New Frontiers in Black Hole Astrophysics
Proceedings IAU Symposium No. 324, 2016
Andreja Gomboc, ed.

doi:10.1017/S1743921317000035

A pseudo-Newtonian description of any stationary space-time

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Abstract. Since the first investigations into accretion onto black holes, astrophysicists have proposed effective Newtonian-like potentials to mimic the strong-field behavior of matter near a Schwarzschild or Kerr black hole. On the other hand, the fields of neutron stars or black holes in many of the alternative gravity theories differ from the idealized Schwarzschild or Kerr field which would require a number of new potentials. To resolve this, we give a Newtonian-like Hamiltonian which almost perfectly mimics the behavior of test particles in any given stationary space-time. The properties of the Hamiltonian are excellent in static space-times such as the Schwarzschild black hole, but become worse for space-times with gravito-magnetic or dragging effects such as near the Kerr black hole.

Keywords. gravitation, relativity, black hole physics, accretion, gravitational lensing

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1. Introduction

In the description of accretion onto compact bodies and violent relativistic events such as tidal disruptions either through analytical or numerical means one must employ a number of approximations to be able to give any observational predictions. One of the popular ways how to simplify the description of all the complicated physics of the matter, electromagnetic fields, and radiation in such processes is to use a completely Newtonian framework and only modify the gravitational field to somehow mimic the relativistic gravitational field of a compact object (e.g. Paczyńsky & Wiita (1980)). This, typically *ad hoc*, simplification is called the pseudo-Newtonian approach.

Here, we present an approach to "pseudo-Newtonization" which is systematic, taking only the space-time metric in a convenient set of coordinates and giving a pseudo-Newtonian prescription without prior knowledge of the structure of orbits which are supposed to be mimicked by the pseudo-Newtonian dynamics. This approach is inspired by previous works of Tejeda & Rosswog (2013, 2015), Sarkar *et al.* (2014), and Ghosh *et al.* (2014), but expands the stated works in generality and elucidates the pattern and theoretical foundation of these pseudo-Newtonian prescriptions. The following is only an overview of the results and a full description can be found in Witzany & Lämmerzahl (2016). In the following we use the $G = c = 1$ geometrized units

2. Overview

We can formally "Newtonize" a space-time by taking every orbit of a particle in it and rescaling its velocity by its individual γ-factor, $\gamma = 1/\sqrt{1-v^2}$. In return, the trajectories approaching the speed of light in the original, relativistic picture approach infinite velocities in the new, "Newtonized" picture.

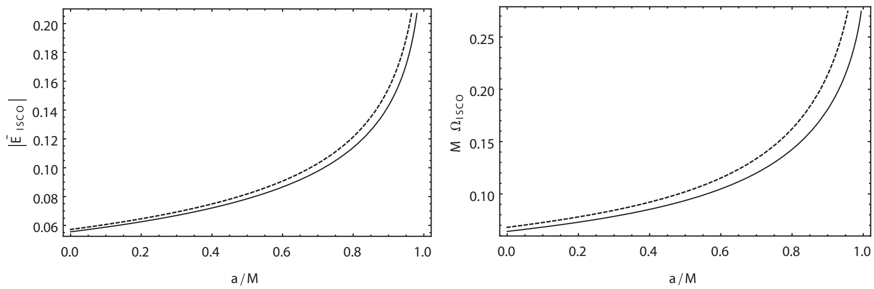

Figure 1. Specific binding energy $\tilde{E}_{\rm ISCO}$ and angular frequency $\Omega_{\rm ISCO}$ in Kerr space-time (dashed) and the pseudo-Newtonian dynamics (full line) as a function of black hole spin a/M.

In curved space-time, the quantity generalizing the notion of the γ-factor is the specific energy $-u_t$, where $u^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}\tau$ is the particle four-velocity and $t = x^0$ is the Killing time (the time with respect to which the space-time is stationary). The "Newtonization" in curved space-time can then be done by using the following Hamiltonian as the new fundamental test-particle Hamiltonian

$$H_{\rm PN} = -\frac{1}{2}\frac{g^{ij}}{g^{00}}u_i u_j - \frac{1}{2}\left(\frac{1}{g^{00}} + 1\right) + \omega^i u_i (\omega^i u_i - \sqrt{\mathcal{D}})\,, \tag{2.1}$$

where $\mathcal{D} = (\omega^i u_i)^2 - (g^{ij}u_i u_j + 1)/g^{00}$ and $\omega^i \equiv g^{0i}/g^{00}$ is the dragging velocity. In static space-times ($g^{0i} = 0$) we obtain the Lagrangian

$$L_{\rm PN} = -\frac{1}{2}\frac{g_{ij}}{g_{00}}\dot{x}^i\dot{x}^j + \frac{1}{2}(g_{00} + 1), \tag{2.2}$$

which generalizes the previous results of Tejeda & Rosswog (2013, 2015) and Sarkar *et al.* (2014). Furthermore, the theoretical construction also gives direct estimates of various errors in frequency and energy with respect to the relativistic counter-part, allows to construct a description of light-rays, and leads to an improved set of pseudo-Newtonian fluid equations in the Schwarzschild space-time (see Witzany & Lämmerzahl (2016) for details).

An astrophysically important application of this formalism is in the Kerr space-time where the formula (2.1) yields the distribution of angular momenta over circular orbits identical to the Kerr space-time, and the energies and frequencies are given within a few-percent error. This is shown in Fig. 1 on the properties of the innermost stable circular orbit (ISCO) at different black-hole spins a.

3. Conclusion

The herein presented systematic pseudo-Newtonization of stationary space-times generally works very well for the description of test particles; especially in the case of static space-times, and with promising features in the Kerr space-time.

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https://doi.org/10.1017/S1743921317000035 Published online by Cambridge University Press