) $\Omega$ has no non-trivial closed normal right $R$-subgroups; and

(ii) $(\Omega,\Lambda)$ has no non-trivial d.g. right $R$-subgroups.

DEFINITION 4.2. A non-zero subgroup $\rho$ of the topological d.g. right $(R,S)$-group $(R^+,S)$ is said to be an irreducible d.g. right $(R,S)$-module if $(\rho,\rho \cap S)$ is an irreducible d.g. right $(R,S)$-group.

DEFINITION 4.3. A topological d.g. near-ring $R$ is said to be right primitive if there exists a faithful, irreducible topological d.g. right $(R,S)$-group $(\Omega,\Lambda)$ for some distributive semigroup $S$ generating $R^+$ topologically.

DEFINITION 4.4. The centraliser $S_1$ of the topological d.g. right $(R,S)$-group $(\Omega,\Lambda)$ is the set of continuous d.g. right $R$-endomorphisms of $(\Omega,\Lambda)$.

Now proceeding in a manner analogous to that in [6] we can obtain the following results:

THEOREM 1. *Let* $R$ *be a topological d.g. near-ring having a discrete faithful, irreducible d.g. right* $(R,S)$*-group* $(\Omega,\Lambda)$. *If* $\overline{R}_\rho$ *is the completion of* $R$ *under the finite topology induced by* $\Lambda$ *on* $R$ *we have*

(i) $\overline{R}_\rho$ *is a complete topological d.g. near-ring having a closed irreducible d.g. right* $(\overline{R}_\rho,S)$*-module;*

(ii) $\overline{R}_\rho$ *is right primitive;*

(iii) $\overline{R}_\rho$ *is a simple topological d.g. near-ring;*

(iv) *there exists a division d.g. near-ring* $(R\ ,S\ )$ *such that* $\Omega$

*is a free left* $(R_1,S_1)$*-group of the variety* $v(R_1^+)$ *of left* $(R_1,S_1)$*-groups generated by the left* $(R_1,S_1)$*-group* $R_1^+$ *and* $\overline{R}_\rho$ *is topological near-ring isomorphic to the topological d.g. near-ring of* $R_1$*-endomorphisms of* $\Omega$ *;*

(v) $\overline{R}_\rho$ *has idempotents* $\overline{e}$ *such that* $R_1$ *is near-ring isomorphic to* $\overline{e}\,\overline{R}_\rho\,\overline{e}$ .

THEOREM 2. *If* $R$ *is a simple topological d.g. near-ring with an irreducible d.g. right* $(R,S)$*-module* $J$ *for some* $S$ *then* $R$ *is right primitive.*

THEOREM 3. *If* $R$ *is a topological d.g. near-ring and* $J$ *an irreducible d.g. right* $(R,S)$*-module for some* $S$ *such that* $J^2 \neq 0$ *then there exists* $e \in J \cap S$ *such that* $e^2 = e$ *and* $J = eR$ .

THEOREM 4. *Suppose* $R$ *is a simple topological d.g. near-ring and* $e$ *an idempotent in some* $S$ *such that* $eR$ *is a discrete d.g. right* $(R,S)$*-module. Then* $eR$ *is an irreducible d.g. right* $(R,S)$*-module if and only if* $eRe$ *is a division discrete d.g. near-ring.*

THEOREM 5. *Suppose* $(R_1,S_1)$ *is a division discrete d.g. near-ring and* $R$ *is the topological d.g. near-ring of* $R_1$*-endomorphisms of a* $v(R_1^+)$*-free left* $(R_1,S_1)$*-group* $\Omega$ *with basis* $\Lambda$ . *Then* $R$ *is right primitive.*

THEOREM 6. *If* $R$ *is a discrete d.g. near-ring satisfying the descending chain condition for right ideals then the following three statements are equivalent:*

(i) $R$ *is simple and has an irreducible d.g. right* $(R,S)$*-module for some* $S$*;*

(ii) $R$ *is right primitive;*

(iii) $R$ *is near-ring isomorphic to the endomorphism d.g. near-ring of a* $v(R_1^+)$*-free left* $(R_1,S_1)$*-group with a finite basis, where* $(R_1,S_1)$ *is a division discrete d.g. near-ring.*

## References

[1] A. Fröhlich, "The near-ring generated by the inner automorphisms of of a finite simple group", *J. London Math. Soc.* 33 (1958) 95-107.

[2] R.R. Laxton, "Primitive distributively generated near-rings", *Mathematika* 8 (1961), 142-158.

[3] S. Ligh, "On division near-rings", *Canad. J. Math.* 21 (1969), 1366-1371.

[4] V. Tharmaratnam, "Complete primitive distributively generated near-rings", *Quart. J. Math. (Oxford)* 18 (1967), 293-313.

[5] V. Tharmaratnam, "Endomorphism near-ring of a relatively free group", *Math. Z.* 113 (1970), 119-135.

[6] V. Tharmaratnam, "Division d.g. near-rings", *J. London Math. Soc.* (2), 14 (1976), 135-147.

Department of Mathematics and Statistics,
University of Jaffna
Thirunelvely
Sri Lanka.

https://doi.org/10.1017/S0004972700013058 Published online by Cambridge University Press