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$A^{-\infty}$-INTERPOLATION IN THE BALL

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We give a necessary and sufficient condition for a sequence $\{a_k\}_k$ in the unit ball of $\mathbb{C}^n$ to be interpolating for the class $A^{-\infty}$ of holomorphic functions with polynomial growth. The condition, which goes along the lines of the ones given by Berenstein and Li for some weighted spaces of entire functions and by Amar for H^∞ functions in the ball, is given in terms of the derivatives of $m \geq n$ functions $F_1, \dots, F_m \in A^{-\infty}$ vanishing on $\{a_k\}_k$.

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1. Introduction and main result

Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$ and S^n its boundary. The space of holomorphic functions with polynomial growth, defined as

$$A^{-\infty} = \left\{ f \in H(\mathbb{B}_n) : \sup_{z \in \mathbb{B}_n} \frac{\log |f(z)|}{\log\left(\frac{e}{1-|z|}\right)} < \infty \right\},$$

is the smallest algebra of holomorphic functions that contains the class H^∞ of bounded functions and is closed by differentiation. $A^{-\infty}$ can be thought of as the union of the spaces

$$A^{-p} = \left\{ f \in H(\mathbb{B}_n) : \|f\|_{A^{-p}} =: \sup_{z \in \mathbb{B}_n} (1-|z|)^p |f(z)| < \infty \right\}, \quad p > 0,$$

as well as the union of the weighted Bergman spaces

$$B^p_\alpha = \left\{ f \in H(\mathbb{B}_n) : \int_{\mathbb{B}_n} |f(z)|^p (1-|z|)^{\alpha-1} dm(z) < \infty \right\} \quad p, \alpha > 0.$$

The norms $\|\cdot\|_{A^{-p}}$ provide a structure of Fréchet space for $A^{-\infty}$.

Given the characteristic growth of the $A^{-\infty}$ functions, a sequence of different points $\{a_k\}_k$ in $\mathbb{B}_n$ is called *$A^{-\infty}$-interpolating* when for any sequence $\{v_k\}_k$ in some space

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$$\ell^{-p}(\{a_k\}) = \left\{\{v_k\}_k \subset \mathbb{C} : \|\{v_k\}\|_{\ell^{-p}} =: \sup_{k\in\mathbb{N}}(1-|a_k|)^p|v_k| < \infty\right\}, \quad p > 0$$

there exist $m > 0$ and $f \in A^{-m}$ such that $f(a_k) = v_k$ for all k.

It is important to stress that in the previous definition the values p and m such that $\{v_k\} \in \ell^{-p}$ and $f \in A^{-m}$ need not be the same. If we require $m = p$ we obtain the more restrictive notion of A^{-p}*-interpolating* sequence, which here we will not discuss. In the unit disk Seip obtained a characterization of A^{-p}-interpolating sequences in terms of a Nyquist density measured with the hyperbolic metric **[13]**. Some partial results for higher dimensions can be found in **[10]** and [7].

Bruna and Pascuas characterized the $A^{-\infty}$-interpolating sequences in dimension 1 by a condition which is essentially Korenblum's condition for the $A^{-\infty}$ zero-sequences made invariant by automorphisms **[4]**. The techniques involved in the proofs, which rely on Korenblum's work **[9]**, are specific of the one-variable analysis, and cannot be adapted to higher dimensions. Some qualitative properties of the $A^{-\infty}$-interpolating sequences in case $n > 1$ are described in **[11]**.

In this paper we obtain a necessary and sufficient condition for a sequence to be $A^{-\infty}$-interpolating in the unit ball. The condition obtained, which goes along the lines of the results given by Berenstein-Li [2] for weighted spaces of entire functions and by Amar [1] for bounded holomorphic functions in the ball, requires a certain growth of the derivatives of a $\mathbb{C}^m$ $(m \geq n)$ valued holomorphic function vanishing on the sequence. The proofs are modelled after **[2]**, and the fact that $A^{-\infty}$ is closed by differentiation plays a crucial role.

Given a $\mathbb{C}^m$ valued holomorphic function $F = (F_1, \ldots, F_m)$ let $Z(F)$ denote the common zero set of the F_j. Also, D_vF_j stands for the derivative of F_j in the direction given by v.

Main Theorem. *Let $\{a_k\}_k$ be a sequence in the ball and let $m \geq n$. Then $\{a_k\}_k$ is $A^{-\infty}$-interpolating if and only if there exist constants $c, \varepsilon > 0$ and a function $F : \mathbb{B}_n \to \mathbb{C}^m$ such that $F_j \in A^{-\infty}$, $j = 1, \ldots, m$, and*

$$\sum_{j=1}^{m} |D_vF_j(a_k)| \geq \varepsilon(1-|a_k|)^c \qquad \forall k \in \mathbb{N} \quad \forall v \in S^n. \tag{1}$$

It is convenient to allow a number $m \geq n$ of function if we don't want to restrict ourselves to sequences $\{a_k\}_k$ which are complete intersections.

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2. Proofs and corollaries

Proof of the Main Theorem. Suppose that $\{a_k\}_k$ is $A^{-\infty}$-interpolating and take $M > 0$ so that $\sum_k (1-|a_k|^2)^M < \infty$. Such an M exists because for some $p, c > 0$ the hyperbolic balls

$$E_k = \{z \in \mathbb{B}_n : |\phi_{a_k}(z)| \leq c(1-|a_k|^2)^p\}$$

are pairwise disjoint ([**11**, Theorem 2]), and therefore the sum of their volumes is finite. Here ϕ_{a_k} denotes the automorphism of $\mathbb{B}_n$ exchanging 0 and a_k.

The structure of Fréchet space enjoyed by $A^{-\infty}$ implies that given M there exist constants N and K such that for every $\{v_k\}_k$ with $\|\{v_k\}\|_{\ell^{-M}} \leq 1$ there is $f \in A^{-N}$ with $\|f\|_{A^{-N}} \leq K$ and $f(a_k) = v_k$ (see for example [**11**, Lemma 2.1]). Consider then $g_k \in A^{-N}$ with $\|g_k\|_{A^{-N}} \leq K$, $g_k(a_j) = \delta_{jk}(1-|a_k|^2)^{-M}$, and define

$$F_j(z) = \sum_{k=1}^{\infty} (1-|a_k|^2)^{2M} g_k^2(z)(z^j - a_k^j) \qquad j = 1, \ldots, n,$$

where z^j denotes the j-coordinate of z. It is clear that each F_j vanishes on $\{a_k\}_k$ and $F_j \in A^{-2N}$. Taking derivatives we also see that

$$\frac{\partial F_j}{\partial z_l}(a_k) = \delta_{jl}\, g_k^2(a_k)(1-|a_k|^2)^M = \delta_{jl},$$

and therefore (1) holds:

$$\sum_{j=1}^{n} |D_v F_j(a_k)| \geq \max_{j=1,\ldots,m} |D_v F_j(a_k)| = \max_{j=1,\ldots,m} |v_j| > 1/\sqrt{n}.$$

Assume now (1). For any $j = 1, \ldots, m$ and $v \in S^n$ consider the one variable function.

$$F_{j,v}(\lambda) := F_j(a_k + \delta(1-|a_k|)v\lambda),$$

where $\delta \in (0, 1)$ is some fixed constant.

Lemma 1. *There exist $\varepsilon_1, \varepsilon_2, c_1, c_2 > 0$ such that for all λ with $|\lambda| = \varepsilon_1(1-|a_k|^2)^{c_1}$ and for all $v \in S^n$:*

$$\sum_{j=1}^{m} |F_j(a_k + \delta(1-|a_k|)v\lambda)| \geq \varepsilon_2(1-|a_k|^2)^{c_2}.$$

Proof. Since $F'_{j,v}(0) = \delta(1-|a_k|)D_vF_j(a_k)$, the hypotheses imply that there exist constants $c, \varepsilon > 0$ and an index j_v such that

$$|F'_{j_v,v}(0)| \geq \varepsilon(1-|a_k|^2)^c. \tag{2}$$

Also, there exist $C, p > 0$ for which

$$|F_{j_v,v}(\lambda)| \leq C(1-|a_k|^2)^{-p} \qquad |\lambda| \leq 1,$$

because $F_j \in A^{-\infty}$.

An application of the maximum principle shows then that the function $g_v(\lambda) := F_{j_v,v}(\lambda)/\lambda$ also satisfies

$$|g_v(\lambda)| \leq C(1-|a_k|^2)^{-p} \qquad |\lambda| \leq 1.$$

Hence the function defined as

$$G_v(\lambda) := \frac{(1-|a_k|^2)^p}{3C}(g_v(\lambda) - g_v(0))$$

has $G_v(0) = 0$ and $|G_v(\lambda)| < 1$ for $|\lambda| < 1$, so by the Schwarz lemma

$$|G_v(\lambda)| \leq |\lambda| \qquad \text{for} \qquad |\lambda| \leq 1.$$

This implies that no zero $a \neq 0$ of $F_{j_v,v}$ is inside the disc of radius $\frac{\varepsilon}{3C}(1-|a_k|^2)^{c+p}$ centred at 0:

$$|a| \geq |G_v(a)| = \frac{(1-|a_k|^2)^p}{3C}|F'_{j_v,v}(0)| \geq \frac{\varepsilon}{3C}(1-|a_k|^2)^{c+p}. \tag{3}$$

An application of Taylor's formula together with (2) and this last inequality shows finally that for some $c_2, \varepsilon_2 > 0$:

$$|F_{j_v,v}(\lambda)| \geq \varepsilon_2(1-|a_k|^2)^{c_2} \qquad \text{for} \qquad |\lambda| = \frac{c_k}{2}. \qquad \square$$

Call now $\alpha_k = \delta(1-|a_k|)$ and $c_k = \varepsilon_1(1-|a_k|^2)^{c_1}$, where ε_1 and c_1 are given by Lemma 1. Note that inequality (3) in the proof of Lemma 1 shows that $|a_j - a_k| \geq c_k$ for every $a_j \neq a_k$, so the balls

$$B_k = \{z \in \mathbb{B}_n : |z - a_k| \leq 1/2\alpha_k c_k\}.$$

are pairwise disjoint.

Let's now turn to solve the interpolation problem. We do it in a standard way, first finding a smooth interpolating function with the required growth and afterwards solving a $\bar{\partial}$-problem in order to make the smooth solution holomorphic.

Let $\mathcal{X} \in \mathcal{C}^{\infty}$ be a cut-off function such that $\mathcal{X} = 1$ for $|z| < 1$, $\mathcal{X} = 0$ for $|z| > 2$ and $|\nabla \mathcal{X}| \leq C$. Given $\{v_k\} \in \ell^{-p}(\{a_k\})$ consider the smooth interpolating function defined by

$$E(z) = \sum_{k=1}^{\infty} v_k \mathcal{X}\left(\frac{|z - a_k|^2}{1/4\alpha_k^2 c_k^2}\right).$$

Since $\bar{\partial}E$ has the support in $\cup_k\{z : 1/4\,\alpha_k c_k \leq |z - a_k| \leq 1/2\,\alpha_k c_k\}$, there exist $\varepsilon', c' > 0$ such that $|F(z)| \geq \varepsilon'(1 - |z|^2)^{c'}$ for $z \in supp(\bar{\partial}E)$. Hence for every $M > 0$ there is $K > 0$ such that

$$\int_{\mathbf{B}_n} \frac{|\bar{\partial}E|^2}{|F(z)|^M}(1 - |z|^2)^K dm(z) < \infty,$$

and we can apply Theorem 2.6 in [**8**]: there are $\bar{\partial}$-closed $(0, 1)$-forms $\omega_1, \ldots, \omega_m$ and some $K > 0$ such that $\bar{\partial}E = \omega_2 F_1 + \cdots + \omega_m F_m$ and:

$$\int_{\mathbf{B}_n} |\omega_i(z)|^2(1 - |z|^2)^K dm(z) < \infty.$$

This is a result on the Koszul complex induced by $F_1, \ldots, F_m$ which as far as we know goes back ultimately to Hörmander [**5**].

We now take the solutions u_i to the equation $\bar{\partial}u = \omega_i$ provided by [**6**], which satisfy the growth condition:

$$\int_{\mathbf{B}_n} |u_i(z)|^2(1 - |z|^2)^K dm(z) < \infty.$$

Finally define

$$H = E - u_i \cdot F_1 - \cdots - u_m \cdot F_m.$$

This is a holomorphic function with $H(a_k) = v_k$ for all k, and since there exists $K > 0$ with

$$\int_{\mathbf{B}_n} |H(z)|^2(1 - |z|^2)^K dm(z) < \infty$$

it belongs to some weighted Bergman space, and therefore to $A^{-\infty}$ ([**5**, Lemma 3]). $\square$

As in [**2**], some corollaries can be obtained from the Main Theorem.

Corollary 1. *Let $\{a_k\}_k$ be a sequence in the ball and let $m \geq n$. Then $\{a_k\}_k$ is $A^{-\infty}$-interpolating if and only if there exists a function $F : \mathbb{B}_n \to \mathbb{C}^m$ such that $F_j \in A^{-\infty}$, $j = 1, \ldots, m$, $\{a_k\} \subset Z(F)$, and there exist constants $c, \varepsilon > 0$ such that for all k there is a $n \times n$ minor $\mathcal{J}$ of the Jacobian matrix $J_{F_1 \ldots F_m}$ with*

$$|\det \mathcal{J}(a_k)| \geq \varepsilon(1 - |a_k|)^c \qquad \forall k \in \mathbb{N}. \tag{4}$$

In particular, for the case of complete intersection:

Corollary 2. *Let $\{a_k\}_k$ be a sequence in the ball. Then $\{a_k\}_k$ is $A^{-\infty}$-interpolating if and only if there exist a function $F : \mathbb{B}_n \to \mathbb{C}^n$ and constants $c, \varepsilon > 0$ such that $F_j \in A^{-\infty}$, $j = 1, \ldots, n$, $\{a_k\} \subset Z(F)$, and*

$$|\det J_{F_1 \ldots F_n}(a_k)| \geq \varepsilon(1 - |a_k|)^c \qquad \forall k \in \mathbb{N}.$$

Although Corollary 1 can be obtained from the Main Theorem in an analogous way as Corollary 2.6 is obtained from the main result in **[2]**, we prefer to exhibit a simplified proof. First we have the following lemma.

Lemma 2 (**[14]**). *Let $m \geq n$ and let $\mathcal{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$. There exists a positive constant $\delta = \delta(m, n)$ such that for every $L : \mathbb{R}^n \to \mathbb{R}^m$ linear with $\|L(x)\| \geq 1$ $\forall x \in \mathcal{S}^{n-1}$ there exists a $n \times n$ minor $\mathcal{J}$ of the matrix of L such that $|\det \mathcal{J}| \geq \delta$.*

Proof. Assume $L : \mathbb{R}^n \to \mathbb{R}^m$ linear with $\|L(x)\| \geq 1$ for all $x \in \mathcal{S}^{n-1}$.

Claim: There exist $\alpha = \alpha(m, n) > 0$ and a set of n coordinates $E \subset \{1, \ldots, m\}$ such that

$$\|\pi_E \circ L(x)\| \geq \alpha \qquad x \in \mathcal{S}^{n-1},$$

being π_E the canonical projection from $\mathbb{R}^m$ to $\mathbb{R}^n$ induced by the coordinates E.

In order to see that the claim implies the lemma consider the minor $\mathcal{J}$ corresponding to the coordinates E. Denote by J the linear application from $\mathbb{R}^n$ to $\mathbb{R}^n$ whose matrix is $\mathcal{J}$, that is $J = \pi_E \circ L$, and consider a Jordan basis of unitary eigenvectors v_j of eigenvalue λ_j. Each λ_j is an eigenvalue of at least one element v_j° of the Jordan basis. The claim yields then

$$\alpha \geq \|J(v_j^\circ)\| = \|\lambda_j v_j^\circ\| = |\lambda_j|,$$

and therefore

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$$|\det \mathcal{J}| = \prod_{j=1}^{n} |\lambda_j| \geq \alpha^n.$$

Let's now prove the claim. Let $\|\pi_{E_{|L(\mathbb{R}^n)}}\|_{op}$ denote the norm of $\pi_{E_{|L(\mathbb{R}^n)}}$ as operator from $L(\mathbb{R}^n)$ to $\mathbb{R}^n$. It's enough to see that there exists a set E of n coordinates such that $\|\pi^{-1}_{E_{|L(\mathbb{R}^n)}}\|_{op} \leq \delta^{-1}$, since then

$$\|y\| \leq \delta^{-1} \|\pi_{E_{|L(\mathbb{R}^n)}}(y)\| \qquad \forall y \in L(\mathbb{R}^n)$$

and the conclusion is obvious.

Consider thus for any σ in the symmetric group $\mathcal{S}_m$ the set of coordinates $E_\sigma = \{\sigma(1), \dots, \sigma(n)\}$. Let $Gr(n; \mathbb{R}^m)$ denote the Grassmanian of n-dimensional linear sub-varieties of $\mathbb{R}^m$ and define the function $\Phi : Gr(n; \mathbb{R}^m) \to \mathbb{R}_+$ by

$$\Phi(V) = \min_{\sigma \in \mathcal{S}_m} \|\pi^{-1}_{E_{\sigma|V}}\|_{op}.$$

Φ is continuous, since it is the minimum of a finite number of continuous functions. Thus, by the compacity of the Grassmanian, Φ is bounded and the claim is proved. □

In the proof of Corollary 1 we will use, for a matrix $A = (a_{ij})_{i,j}$ the norm $\|A\|_1 = \sum_{i,j} |a_{ij}|$. For vectors v in $\mathbb{C}^n$ the norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent: there exists a constant $c_n > 0$ depending only on n such that $\|\cdot\| \leq \|\cdot\|_1 \leq c_n \|\cdot\|$.

Proof of Corollary 1. Assume first that $\{a_k\}_k$ is $A^{-\infty}$-interpolating. By the Main Theorem there exist $c, \varepsilon > 0$ and $F : \mathbb{B}_n \to \mathbb{C}^m$, $F_j \in A^{-\infty}$ satisfying (1), and the conclusion is immediately obtained by applying Lemma 2 to

$$L(v) = \frac{1}{\varepsilon(1 - |a_k|)^c} (D_v F_1(a_k), \dots, D_v F_m(a_k)).$$

For the sufficiency proceed as in [2]. Denote by $\mathcal{J}^*(a_k)$ the adjoint of $\mathcal{J}(a_k)$. Since $\mathcal{J}^{-1}(a_k) = (\det \mathcal{J}(a_k))^{-1} \mathcal{J}^*(a_k)$ and $A^{-\infty}$ is closed by differentiation, (4) implies that for some $\delta, p > 0$ independent of k

$$\|\mathcal{J}^{-1}(a_k)\| \leq \delta(1 - |a_k|)^{-p}.$$

Given $v \in S^n$ and $\omega = \mathcal{J}(a_k) \cdot v$ we have

$$\sum_{j=1}^{m} |D_v F_j(a_k)| \geq \|\mathcal{J} \cdot v\|_1 = \|\omega\|_1 \geq \frac{\|v\|_1}{\|\mathcal{J}^{-1}(a_k)\|_1} \geq \frac{1}{\delta\sqrt{n}} (1 - |a_k|)^p,$$

which by the Main Theorem implies that $\{a_k\}_k$ is $A^{-\infty}$-interpolating. □

3. Final remarks

1. Using again that $A^{-\infty}$ is closed by differentiation it is not difficult to see that Corollary 2 can also be directly obtained from the weighted Jacobi interpolation formula (see [3]):

$$H(z)=\sum_{k=1}^{\infty} v_k\left(\frac{1-|a_k|^2}{1-\bar{a}_k z}\right)^M \frac{\det g_i^j(a_k,z)}{\det \frac{\partial F_i}{\partial z_j}(a_k)},$$

where the functions $g_i^j(\zeta,z)$ solve the division problem

$$F_j(z)-F_j(\zeta)=\sum_{i=1}^{n} g_i^j(\zeta,z)(z^i-\zeta^i).$$

2. The same type of results hold for any space of *slowly increasing functions*

$$A^\lambda=\left\{f\in H(\mathbb{B}_n):\sup_{z\in\mathbf{B}_n}\frac{\log|f(z)|}{\lambda(|z|)}<\infty\right\},$$

being $\lambda:[0,1)\to\mathbb{R}^+$ an increasing function such that

(a) $\lim_{r\to 1}\lambda(r)=+\infty$

(b) $(1-r)\lambda(r)$ is decreasing

(c) there exists $c_\lambda>1$ such that $\lambda(1-r^2)\le c_\lambda\lambda(1-r)$

(d) $(1-r)\lambda'(r)$ is decreasing

These properties ensure that A^λ is closed by differentiation and the analogous computations to the ones above can be carried out similarly. Thus we obtain the following result and the corresponding corollaries:

Theorem. *Let $\{a_k\}_k$ be a sequence in the ball and let $m\ge n$. Then $\{a_k\}_k$ is A^λ-interpolating if and only if there exist constants $c,\varepsilon>0$ and a function $F:\mathbb{B}_n\to\mathbb{C}^m$ such that $F_j\in A^\lambda, j=1,\dots,m$, and*

$$\sum_{j=1}^{m}|D_v F_j(a_k)|\ge \varepsilon e^{-c\lambda(|a_k|)}\qquad \forall k\in\mathbb{N}\quad \forall v\in S^n.$$

REFERENCES

1. E. Amar, *Interpolating sequences in the ball of* $\mathbb{C}^n$, preprint (1996).

2. C. A. Berenstein and H. Q. Li, Interpolating varieties for weighted spaces of entire functions in $\mathbb{C}^n$, *Publ. Mat.* **38** (1994), 157–173.

3. B. Berndtsson, A formula for interpolation and division in $\mathbb{C}^n$, *Math. Ann.* **263** (1983), 399–418.

4. J. Bruna and D. Pascuas, Interpolation in $A^{-\infty}$, *J. London Math. Soc.* **40** (1989), 452–466.

5. L. Hörmander, Generators for some rings of analytic functions, *Bull. Amer. Math. Soc.* (1967), 943–949.

6. L. Hörmander, *An introduction to complex analysis in several variables. 2nd ed.* (North Holland Publishing Co., Amsterdam, 1973).

7. M. Jevtić, X. Massaneda and P. Thomas, Interpolating sequences for weighted Bergman spaces of the ball, *Michigan Math. J.* **43** (1996), 495–517.

8. J. Kelleher and B. A. Taylor, Finitely generated ideals in rings of analytic functions, *Math. Ann.* **193** (1971), 225–237.

9. B. Korenblum, An extension of the Nevanlinna theory, *Acta Math.* **135** (1975), 187–219.

10. X. Massaneda, A^{-p}-interpolation in the unit ball, *J. London Math. Soc.* **52** (1995), 391–401.

11. X. Massaneda, Interpolation by holomorphic functions in the unit ball with polynomial growth, *Ann. Fac. Sci. Toulouse*, to appear.

12. W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$ (Springer Verlag, Berlin, 1980).

13. K. Seip, Beurling type density theorems in the unit disk, *Invent. Math.* **113** (1993), 21–39.

14. P. J. Thomas, *Oral Communication* (1996).

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